

Homotopy Theory of Schemes

Oberseminar Algebraic Geometry – Summer 2005

Prerequisites: Basic knowledge algebraic topology (e.g. definition of homotopies, homotopy groups, Serre fibrations,...), simplicial sets, Nisnevich topology (see Oberseminar Winter 04/05).

The talks:

1. **Introduction Algebraic Topology:** Standard textbooks on algebraic topology are [Hat02, Spa66, Swi75]. *Focus on stating definitions and examples.* Definition of CW-complexes, repeat definition of singular/cellular (co-)homology, e.g. from [Hat02, Chapter 0, Chapter 2.2 resp. Appendix]. Definition of suspension and loop spaces. Explicit construction of $K(G, n)$, e.g. [Hat02, Chapter 4.2].
2. **Generalised Cohomology:** *Complementary talk providing necessary background.* Eilenberg-Steenrod axioms, universality of $H^*(-, \mathbb{Z})$, perhaps mention the Atiyah-Hirzebruch spectral sequence. [ES52, Ada74].
3. **The Steenrod Algebra and Operations:** *Focus on proofs to see how the machinery works.* Shortly state the theorem, Adém relations. As-full-as-can-be-presented-in-90-minutes construction of Steenrod operations (mod 2). [Hat02, Chapter 4.L], [Swi75, 17.1, 17.9], [Eps62].
4. **Model Categories:** References for homotopical algebra and model categories are [Hov98, Qui67, Hir03, DS95]. *Focus on stating definitions and presenting examples.* Definition of model categories, only state examples (simplicial sets, chain complexes, CW-complexes...). Quillen adjunctions and equivalences of model categories. Simplicial proper model categories, see also [GJ99].
5. **Some Consequences I + II (two talks):** *Focus on proofs to see how the machinery works.* Existence of localisations via right and left homotopies [DS95, Section 4], existence of factorisations, the small object argument [Qui67] (complete). Long exact homotopy sequences, and why certain homotopy classes of maps have (abelian) group structure [Hov98, Chapter 6]. Quillen's total derived functor theorem [GJ99, Chapter II.7], and definition of homotopy colimits and limits, see also [BK72].
6. **Simplicial Homotopy of Schemes:** Definition of Morel-Voevodsky simplicial model structure, following [MV99, Jar00]. Perhaps explain that this works for category of (pre-)sheaves on a site [Jar87].
7. **Bousfield Localisation:** Definition of Bousfield localisation [BK72, Chapter V.3+4], [Hir03], examples p -localisation, rational homotopy type, the spectral sequence?
8. **Localising $*$ $\rightarrow \mathbb{A}^1$:** Application of Bousfield localisation, definition of $\mathbb{A}^1$ -model structure, [MV99, Voe98, Jar00]. Mention homotopy theory of sites with intervals [MV99].

Now comes the wishlist for topics of invited talks, *subject to change*.

9. **Stable Homotopy of Schemes:** This might split: For the purely topological part of stable homotopy theory, it has become a fruitful analogy to think of the stable model category as a category of modules over the sphere spectrum, where ring spectra are the algebras. This allows to talk e.g. about Galois theory of commutative ring spectra. We could invite e.g. Stefan Schwede to talk about how far that analogy works.

For the other part, one could consider is the geometric one. Given the $\mathbb{A}^1$ -model category, we can construct spectra to represent generalised cohomology theories on it.

10. **Mixed Motives and $\mathbb{A}^1$ -Homotopy:** A presentation of the link (i.e. in Morel's lecture notes) that the heart of the simplicial homotopy t-structure is the category of $\mathbb{A}^1$ -invariant Nisnevich-sheaves of abelian groups on Sm/k . Rost's cycle modules as heart of the homotopy t-structure of the $\mathbb{P}^1$ -stable theory. Are motives the right coefficient systems for $\mathbb{A}^1$ -homotopy theory?
11. **The Motivic Steenrod Algebra:** Construction of Steenrod Operations and Steenrod Algebra as in [Voe03].

For questions contact:

Annette Huber-Klawitter, huber@math.uni-leipzig.de, tel. 9732185, room 3-20
Matthias Wendt, wendt@math.uni-leipzig.de, tel. 9732144, room 5-49

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