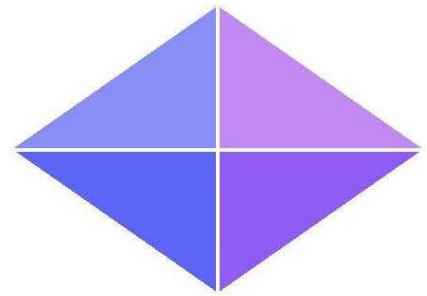


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**On a noncommutative Iwasawa main
conjecture for varieties over finite
fields**

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ON A NONCOMMUTATIVE IWASAWA MAIN CONJECTURE FOR VARIETIES OVER FINITE FIELDS

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ABSTRACT. We formulate and prove an analogue of the noncommutative Iwasawa main conjecture for ℓ -adic Lie extensions of a separated scheme X of finite type over a finite field of characteristic prime to ℓ .

1. INTRODUCTION

In [CFK⁺05], Coates, Fukaya, Kato, Sujatha and Venjakob formulate a non-commutative Iwasawa main conjecture for ℓ -adic Lie extensions of number fields. Other, partly more general versions are formulated in [HK02], [RW04], and [FK06]. To provide evidence for these Iwasawa main conjectures we formulate and prove below an analogous statement for ℓ -adic Lie extensions of a separated scheme X of finite type over a finite field $\mathbb{F}_q$ with q elements, where ℓ does not divide q .

Assume for the moment that X is a geometrically connected. Let G be a factor group of the fundamental group of X and assume that $G = H \rtimes \Gamma$ where H is a compact ℓ -adic Lie group and $\Gamma = \text{Gal}(\mathbb{F}_{q^{\ell^\infty}}/\mathbb{F}_q) \cong \mathbb{Z}_\ell$. Recall that every continuous representation ρ of the fundamental group of X on a finitely generated, projective $\mathbb{Z}_\ell$ -module gives rise to a flat and smooth $\mathbb{Z}_\ell$ -sheaf $\mathcal{M}(\rho)$ on X .

Let $R\Gamma_c(X, \mathcal{F})$ and $R\Gamma_c(\bar{X}, \mathcal{F})$ be the compact cohomology of a flat constructible $\mathbb{Z}_\ell$ -sheaf $\mathcal{F}$ on X and on the base change $\bar{X}$ of X to the algebraic closure of $\mathbb{F}_q$, respectively. Furthermore, let $\mathfrak{F}_{\bar{\mathbb{F}}_q} \in \text{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$ denote the geometric Frobenius. The Grothendieck trace formula

$$L(\mathcal{F}, T) = \prod_{i \in \mathbb{Z}} \det(1 - \mathfrak{F}_{\bar{\mathbb{F}}_q} T : H_c^i(\bar{X}, \mathcal{F}))^{(-1)^{i+1}}$$

implies that the L -function $L(\mathcal{F}, T)$ of $\mathcal{F}$ is in fact a rational function.

We write

$$\mathbb{Z}_\ell[[G]] = \varprojlim \mathbb{Z}_\ell[G/U]$$

for the Iwasawa algebra of G . Moreover, let

$$S = \{f \in \mathbb{Z}_\ell[[G]] : \mathbb{Z}_\ell[[G]]/\mathbb{Z}_\ell[[G]]f \text{ is finitely generated as } \mathbb{Z}_\ell[[H]]\text{-module}\}$$

denote Venjakob's canonical Ore set and write $\mathbb{Z}_\ell[[G]]_S$ for the localisation of $\mathbb{Z}_\ell[[G]]$ at S . We turn $\mathbb{Z}_\ell[[G]]$ into a smooth $\mathbb{Z}_\ell[[G]]$ -sheaf $\mathcal{M}(G)$ on X by letting the fundamental group of X act contragrediently on $\mathbb{Z}_\ell[[G]]$.

Recall that there exists an exact localisation sequence of algebraic K-groups

$$K_1(\mathbb{Z}_\ell[[G]]) \rightarrow K_1(\mathbb{Z}_\ell[[G]]_S) \xrightarrow{d} K_0(\mathbb{Z}_\ell[[G]], \mathbb{Z}_\ell[[G]]_S) \rightarrow 0.$$

Any endomorphism of perfect complexes of $\mathbb{Z}_\ell[[G]]_S$ -modules which is a quasi-isomorphism gives rise to an element in the group $K_1(\mathbb{Z}_\ell[[G]]_S)$ and the relative K-group $K_0(\mathbb{Z}_\ell[[G]], \mathbb{Z}_\ell[[G]]_S)$ is generated by perfect complexes of $\mathbb{Z}_\ell[[G]]$ -modules whose cohomology groups are S -torsion. Moreover, recall that for every $\mathbb{Z}_\ell$ -representation ρ of G , there exists a homomorphism

$$\rho : K_1(\mathbb{Z}_\ell[[G]]_S) \rightarrow Q(\mathbb{Z}_\ell[[\Gamma]])^\times$$

induced by sending $g \in G$ to $\det([g]\rho(g)^{-1})$, with $[g]$ denoting the image of g in Γ .

The following theorem is our analogue of the noncommutative Iwasawa main conjecture in the special situation described above:

Theorem 1.1.

- (1) $R\Gamma_c(X, \mathcal{M}(G) \otimes_{\mathbb{Z}_\ell} \mathcal{F})$ is a perfect complex of $\mathbb{Z}_\ell[[G]]$ -modules whose cohomology groups are S -torsion. Moreover, the endomorphism $\text{id} - \mathfrak{F}_{\mathbb{F}_q}$ of the complex $R\Gamma_c(\bar{X}, \mathcal{M}(G)_S \otimes_{\mathbb{Z}_\ell} \mathcal{F})$ is a quasi-isomorphism and hence, it gives rise to an element

$$\mathcal{L}_G(X/\mathbb{F}_q, \mathcal{F}) = [\text{id} - \mathfrak{F}_{\mathbb{F}_q}]^{-1} \in K_1(\mathbb{Z}_\ell[[G]]_S).$$

- (2) $d\mathcal{L}_G(X/\mathbb{F}_q, \mathcal{F}) = [R\Gamma_c(X, \mathcal{M}(G) \otimes_{\mathbb{Z}_\ell} \mathcal{F})]^{-1}$.
(3) Assume that ρ is a continuous representation of G . Then

$$\rho(\mathcal{L}_G(X/\mathbb{F}_q, \mathcal{F})) = L(\mathcal{M}(\rho) \otimes_{\mathbb{Z}_\ell} \mathcal{F}, [\mathfrak{F}_{\mathbb{F}_q}]^{-1})$$

in $Q(\mathbb{Z}_\ell[[\Gamma]])^\times$.

Let ϵ denote the cyclotomic character and decompose $\epsilon = \epsilon_f \times \epsilon_\infty$ according to the decomposition $\text{Gal}(\mathbb{F}_q(\zeta_{\ell^\infty})/\mathbb{F}_q) = \Delta \times \Gamma$. Then Theorem 1.1.(3) implies that for every $n \in \mathbb{Z}$ the leading term of the image of $\epsilon_\infty^n \rho(\mathcal{L}_G(X/\mathbb{F}_q, \mathcal{F}))$ under the isomorphism

$$Q(\mathbb{Z}_\ell[[\Gamma]]) \rightarrow Q(\mathbb{Z}_\ell[[X]]), \quad [\mathfrak{F}_{\mathbb{F}_q}]^{-1} \mapsto X + 1$$

agrees with the leading term of $L(\mathcal{M}(\epsilon_f^{-n} \rho) \otimes_{\mathbb{Z}_\ell} \mathcal{F}, q^{-n}T)$ at $T = 1$.

The central idea of the proof of the above theorem is fairly simple. The main step is to show that the cohomology groups of $R\Gamma_c(X, \mathcal{M}(G) \otimes_{\mathbb{Z}_\ell} \mathcal{F})$ are S -torsion. This can be done by reducing to the case that $G = H \times \mathbb{Z}_\ell$ with H finite. In this case, one easily verifies that

$$H_c^i(X, \mathcal{M}(G) \otimes_{\mathbb{Z}_\ell} \mathcal{F}) \cong \varprojlim_n H_c^{i-1}(\bar{Y}, \mathcal{F}) / (1 - \mathfrak{F}_{\mathbb{F}_q}^{\ell^n}) H_c^{i-1}(\bar{Y}, \mathcal{F})$$

where Y is the Galois covering of X corresponding to H . Since $H_c^{i-1}(\bar{Y}, \mathcal{F})$ is finitely generated as $\mathbb{Z}_\ell$ -module, it follows that $H_c^i(X, \mathcal{M}(G) \otimes_{\mathbb{Z}_\ell} \mathcal{F})$ is S -torsion. All other assertions of the theorem are more or less formal consequences of this fact.

More generally, we will formulate and prove the noncommutative Iwasawa main conjecture for not necessarily connected principal coverings with a profinite Galois group G which satisfies a slightly weaker condition than being an ℓ -adic Lie group. Furthermore, we will replace $\mathbb{Z}_\ell$ by an arbitrary adic $\mathbb{Z}_\ell$ -algebra.

The article is structured as follows. In Section 2 we develop the necessary terminology of infinite principal coverings. Section 3 contains a brief account on adic rings and their K-theory. In Section 4 we explain how one extend the definition and properties of $K_1(\mathbb{Z}_\ell[[G]]_S)$ to our more general setting. In Section 5 we recall how to generalise the definition of $\mathbb{Z}_\ell$ -sheaves to perfect complexes of sheaves with adic coefficients. This construction is then used in Section 6 to define an analogue of $\mathcal{M}(G)$ for arbitrary principal coverings. In Section 7 we consider the special case $G = \Gamma$. Putting everything together, we give the precise formulations and the proofs of our main results in Section 8. The proof needs an explicit description of the rightmost connecting homomorphism of the localisation sequence in a very general setting. In the appendix we derive this description under the same assumptions under which the localisation sequence is known to exist.

An analogue of the noncommutative Iwasawa main conjecture for elliptic curves over function fields in the case that ℓ is equal to the characteristic p of the field in question has been considered in [OT09]. Both F. Trihan and D. Burns have recently announced proofs of more general main conjectures in the case $\ell = p$. A proof of a

function field analogue of the related equivariant Tamagawa number conjecture is given in [Bur10]. We also point out that tremendous progress towards a proof of the noncommutative Iwasawa main conjecture for totally real fields has been achieved in [Kat06], [Har08], [Kak08], and in [RW09].

2. PRINCIPAL COVERINGS

If A is either a commutative ring or a scheme, we let $\mathbf{Sch}_A$ denote the category of schemes of finite type over A . Recall the concept of principal coverings with finite Galois group from [Gro03, Def. 2.8]. We extend this concept to a profinite setting as follows: If G is any profinite group, we write $\mathbf{NS}(G)$ for the collection of open normal subgroups of G viewed as a category with the natural inclusion maps as morphisms.

Definition 2.1. Let G be a profinite group and X a locally noetherian scheme. A *principal covering* $(f: Y \rightarrow X, G)$ of X with *Galois group* G is a covariant functor

$$F: \mathbf{NS}(G) \rightarrow \mathbf{Sch}_X, \quad U \mapsto (f_U: Y_U \rightarrow X),$$

together with a right operation of G on F such that for any U in $\mathbf{NS}(G)$,

- (1) f_U is finite, étale, and surjective,
- (2) the operation of U on the scheme Y_U is trivial,
- (3) the natural morphism

$$\bigsqcup_{\sigma \in G/U} Y_U \xrightarrow{\bigsqcup \text{id}_{Y_U} \times \sigma} Y_U \times_X Y_U$$

is an isomorphism.

In other words, a principal covering with Galois group G may be viewed as an inverse system $(f_U: Y_U \rightarrow X)_{U \in \mathbf{NS}(G)}$, indexed by the open normal subgroups of G .

For any profinite group G and a locally noetherian scheme X , there is always the trivial principal covering $(X \times G \rightarrow X, G)$ given by

$$(X \times G)_U = \bigsqcup_{\sigma \in G/U} X$$

for any open normal subgroup U of G . If X is connected and x is a geometric point of X , then there exists a distinguished principal covering $(f: \tilde{X} \rightarrow X, \pi_1^{\text{ét}}(X, x))$ whose Galois group is the étale fundamental group $\pi_1^{\text{ét}}(X, x)$ of X . It is characterised by the property that there exists a canonical bijection

$$\lim_{U \in \mathbf{NS}(\pi_1^{\text{ét}}(X, x))} \text{Hom}_X(\tilde{X}_U, Z) \rightarrow \text{Hom}_X(x, Z)$$

for any finite étale X -scheme Z . Moreover, the schemes $\tilde{X}_U$ are connected.

If $F = (f: Y \rightarrow X, G)$ is a principal covering and $X' \rightarrow X$ is a locally noetherian X -scheme, then we obtain by base change a principal covering

$$(f \times_X X': Y \times_X X' \rightarrow X', G)$$

of X' , i. e. $(Y \times_X X')_U = Y_U \times_X X'$.

If V is an open (not necessarily normal) subgroup of G and $U \subset V$ is an open normal subgroup of G then the quotient scheme

$$Y_V := Y_U/(V/U)$$

exists and is independent of the choice of U . Moreover, we obtain a principal covering

$$F' = (f^V: Y \rightarrow Y_V, V)$$

of Y_V by setting $F'(U) = Y_U$ for any open normal subgroup U of V .

If $\alpha: G \rightarrow G'$ is a continuous group homomorphism and $F = (f: Y \rightarrow X, G)$ is a principal covering with Galois group G , we obtain a functor

$$\alpha_* F: \mathbf{NS}(G') \rightarrow \mathbf{Sch}_X, \quad U \mapsto F(\alpha^{-1}(U)).$$

If α is surjective with kernel H , then we define

$$(f_H: Y_H \rightarrow X, G/H) := \alpha_* F,$$

which is a principal covering with Galois group G/H .

Definition 2.2. Let $F = (f: Y \rightarrow X, G)$ and $F' = (f': Y' \rightarrow X, G')$ be two principal coverings of X . A *morphism*

$$a: F \rightarrow F'$$

is a continuous group homomorphism $\alpha: G \rightarrow G'$ together with a G -equivariant functorial transformation $a: \alpha_* F \rightarrow F'$.

Lemma 2.3. Let $F = (f: Y \rightarrow X, G)$ and $F' = (f': Y' \rightarrow X, G')$ be two principal coverings of X . Then a morphism $a: F \rightarrow F'$ is an isomorphism if and only if the associated homomorphism of groups $\alpha: G \rightarrow G'$ is an isomorphism.

Proof. We may assume that $G = G'$ and that α is the identity. We may then reduce to the case that G is finite and that X is the spectrum of a local ring A . Then Y and Y' are the spectra of finite flat A -algebras B and B' , respectively. The rank of both B and B' as free A -modules is equal to the cardinality of G . Since $a: Y \rightarrow Y'$ is finite étale, it follows that B is a finitely generated, projective B' -module of constant rank 1. Hence, $B \cong B'$. $\square$

We will need further restrictions on our group G to go on:

Definition 2.4. Let ℓ be a prime. We call a profinite group G *virtually pro- ℓ* if its ℓ -Sylow subgroups are of finite index. Without further mentioning, we require all virtually pro- ℓ groups appearing in this article to be topologically finitely generated.

A principal covering is called *virtually pro- ℓ* if its Galois group is virtually pro- ℓ .

Note that all compact ℓ -adic Lie groups are virtually pro- ℓ in the above sense, but the converse does not hold: The free pro- ℓ -group on two topological generators is a counterexample. A simple but important example of an virtually pro- ℓ principal covering is the following: Let $\mathbb{F}_q$ be finite field with q elements, let ℓ be any prime, k an integer prime to ℓ and set

$$\mathbb{F}_{q^{k\ell\infty}} = \bigcup_{n \geq 0} \mathbb{F}_{q^{k\ell n}}.$$

This gives rise to a principal covering $(\mathrm{Spec} \mathbb{F}_{q^{k\ell\infty}} \rightarrow \mathrm{Spec} \mathbb{F}_q, \Gamma_{k\ell\infty})$ with Galois group $\Gamma_{k\ell\infty} \cong \mathbb{Z}/k\mathbb{Z} \times \mathbb{Z}_\ell$.

Definition 2.5. Let X be a scheme of finite type over the finite field $\mathbb{F}_q$. The principal covering

$$(X_{k\ell\infty} = X \times_{\mathrm{Spec} \mathbb{F}} \mathrm{Spec} \mathbb{F}_{q^{k\ell\infty}} \rightarrow X, \Gamma_{k\ell\infty})$$

will be called the *cyclotomic $\Gamma_{k\ell\infty}$ -covering* of X .

We point out that with this definition, $X_{k\ell\infty}$ is not necessarily connected, even if X itself is connected.

Definition 2.6. Let X be a scheme of finite type over a finite field $\mathbb{F}$, ℓ an arbitrary prime number. We call a principal covering $(f: Y \rightarrow X, G)$ an *admissible covering* if

- (1) $G = H \rtimes \Gamma_{\ell^\infty}$ is the semidirect product of a closed normal subgroup H which is virtually pro- ℓ and the group Γ_{ℓ^∞} ,
- (2) $(f_H: Y_H \rightarrow X, \Gamma_{\ell^\infty})$ is isomorphic to the cyclotomic Γ_{ℓ^∞} -covering of X .

Note that the semidirect product $H \rtimes \Gamma_{\ell^\infty}$ of a virtually pro- ℓ group H and $\Gamma_{\ell^\infty} \cong \mathbb{Z}_\ell$ is itself virtually pro- ℓ .

3. THE K-THEORY OF ADIC RINGS

In this section, we recall some facts about adic rings and their K-theory. Properties of these rings have previously been studied in [War93] and in [FK06]. We also refer to [Wit08, Section 5.1-2] for a more complete treatment.

All rings will be associative with unity, but not necessarily commutative. For any ring R , we let

$$\text{Jac}(R) = \{x \in R \mid 1 - rx \text{ is invertible for any } r \in R\}$$

denote the *Jacobson radical* of R . The ring R is called *semilocal* if $R/\text{Jac}(R)$ is artinian.

Definition 3.1. A ring Λ is called an *adic ring* if for each integer $n \geq 1$, the ideal $\text{Jac}(\Lambda)^n$ is of finite index in Λ and

$$\Lambda = \varprojlim_n \Lambda / \text{Jac}(\Lambda)^n.$$

Note that Λ is adic precisely if it is compact, semilocal and the Jacobson radical is finitely generated [War93, Theorem 36.39].

Definition 3.2. For any adic ring Λ we denote by $\mathfrak{I}_\Lambda$ the set of open two-sided ideals of Λ , partially ordered by inclusion.

Proposition 3.3. Let Λ be an adic $\mathbb{Z}_\ell$ -algebra and let G be a virtually pro- ℓ group. Then the profinite group ring

$$\Lambda[[G]] = \varprojlim_{J \in \mathfrak{I}_\Lambda} \varprojlim_{U \in \mathbf{NS}(G)} \Lambda/J[G/U]$$

is an adic $\mathbb{Z}_\ell$ -algebra. Moreover, if U is any open normal pro- ℓ -subgroup of G , then the kernel of

$$\Lambda[[G]] \rightarrow \Lambda/\text{Jac}(\Lambda)[G/U]$$

is contained in $\text{Jac}(\Lambda[[G]])$.

Proof. We begin by proving the assertion about the Jacobson radical. Clearly, $\Lambda[[G]]$ is a compact ring. Hence,

$$\text{Jac}(\Lambda[[G]]) = \varprojlim_{V \in \mathbf{NS}(G), n \geq 0} \text{Jac}(\Lambda/\text{Jac}(\Lambda)^n[G/V]).$$

We may thus assume that Λ and G are finite. A direct calculation shows that the two-sided ideal $\text{Jac}(\Lambda)\Lambda[G]$ is nilpotent and therefore, it is contained in the Jacobson radical of $\Lambda[G]$. Consequently, we may assume that $n = 1$, i. e. Λ is a finite product of full matrix rings over finite fields of characteristic ℓ . Considering each factor of Λ separately and using that

$$\text{Jac}(M_{k,k}(\Lambda)[G]) = \text{Jac}(M_{k,k}(\Lambda[G])) = M_{k,k}(\text{Jac}(\Lambda[G])),$$

we can restrict to Λ itself being a finite field of characteristic ℓ . We are thus reduced to the classical case treated in [CR90a, Prop. 5.26].

Hence, returning to the general situation, we find an open normal pro- ℓ subgroup U of G such that the kernel of

$$\Lambda[[G]] \rightarrow \Lambda/\text{Jac}(\Lambda)[G/U]$$

is contained in $\text{Jac}(\Lambda[[G]])$. This kernel is an open ideal of $\Lambda[[G]]$ generated by a system of generators of $\text{Jac}(\Lambda)$ over Λ together with the elements $1 - u_i$ for a system of topological generators (u_i) of U . Thus, $\text{Jac}(\Lambda[[G]])$ is also open and finitely generated. Therefore, we conclude that $\Lambda[[G]]$ is an adic ring. $\square$

We will now examine the algebraic K-groups of Λ . For this, we will follow Waldhausen's approach [Wal85] which is more flexible than Quillen's original construction. Recall that a *Waldhausen category* is a category $\mathbf{W}$ with zero object together with two classes of morphisms, called *cofibrations* and *weak equivalences*, that satisfy a certain set of axioms. Using Waldhausen's S -construction one can associate to each such category in a functorial manner a connected topological space $X(\mathbf{W})$. By definition, the n -th K-group of $\mathbf{W}$ is the $(n+1)$ -th homotopy group of this space:

$$K_n(\mathbf{W}) = \pi_{n+1}(X(\mathbf{W})).$$

Waldhausen exact functors are functors that respect the additional structure of a Waldhausen category. Each such functor $F: \mathbf{W} \rightarrow \mathbf{W}'$ induces a continuous map between the associated topological spaces and hence, a homomorphism

$$K_n(F): K_n(\mathbf{W}) \rightarrow K_n(\mathbf{W}').$$

We refer to [TT90] for more thorough introduction to the topic.

Definition 3.4. Let R be any ring. A complex $M^\bullet$ of left R -modules is called *strictly perfect* if it is strictly bounded and for every n , the module M^n is finitely generated and projective. The complex $M^\bullet$ is called *perfect* if it quasi-isomorphic to a strictly perfect complex in the category of all complexes of left R -modules. We let $\mathbf{SP}(R)$ denote the Waldhausen category of strictly perfect complexes, $\mathbf{P}(R)$ the Waldhausen category of perfect complexes, with quasi-isomorphisms as weak equivalences and injective complex morphisms as cofibrations.

By the Gillet-Waldhausen Theorem we know that the Waldhausen K-theory of $\mathbf{SP}(R)$ and of $\mathbf{P}(R)$ coincide with the Quillen K-theory of R :

$$K_n(\mathbf{P}(R)) = K_n(\mathbf{SP}(R)) = K_n(R).$$

Definition 3.5. Let R and S be two rings. We denote by $R^{\mathfrak{p}}\text{-}\mathbf{SP}(S)$ the Waldhausen category of complexes of S - R -bimodules (with S acting from the left, R acting from the right) which are strictly perfect as complexes of S -modules. The weak equivalences and cofibrations are the same as in $\mathbf{SP}(S)$.

For complexes $M^\bullet$ and $N^\bullet$ of right and left R -modules, respectively, we let

$$(M \otimes_R N)^\bullet$$

denote the total complex of the bicomplex $M^\bullet \otimes_R N^\bullet$. Any complex $M^\bullet$ in $R^{\mathfrak{p}}\text{-}\mathbf{SP}(S)$ clearly gives rise to a Waldhausen exact functor

$$(M \otimes_R (-))^\bullet: \mathbf{SP}(R) \rightarrow \mathbf{SP}(S).$$

and hence, to homomorphisms $K_n(R) \rightarrow K_n(S)$.

Let now Λ be an adic ring. The first algebraic K-group of Λ has the following useful property.

Proposition 3.6 ([FK06], Prop. 1.5.3). *Let Λ be an adic ring. Then*

$$K_1(\Lambda) = \varprojlim_{I \in \mathfrak{I}_\Lambda} K_1(\Lambda/I)$$

In particular, $K_1(\Lambda)$ is a profinite group.

It will be convenient to introduce another Waldhausen category that computes the K-theory of Λ .

Definition 3.7. Let R be any ring. A complex $M^\bullet$ of left R -modules is called *DG-flat* if every module M^n is flat and for every acyclic complex $N^\bullet$ of right R -modules, the complex $(N \otimes_R M)^\bullet$ is acyclic.

Definition 3.8. Let Λ be an adic ring. We denote by $\mathbf{PDG}^{\text{cont}}(\Lambda)$ the following Waldhausen category. The objects of $\mathbf{PDG}^{\text{cont}}(\Lambda)$ are inverse system $(P_I^\bullet)_{I \in \mathfrak{I}_\Lambda}$ satisfying the following conditions:

- (1) for each $I \in \mathfrak{I}_\Lambda$, $P_I^\bullet$ is a DG-flat perfect complex of left Λ/I -modules,
- (2) for each $I \subset J \in \mathfrak{I}_\Lambda$, the transition morphism of the system

$$\varphi_{IJ} : P_I^\bullet \rightarrow P_J^\bullet$$

induces an isomorphism

$$\Lambda/J \otimes_{\Lambda/I} P_I^\bullet \cong P_J^\bullet.$$

A morphism of inverse systems $(f_I : P_I^\bullet \rightarrow Q_I^\bullet)_{I \in \mathfrak{I}_\Lambda}$ in $\mathbf{PDG}^{\text{cont}}(\Lambda)$ is a weak equivalence if every f_I is a quasi-isomorphism. It is a cofibration if every f_I is injective.

Proposition 3.9. *The Waldhausen exact functor*

$$F : \mathbf{SP}(\Lambda) \rightarrow \mathbf{PDG}^{\text{cont}}(\Lambda), \quad P^\bullet \rightarrow (\Lambda/I \otimes_\Lambda P^\bullet)_{I \in \mathfrak{I}_\Lambda}$$

identifies $\mathbf{SP}(\Lambda)$ with a full Waldhausen subcategory of $\mathbf{PDG}^{\text{cont}}(\Lambda)$ such that for every $Q^\bullet$ in $\mathbf{PDG}^{\text{cont}}(\Lambda)$ there exists a complex $P^\bullet$ in $\mathbf{SP}(\Lambda)$ and a quasi-isomorphism $F(P^\bullet) \xrightarrow{\sim} Q^\bullet$. Moreover, F induces isomorphisms

$$K_n(\mathbf{SP}(\Lambda)) \cong K_n(\mathbf{PDG}^{\text{cont}}(\Lambda)).$$

Proof. The main step is to show that for every object $(Q_I^\bullet)_{I \in \mathfrak{I}_\Lambda}$ in $\mathbf{PDG}^{\text{cont}}(\Lambda)$, the complex

$$\varprojlim_{I \in \mathfrak{I}_\Lambda} Q_I^\bullet$$

is a perfect complex of Λ -modules. This is proved using the argument of [FK06, Proposition 1.6.5]. The assertion about the K-theory is then an easy consequence of the Waldhausen approximation theorem. We refer to [Wit08, Proposition 5.2.5] for the details. $\square$

Remark 3.10. Definition 3.8 makes sense for any compact ring Λ . However, we do not expect Proposition 3.9 to be true in this generality. The argument of [FK06, Proposition 1.6.5] uses in an essential way that Λ is compact for its $\text{Jac}(\Lambda)$ -adic topology.

We can extend the definition of the tensor product to $\mathbf{PDG}^{\text{cont}}(\Lambda)$ as follows.

Definition 3.11. For $(P_I^\bullet)_{I \in \mathfrak{I}_\Lambda} \in \mathbf{PDG}^{\text{cont}}(\Lambda)$ and $M^\bullet \in \Lambda^\varphi\text{-}\mathbf{SP}(\Lambda')$ we define a Waldhausen exact functor

$$\Psi_M : \mathbf{PDG}^{\text{cont}}(\Lambda) \rightarrow \mathbf{PDG}^{\text{cont}}(\Lambda'), \quad P^\bullet \rightarrow (\varprojlim_{J \in \mathfrak{I}_\Lambda} \Lambda'/I \otimes_{\Lambda'} (M \otimes_\Lambda P_J)^\bullet)_{I \in \mathfrak{I}_{\Lambda'}}$$

Note that for every $I \in \mathfrak{I}_{\Lambda'}$ there exists a $J_0 \in \mathfrak{I}_\Lambda$ such that

$$\varprojlim_{J \in \mathfrak{I}_\Lambda} \Lambda'/I \otimes_{\Lambda'} (M \otimes_\Lambda P_J)^\bullet = (M/IM \otimes_{\Lambda/J_0} P_{J_0})^\bullet.$$

One checks easily that this definition is compatible with the usual tensor product with $M^\bullet$ on $\mathbf{SP}(\Lambda)$.

From [MT07] we deduce the following properties of the group $K_1(\Lambda)$.

Proposition 3.12. *The group $K_1(\Lambda)$ is generated by quasi-isomorphisms*

$$(f_I : P_I^\bullet \xrightarrow{\sim} P_I^\bullet)_{I \in \mathfrak{I}_\Lambda}$$

in $\mathbf{PDG}^{\text{cont}}(\Lambda)$. Moreover, the following relations are satisfied:

- (1) $[(f_I: P_I^\bullet \xrightarrow{\sim} P_I^\bullet)_{I \in \mathfrak{I}_\Lambda}] = [(g_I: P_I^\bullet \xrightarrow{\sim} P_I^\bullet)_{I \in \mathfrak{I}_\Lambda}][h_I: P_I^\bullet \xrightarrow{\sim} P_I^\bullet]_{I \in \mathfrak{I}_\Lambda}$ if for each $I \in \mathfrak{I}_\Lambda$, one has $f_I = g_I \circ h_I$,
- (2) $[(f_I: P_I^\bullet \xrightarrow{\sim} P_I^\bullet)_{I \in \mathfrak{I}_\Lambda}] = [(g_I: Q_I^\bullet \xrightarrow{\sim} Q_I^\bullet)_{I \in \mathfrak{I}_\Lambda}]$ if for each $I \in \mathfrak{I}_\Lambda$, there exists a quasi-isomorphism $a_I: P_I^\bullet \xrightarrow{\sim} Q_I^\bullet$ such that the square

$$\begin{array}{ccc} P_I^\bullet & \xrightarrow{f_I} & P_I^\bullet \\ \downarrow a_I & & \downarrow a_I \\ Q_I^\bullet & \xrightarrow{g_I} & Q_I^\bullet \end{array}$$

commutes up to homotopy,

- (3) $[(g_I: P'_I^\bullet \xrightarrow{\sim} P'_I^\bullet)_{I \in \mathfrak{I}_\Lambda}] = [(f_I: P_I^\bullet \xrightarrow{\sim} P_I^\bullet)_{I \in \mathfrak{I}_\Lambda}][h_I: P''_I^\bullet \xrightarrow{\sim} P''_I^\bullet]_{I \in \mathfrak{I}_\Lambda}$ if for each $I \in \mathfrak{I}_\Lambda$, there exists an exact sequence $P^\bullet \rightarrow P'^\bullet \rightarrow P''^\bullet$ such that the diagram

$$\begin{array}{ccccc} P_I^\bullet & \longrightarrow & P'_I^\bullet & \twoheadrightarrow & P''_I^\bullet \\ \downarrow f_I & & \downarrow g_I & & \downarrow h_I \\ P_I^\bullet & \longrightarrow & P'_I^\bullet & \twoheadrightarrow & P''_I^\bullet \end{array}$$

commutes in the strict sense.

Proof. The description of $K_1(\mathbf{PDG}^{\text{cont}}(\Lambda))$ as the kernel of

$$\mathcal{D}_1 \mathbf{PDG}^{\text{cont}}(\Lambda) \xrightarrow{\partial} \mathcal{D}_0 \mathbf{PDG}^{\text{cont}}(\Lambda)$$

given in [MT07] shows that all endomorphisms which are quasi-isomorphisms do indeed give rise to elements of $K_1(\mathbf{PDG}^{\text{cont}}(\Lambda))$. Together with Proposition 3.6 this description also implies that relations (1) and (3) are satisfied. For relation (2), one can use [Wit08, Lemma 3.1.6]. Finally, the classical description of $K_1(\Lambda)$ implies that $K_1(\mathbf{PDG}^{\text{cont}}(\Lambda))$ is already generated by isomorphisms of finitely generated, projective modules viewed as strictly perfect complexes concentrated in degree 0. $\square$

4. LOCALISATION

Localisation is considered a difficult topic in noncommutative ring theory. To be able to localise at a set of elements S in a noncommutative ring R one needs to show that this set is a denominator set. In particular, one must verify the Ore condition which is often a tedious task. We can avoid this topic by localising the associated Waldhausen category of perfect complexes instead of the ring itself.

Let Λ be an adic $\mathbb{Z}_\ell$ -algebra, H a closed subgroup of a profinite group G and assume that both G and H are virtually pro- ℓ . We define the following Waldhausen categories.

Definition 4.1. We write $\mathbf{PDG}^{\text{cont}, w_H}(\Lambda[[G]])$ for the full Waldhausen subcategory of $\mathbf{PDG}^{\text{cont}}(\Lambda[[G]])$ of objects $(P_J^\bullet)_{J \in \mathfrak{I}_{\Lambda[[G]]}}$ such that

$$\varprojlim_{J \in \mathfrak{I}_{\Lambda[[G]]}} P_J^\bullet$$

is a perfect complex of $\Lambda[[H]]$ -modules.

We write $w_h \mathbf{PDG}^{\text{cont}}(\Lambda[[G]])$ for the Waldhausen category with the same objects, morphisms and cofibrations as $\mathbf{PDG}^{\text{cont}}(\Lambda[[G]])$, but with a new set of weak equivalences given by those morphisms whose cone is an object of $\mathbf{PDG}^{\text{cont}, w_H}(\Lambda[[G]])$.

Note that $\mathbf{PDG}^{\text{cont}, w_H}(\Lambda[[G]])$ is a full additive subcategory of $\mathbf{PDG}^{\text{cont}}(\Lambda[[G]])$ and that it is closed under weak equivalences, shifts, and extensions. This implies immediately that both $\mathbf{PDG}^{\text{cont}, w_H}(\Lambda[[G]])$ and $w_H \mathbf{PDG}^{\text{cont}}(\Lambda[[G]])$ are indeed Waldhausen categories (see e.g. [HM08, Section 3]) and that the natural functors

$$\mathbf{PDG}^{\text{cont}, w_H}(\Lambda[[G]]) \rightarrow \mathbf{PDG}^{\text{cont}}(\Lambda[[G]]) \rightarrow w_H \mathbf{PDG}^{\text{cont}}(\Lambda[[G]])$$

induce a cofibre sequence of the associated K-theory spaces and hence, a long exact localisation sequence

$$\begin{aligned} \cdots \rightarrow K_i(\mathbf{PDG}^{\text{cont}, w_H}(\Lambda[[G]])) &\rightarrow K_i(\mathbf{PDG}^{\text{cont}}(\Lambda[[G]])) \rightarrow \\ K_i(w_H \mathbf{PDG}^{\text{cont}}(\Lambda[[G]])) &\rightarrow K_{i-1}(\mathbf{PDG}^{\text{cont}, w_H}(\Lambda[[G]])) \rightarrow \cdots \end{aligned}$$

[TT90, Theorem 1.8.2].

Assume for the moment that $\Lambda = \mathbb{Z}_\ell$ and that $G = H \rtimes \Gamma_{\ell^\infty}$ with H a compact ℓ -adic Lie group and $\Gamma_{\ell^\infty} \cong \mathbb{Z}_\ell$. Then Venjakob constructed a left and right Ore set S of non-zero divisors in $\mathbb{Z}_\ell[[G]]$. In particular, the quotient ring $S^{-1}\mathbb{Z}_\ell[[G]]$ exists and is flat as right $\mathbb{Z}_\ell[[G]]$ -module. Moreover, a finitely generated $\mathbb{Z}_\ell[[G]]$ -module is S -torsion if and only if it is finitely generated as $\mathbb{Z}_\ell[[H]]$ -module [CFK⁺05, Section 2].

Since $\mathbb{Z}_\ell[[G]]$ is a skew power series ring over the noetherian ring $\mathbb{Z}_\ell[[H]]$, we see that $\mathbb{Z}_\ell[[G]]$ is flat as $\mathbb{Z}_\ell[[H]]$ -module. In particular, a complex $(P_J^\bullet)_{J \in \mathfrak{J}_{\mathbb{Z}_\ell[[G]]}}$ in $\mathbf{PDG}^{\text{cont}}(\mathbb{Z}_\ell[[G]])$ is in $\mathbf{PDG}^{\text{cont}, w_H}(\mathbb{Z}_\ell[[G]])$ if and only if

$$S^{-1}\mathbb{Z}_\ell[[G]] \otimes_{\mathbb{Z}_\ell[[G]]} \varprojlim_{J \in \mathfrak{J}_{\mathbb{Z}_\ell[[G]]}} P_J^\bullet$$

is acyclic.

From the localisation theorem in [WY92] we conclude that in this case,

$$K_i(\mathbf{PDG}^{\text{cont}, w_H}(\mathbb{Z}_\ell[[G]])) = K_i(\mathbb{Z}_\ell[[G]], S^{-1}\mathbb{Z}_\ell[[G]])$$

is the relative K-group and that the functor

$$\begin{aligned} w_H \mathbf{PDG}^{\text{cont}}(\mathbb{Z}_\ell[[G]]) &\rightarrow \mathbf{P}(S^{-1}\mathbb{Z}_\ell[[G]]), \\ (P_J^\bullet)_{J \in \mathfrak{J}_{\mathbb{Z}_\ell[[G]]}} &\mapsto S^{-1}\mathbb{Z}_\ell[[G]] \otimes_{\mathbb{Z}_\ell[[G]]} \varprojlim_{J \in \mathfrak{J}_{\mathbb{Z}_\ell[[G]]}} P_J^\bullet \end{aligned}$$

induces isomorphisms

$$K_i(w_H \mathbf{PDG}^{\text{cont}}(\mathbb{Z}_\ell[[G]])) = \begin{cases} K_i(S^{-1}\mathbb{Z}_\ell[[G]]) & \text{if } i > 0, \\ \text{im } K_0(\mathbb{Z}_\ell[[G]]) \rightarrow K_0(S^{-1}\mathbb{Z}_\ell[[G]]) & \text{if } i = 0 \end{cases}$$

(see also [Wit08, Prop. 5.3.4]).

We need this more explicit description of $K_1(w_H \mathbf{PDG}^{\text{cont}}(\Lambda[[G]]))$ only in the following situation.

Lemma 4.2. *Let Λ be a commutative adic $\mathbb{Z}_\ell$ -algebra, $H = 1$ and $G = \Gamma_{k\ell^\infty}$ with k prime to ℓ . Set*

$$S = \{f \in \Lambda[[\Gamma_{k\ell^\infty}]] : [f] \in \Lambda / \text{Jac}(\Lambda)[[\Gamma_{k\ell^\infty}]] \text{ is a nonzerodivisor}\}$$

Then

$$K_1(w_1 \mathbf{PDG}^{\text{cont}}(\Lambda[[\Gamma_{k\ell^\infty}]))) = K_1(\Lambda[[\Gamma_{k\ell^\infty}]]_S) = \Lambda[[\Gamma_{k\ell^\infty}]]_S^\times.$$

Proof. We will show that a strictly perfect complex $P^\bullet$ of $\Lambda[[\Gamma_{k\ell^\infty}]]$ -modules is perfect as complex of Λ -modules if and only if its cohomology groups are S -torsion. Then the localisation theorem in [WY92] implies that

$$K_n(w_1 \mathbf{PDG}^{\text{cont}}(\Lambda[[\Gamma_{k\ell^\infty}]))) = K_n(\Lambda[[\Gamma_{k\ell^\infty}]]_S)$$

for $n \geq 1$. Since $\Lambda[[\Gamma_{k\ell^\infty}]]_S$ is clearly a commutative semilocal ring, the determinant map induces an isomorphism

$$K_1(\Lambda[[\Gamma_{k\ell^\infty}]]_S) \cong \Lambda[[\Gamma_{k\ell^\infty}]]_S^\times$$

[CR90b, Theorem 40.31].

Recall that a commutative adic ring is always noetherian and that a bounded complex over a noetherian ring is perfect if and only if its cohomology modules are finitely generated. Hence, it suffices to show that a finitely generated $\Lambda[[\Gamma_{\ell^\infty}]]$ -module M is finitely generated as Λ -module if and only if it is S -torsion. This is accomplished by the same argument as in [CFK⁺05, Proposition 2.3]. $\square$

For general G , H and Λ , it is not difficult to prove that the first K-group of $w_H \mathbf{PDG}^{\text{cont}}(\Lambda[[G]])$ agrees with the corresponding localised K_1 -group defined in [FK06, Def. 1.3.2]:

$$K_1(w_H \mathbf{PDG}^{\text{cont}}(\Lambda[[G]])) = K_1(\mathbf{PDG}^{\text{cont}}(\Lambda[[G]]), \mathbf{PDG}^{\text{cont}, w_H}(\Lambda[[G]])),$$

but we will make no use of this fact in the following.

Next, let Λ and Λ' be two adic $\mathbb{Z}_\ell$ -algebras and G, G', H, H' be virtually pro- ℓ groups. Assume that H and H' are closed subgroups of G and G' , respectively. We want to investigate under which circumstances the Waldhausen exact functor

$$\Psi_K: \mathbf{PDG}^{\text{cont}}(\Lambda[[G]]) \rightarrow \mathbf{PDG}^{\text{cont}}(\Lambda'[[G']])$$

for an object $K^\bullet$ in $\Lambda[[G]]^\varphi - \mathbf{SP}(\Lambda'[[G']])$ restricts to a functor

$$\mathbf{PDG}^{\text{cont}, w_H}(\Lambda[[G]]) \rightarrow \mathbf{PDG}^{\text{cont}, w_{H'}}(\Lambda'[[G']]).$$

Note that if this is the case, then Ψ_K also extends to a functor

$$w_H \mathbf{PDG}^{\text{cont}}(\Lambda[[G]]) \rightarrow w_{H'} \mathbf{PDG}^{\text{cont}}(\Lambda'[[G']]).$$

Both functors will again be denoted by Ψ_K .

We need some preparation. For any compact ring Ω , we let

$$M \hat{\otimes}_\Omega N = \varinjlim_{U, V} M/U \otimes_\Omega N/V$$

denote the *completed tensor product* of the compact right Ω -module M with the compact left Ω -module N . Here, U and V run through the open submodules of M and N , respectively. Note that the completed tensor product $M \hat{\otimes}_\Omega N$ agrees with the usual tensor product $M \otimes_\Omega N$ if either M or N is finitely presented.

Definition 4.3. We call a compact Ω -module P *compact-flat* if the completed tensor product with P preserves continuous injections of compact modules.

If the compact ring Ω is noetherian, then P is compact-flat precisely if it is flat, but in general, the two notions do not need to coincide.

Lemma 4.4. *Let Λ be an adic $\mathbb{Z}_\ell$ -algebra, H a closed subgroup of G , with both G and H virtually pro- ℓ . Then any finitely generated, projective $\Lambda[[G]]$ -module is compact-flat as $\Lambda[[H]]$ -module.*

Proof. It suffices to prove the lemma for the finitely generated, projective $\Lambda[[G]]$ -module $\Lambda[[G]]$. Then the statement follows since for every n and every open normal subgroup U in G , $\Lambda/\text{Jac}^n(\Lambda)[G/U]$ is flat as $\Lambda/\text{Jac}^n(\Lambda)[H/H \cap U]$ -module. $\square$

Lemma 4.5. *Let Λ be an adic ring and $P^\bullet$ a strictly bounded complex of compact-flat left Λ -modules. Then $P^\bullet$ is a perfect complex of Λ -modules if and only if $\Lambda/\text{Jac}(\Lambda) \otimes_\Lambda P^\bullet$ has finite cohomology groups.*

Proof. Assume that $Q^\bullet \xrightarrow{\sim} P^\bullet$ is a quasi-isomorphism with $Q^\bullet$ strictly perfect. Then the quasi-isomorphism

$$\Lambda / \text{Jac}(\Lambda) \hat{\otimes}_\Lambda Q^\bullet = \Lambda / \text{Jac}(\Lambda) \hat{\otimes}_\Lambda Q^\bullet \xrightarrow{\sim} \Lambda / \text{Jac}(\Lambda) \hat{\otimes}_\Lambda P^\bullet$$

shows that $\Lambda / \text{Jac}(\Lambda) \hat{\otimes}_\Lambda P^\bullet$ has finite cohomology groups. Since $\Lambda / \text{Jac}(\Lambda)$ is finite and $\text{Jac}(\Lambda)$ is finitely generated, we can replace the completed tensor product by the usual tensor product.

Conversely, assume that $\Lambda / \text{Jac}(\Lambda) \otimes_\Lambda P^\bullet$ has finite cohomology groups. Without loss of generality we may suppose that $P^k = 0$ for $k < 0$ and $k > n$ with some $n \geq 0$. By assumption, $\Lambda / \text{Jac}(\Lambda) \otimes_\Lambda H^n(P)$ is finite. By the topological Nakayama lemma we conclude that the compact module $H^n(P)$ is finitely generated. We proceed by induction over n to prove the perfectness of $P^\bullet$. If $n = 0$, we see that P^0 is finitely generated. Using the lifting of idempotents in Λ , we conclude that P^0 is also projective. If $n > 0$, we may choose a homomorphism $f: \Lambda^k[n] \rightarrow P^\bullet$ such that $H^n(f)$ is surjective. The cone of this morphism then satisfies the induction hypothesis. Since the category of perfect complexes is closed under extensions we conclude that $P^\bullet$ is perfect. $\square$

We can now state the following criterion:

Proposition 4.6. *Let Λ and Λ' be two adic $\mathbb{Z}_\ell$ -algebras and G, G', H, H' be virtually pro- ℓ groups. Assume that H and H' are closed subgroups of G and G' , respectively. Suppose that $K^\bullet$ is a complex in $\Lambda[[G]]^\mathfrak{p}\text{-SP}(\Lambda'[[G']])$ such that there exists a complex $L^\bullet$ in $\Lambda[[H]]^\mathfrak{p}\text{-SP}(\Lambda'[[H']])$ and a quasi-isomorphism of complexes of $\Lambda'[[H']]\text{-}\Lambda[[G]]$ -bimodules*

$$L^\bullet \hat{\otimes}_{\Lambda[[H]]} \Lambda[[G]] \xrightarrow{\sim} K^\bullet.$$

Then

$$\Psi_K: \mathbf{PDG}^{\text{cont}}(\Lambda[[G]]) \rightarrow \mathbf{PDG}^{\text{cont}}(\Lambda'[[G']])$$

restricts to

$$\Psi_K: \mathbf{PDG}^{\text{cont}, w_H}(\Lambda[[G]]) \rightarrow \mathbf{PDG}^{\text{cont}, w_{H'}}(\Lambda'[[G']]).$$

Proof. According to Proposition 3.9 it suffices to consider a strictly perfect complex $P^\bullet$ of $\Lambda[[G]]$ -modules which is also perfect as complex of $\Lambda[[H]]$ -modules. Hence, there exists a quasi-isomorphism $Q^\bullet \rightarrow P^\bullet$ of complexes of $\Lambda[[H]]$ -modules with $Q^\bullet$ strictly perfect. According to Lemma 4.4, each P^n is compact-flat as $\Lambda[[H]]$ -module. Therefore, there exists a quasi-isomorphism of complexes of $\Lambda'[[H']]\text{-}\Lambda[[G]]$ -modules

$$(L \otimes_{\Lambda[[H]]} Q)^\bullet \xrightarrow{\sim} (L \hat{\otimes}_{\Lambda[[H]]} P)^\bullet \xrightarrow{\sim} (K \otimes_{\Lambda[[G]]} P)^\bullet.$$

Since $(L \otimes_{\Lambda[[H]]} Q)^\bullet$ is strictly perfect as complex of $\Lambda'[[H']]\text{-}\Lambda[[G]]$ -modules, we see that $\Psi_K P^\bullet$ is in $\mathbf{PDG}^{\text{cont}, w_{H'}}(\Lambda'[[G']])$. $\square$

Proposition 4.7. *The following complexes $K^\bullet$ in $\Lambda[[G]]^\mathfrak{p}\text{-SP}(\Lambda'[[G']])$ satisfy the premisses of Proposition 4.6:*

- (1) *Assume $G = G', H = H'$. For any complex $P^\bullet$ in $\Lambda[[G]]^\mathfrak{p}\text{-SP}(\Lambda')$ let $K^\bullet$ be the complex $\Lambda'[[G]] \otimes_{\Lambda'} P^\bullet$ in $\Lambda[[G]]^\mathfrak{p}\text{-SP}(\Lambda'[[G]])$ with the right G -operation given by the diagonal action on both factors. This applies in particular for any complex $P^\bullet$ in $\Lambda^\mathfrak{p}\text{-SP}(\Lambda')$ equipped with the trivial G -operation.*
- (2) *Assume that G' is an open subgroup of G and that $H' = H \cap G'$. Let $\Lambda = \Lambda'$ and let $K^\bullet$ be the complex concentrated in degree 0 given by the $\Lambda[[G']]\text{-}\Lambda[[G]]$ -bimodule $\Lambda[[G]]$.*
- (3) *Assume $\Lambda = \Lambda'$. Let $\alpha: G \rightarrow G'$ be a continuous homomorphism such that α maps H to H' and induces a bijection of the sets $H \setminus G$ and $H' \setminus G'$. Let $K^\bullet$ be the $\Lambda[[G']]\text{-}\Lambda[[G]]$ -bimodule $\Lambda[[G']]$.*

Proof. In the first example, one may choose $L^\bullet = \Lambda'[[H]] \otimes_{\Lambda'} P^\bullet$ with the diagonal right operation of H . The isomorphism $L^\bullet \hat{\otimes}_{\Lambda[[H]]} \Lambda[[G]] \rightarrow K^\bullet$ is then induced by $h \otimes p \otimes g \mapsto hg \otimes pg$ for $h \in H$, $g \in G$ and $p \in P^n$. In the second example, $L^\bullet = \Lambda[[H]]$ will do the job. In the last example, choose $L^\bullet = \Lambda[[H']]$. The inclusion $\Lambda[[H']] \subset \Lambda[[G']]$ and the continuous ring homomorphism $\Lambda[[G]] \rightarrow \Lambda[[G']]$ induced by α give rise to a morphism of $\Lambda[[H']]$ - $\Lambda[[G]]$ -bimodules

$$f: \Lambda[[H']] \hat{\otimes}_{\Lambda[[H]]} \Lambda[[G]] \rightarrow \Lambda[[G']].$$

Let U' be any open normal subgroup of G' , $U = \alpha^{-1}(U')$ its preimage under α . Then, as $\Lambda[H'/H' \cap U']$ -modules, $\Lambda[H'/H' \cap U'] \otimes_{\Lambda[H/H \cap U]} \Lambda[G/U]$ is freely generated by a choice of coset representatives of $UH \setminus G$ and $\Lambda[G'/U']$ is freely generated by the images of these representatives under α . Hence, we conclude that f is an isomorphism. $\square$

Generalising [CFK⁺05, Lemma 2.1], we can give a useful characterisation of the complexes in $\mathbf{PDG}^{\text{cont}, w_H}(\Lambda[[G]])$ if we further assume that H is normal in G . Under this condition, we find an open pro- ℓ -subgroup K in H which is normal in G (take for example the intersection of all ℓ -Sylow subgroups of G with H).

Proposition 4.8. *Let Λ be an adic ring, H be a closed subgroup of G , with both G and H virtually pro- ℓ , K an open pro- ℓ -subgroup of H which is normal in G . For a complex $P^\bullet = (P_j^\bullet)_{j \in \mathbb{Z}}$ in $\mathbf{PDG}^{\text{cont}}(\Lambda[[G]])$, the following assertions are equivalent:*

- (1) $P^\bullet$ is in $\mathbf{PDG}^{\text{cont}, w_H}(\Lambda[[G]])$
- (2) $\Psi_{\Lambda/\text{Jac}(\Lambda)[[G/K]]}(P^\bullet)$ is in $\mathbf{PDG}^{\text{cont}, w_{H/K}}(\Lambda/\text{Jac}(\Lambda)[[G/K]])$,
- (3) $\Psi_{\Lambda/\text{Jac}(\Lambda)[[G/K]]}(P^\bullet)$ has finite cohomology groups.

Proof. Assume that $P^\bullet$ is a strictly perfect complex of $\Lambda[[G]]$ -modules. It is a strictly bounded complex of compact-flat $\Lambda[[H]]$ -modules by Lemma 4.4. Proposition 3.3 implies that

$$\Lambda[[H]]/\text{Jac}(\Lambda[[H]]) \otimes_{\Lambda[[H]]} P^\bullet = R/\text{Jac}(R) \otimes_R \Lambda/\text{Jac}(\Lambda)[[G/K]] \otimes_{\Lambda[[G]]} P^\bullet$$

for the finite ring $R = \Lambda/\text{Jac}(\Lambda)[H/K]$. Now the equivalences in the statement of Proposition 4.8 are an immediate consequence of Lemma 4.5. $\square$

We can use similar arguments to prove the following result, which can be combined with Proposition 4.6.

Proposition 4.9. *Let H be a closed subgroup of G , with both G and H virtually pro- ℓ . Assume that H' is an open subgroup of H . Then*

$$\mathbf{PDG}^{\text{cont}, w_{H'}}(\Lambda) = \mathbf{PDG}^{\text{cont}, w_H}(\Lambda).$$

Proof. Since $\Lambda[[H]]$ is a finitely generated free $\Lambda[[H']]$ module, it is clear that every perfect complex of $\Lambda[[H]]$ -modules is also perfect as complex of $\Lambda[[H']]$ -modules. For the other implication we may shrink H' and assume that it is pro- ℓ , normal and open in H . By Proposition 3.3 we conclude

$$\Lambda[[H']]/\text{Jac}(\Lambda[[H']]) = \Lambda/\text{Jac}(\Lambda),$$

$$\Lambda[[H]]/\text{Jac}(\Lambda[[H]]) = (\Lambda/\text{Jac}(\Lambda)) [H/H']/\text{Jac}((\Lambda/\text{Jac}(\Lambda))[H/H']).$$

Let $P^\bullet$ be a strictly perfect complex of $\Lambda[[G]]$ -modules which is also perfect as complex of $\Lambda[[H']]$ -modules. Then Lemma 4.5 implies that the complexes

$$\Lambda/\text{Jac}(\Lambda) \otimes_{\Lambda[[H']]} P^\bullet \cong \Lambda/\text{Jac}(\Lambda)[H/H'] \otimes_{\Lambda[[H]]} P^\bullet$$

have finite cohomology groups and that $P^\bullet$ is perfect as complex of $\Lambda[[H]]$ -modules. $\square$

5. PERFECT COMPLEXES OF ADIC SHEAVES

We let $\mathbb{F}$ denote a finite field of characteristic p , with $q = p^\nu$ elements. Furthermore, we fix an algebraic closure $\overline{\mathbb{F}}$ of $\mathbb{F}$.

For any scheme X in the category $\mathbf{Sch}_{\mathbb{F}}$ of $\mathbb{F}$ -schemes of finite type and any adic ring Λ we introduced in [Wit08] a Waldhausen category $\mathbf{PDG}^{\text{cont}}(X, \Lambda)$ of perfect complexes of adic sheaves on X . Below, we will recall the definition.

Definition 5.1. Let R be a finite ring and X be a scheme in $\mathbf{Sch}_{\mathbb{F}}$. A complex $\mathcal{F}^\bullet$ of étale sheaves of left R -modules on X is called *strictly perfect* if it is strictly bounded and each $\mathcal{F}^n$ is constructible and flat. A complex is called *perfect* if it is quasi-isomorphic to a strictly perfect complex. It is *DG-flat* if for each geometric point of X , the complex of stalks is DG-flat.

Definition 5.2. Let X be a scheme in $\mathbf{Sch}_{\mathbb{F}}$ and let Λ be an adic ring. The *category of perfect complexes of adic sheaves* $\mathbf{PDG}^{\text{cont}}(X, \Lambda)$ is the following Waldhausen category. The objects of $\mathbf{PDG}^{\text{cont}}(X, \Lambda)$ are inverse system $(\mathcal{F}_I^\bullet)_{I \in \mathfrak{I}_\Lambda}$ such that:

- (1) for each $I \in \mathfrak{I}_\Lambda$, $\mathcal{F}_I^\bullet$ is a perfect and DG-flat complex of Λ/I -modules,
- (2) for each $I \subset J \in \mathfrak{I}_\Lambda$, the transition morphism

$$\varphi_{IJ} : \mathcal{F}_I^\bullet \rightarrow \mathcal{F}_J^\bullet$$

of the system induces an isomorphism

$$\Lambda/J \otimes_{\Lambda/I} \mathcal{F}_I^\bullet \xrightarrow{\sim} \mathcal{F}_J^\bullet.$$

Weak equivalences and cofibrations are those morphisms of inverse systems that are weak equivalences or cofibrations for each $I \in \mathfrak{I}_\Lambda$, respectively.

If $\Lambda = \mathbb{Z}_\ell$, then the subcategory of complexes concentrated in degree 0 of $\mathbf{PDG}^{\text{cont}}(X, \mathbb{Z}_\ell)$ corresponds precisely to the exact category of flat constructible ℓ -adic sheaves on X in the sense of [Gro77, Exposé VI, Definition 1.1.1]. In this sense, we recover the classical theory.

If $f : Y \rightarrow X$ is a morphism of schemes, we define a Waldhausen exact functor

$$f^* : \mathbf{PDG}^{\text{cont}}(X, \Lambda) \rightarrow \mathbf{PDG}^{\text{cont}}(Y, \Lambda), \quad (\mathcal{F}_I^\bullet)_{I \in \mathfrak{I}_\Lambda} \mapsto (f^* \mathcal{F}_I^\bullet)_{I \in \mathfrak{I}_\Lambda}.$$

We will also need a Waldhausen exact functor that computes higher direct images with proper support. For the purposes of this article it suffices to use the following construction.

Definition 5.3. Let $f : X \rightarrow Y$ be a morphism of separated schemes in $\mathbf{Sch}_{\mathbb{F}}$. Then there exists a factorisation $f = p \circ j$ with $j : X \hookrightarrow X'$ an open immersion and $p : X' \rightarrow Y$ a proper morphism. Let $G_{X'}^\bullet \mathcal{K}$ denote the Godement resolution of a complex $\mathcal{K}^\bullet$ of abelian étale sheaves on X' . Define

$$\begin{aligned} R f_! : \mathbf{PDG}^{\text{cont}}(X, \Lambda) &\rightarrow \mathbf{PDG}^{\text{cont}}(Y, \Lambda) \\ (\mathcal{F}_I^\bullet)_{I \in \mathfrak{I}_\Lambda} &\mapsto (f_* G_{X'}^\bullet j_! \mathcal{F}_I^\bullet)_{I \in \mathfrak{I}_\Lambda} \end{aligned}$$

Obviously, this definition depends on the particular choice of the compactification $f = p \circ j$. However, all possible choices will induce the same homomorphisms

$$K_n(R f_!) : K_n(\mathbf{PDG}^{\text{cont}}(X, \Lambda)) \rightarrow K_n(\mathbf{PDG}^{\text{cont}}(Y, \Lambda))$$

and this is all we need.

Definition 5.4. Let X be a separated scheme in $\mathbf{Sch}_{\mathbb{F}}$ and write $h : X \rightarrow \text{Spec } \mathbb{F}$ for the structure map, $s : \text{Spec } \overline{\mathbb{F}} \rightarrow \text{Spec } \mathbb{F}$ for the map induced by the embedding into the algebraic closure. We define the Waldhausen exact functors

$$R \Gamma_c(\overline{X}, -), R \Gamma_c(X, -) : \mathbf{PDG}^{\text{cont}}(X, \Lambda) \rightarrow \mathbf{PDG}^{\text{cont}}(\Lambda)$$

to be the composition of

$$\mathrm{R} h_! : \mathbf{PDG}^{\mathrm{cont}}(X, \Lambda) \rightarrow \mathbf{PDG}^{\mathrm{cont}}(\mathrm{Spec} \mathbb{F}, \Lambda)$$

with the section functors

$$\begin{aligned} \mathbf{PDG}^{\mathrm{cont}}(\mathrm{Spec} \mathbb{F}, \Lambda) &\rightarrow \mathbf{PDG}^{\mathrm{cont}}(\Lambda), \\ (\mathcal{F}_I^\bullet)_{I \in \mathfrak{I}_\Lambda} &\rightarrow (\Gamma(\mathrm{Spec} \overline{\mathbb{F}}, s^* \mathcal{F}_I^\bullet))_{I \in \mathfrak{I}_\Lambda}, \\ (\mathcal{F}_I^\bullet)_{I \in \mathfrak{I}_\Lambda} &\rightarrow (\Gamma(\mathrm{Spec} \mathbb{F}, \mathcal{F}_I^\bullet))_{I \in \mathfrak{I}_\Lambda}, \end{aligned}$$

respectively.

Definition 5.5. We let $\mathfrak{F}_\mathbb{F} \in \mathrm{Gal}(\overline{\mathbb{F}}/\mathbb{F})$ denote the *geometric Frobenius* of $\mathbb{F}$, i. e. if $\mathbb{F}$ has q elements and $x \in \overline{\mathbb{F}}$, then $\mathfrak{F}_\mathbb{F}(x) = x^{\frac{1}{q}}$.

Clearly, $\mathfrak{F}_\mathbb{F}$ operates on $\mathrm{R} \Gamma_c(\overline{X}, \mathcal{F}^\bullet)$.

Proposition 5.6. *Let X be a separated scheme in $\mathbf{Sch}_\mathbb{F}$. The following sequence is exact in $\mathbf{PDG}^{\mathrm{cont}}(X, \Lambda)$:*

$$0 \rightarrow \mathrm{R} \Gamma_c(X, \mathcal{F}^\bullet) \rightarrow \mathrm{R} \Gamma_c(\overline{X}, \mathcal{F}^\bullet) \xrightarrow{\mathrm{id} - \mathfrak{F}_\mathbb{F}} \mathrm{R} \Gamma_c(\overline{X}, \mathcal{F}^\bullet) \rightarrow 0.$$

Proof. See [Wit08, Proposition 6.1.2]. In fact, all that we will need is that the cone of $\mathrm{id} - \mathfrak{F}_\mathbb{F}$ is quasi-isomorphic to $\mathrm{R} \Gamma_c(X, \mathcal{F}^\bullet)$ shifted by one, which is a well-known consequence of the Hochschild-Serre spectral sequence. $\square$

The definition of Ψ_M extends to $\mathbf{PDG}^{\mathrm{cont}}(X, \Lambda)$.

Definition 5.7. For two adic rings Λ and Λ' we let $\Lambda^\mathfrak{p}\text{-}\mathbf{SP}(X, \Lambda')$ the Waldhausen category of strictly bounded complexes $(\mathcal{F}_J^\bullet)_{J \in \mathfrak{I}_{\Lambda'}}$ in $\mathbf{PDG}^{\mathrm{cont}}(X, \Lambda')$ with each $\mathcal{F}_J^n$ a sheaf of Λ'/J - Λ -bimodules, constructible and flat as sheaf of Λ'/J -modules. The transition maps in the system $(\mathcal{F}_J^\bullet)_{J \in \mathfrak{I}_{\Lambda'}}$ and the boundary maps of the complexes are supposed to be compatible with the right Λ -structure.

Definition 5.8. For $(\mathcal{F}_I^\bullet)_{I \in \mathfrak{I}_\Lambda} \in \mathbf{PDG}^{\mathrm{cont}}(X, \Lambda)$ and $\mathcal{K}^\bullet \in \Lambda^\mathfrak{p}\text{-}\mathbf{SP}(X, \Lambda')$ we set

$$\Psi_{\mathcal{K}}((\mathcal{F}_I^\bullet)_{I \in \mathfrak{I}_\Lambda}) = \left(\varprojlim_{J \in \mathfrak{I}_{\Lambda'}} (\mathcal{K}_I \otimes_\Lambda \mathcal{F}_J)^\bullet \right)_{I \in \mathfrak{I}_{\Lambda'}}$$

and obtain a Waldhausen exact functor

$$\Psi_{\mathcal{K}} : \mathbf{PDG}^{\mathrm{cont}}(X, \Lambda) \rightarrow \mathbf{PDG}^{\mathrm{cont}}(X, \Lambda').$$

Obviously, any complex $M^\bullet$ in $\Lambda^\mathfrak{p}\text{-}\mathbf{SP}(\Lambda')$ may be identified with the complex of constant sheaves $(\Lambda'/I \otimes_{\Lambda'} M)_{I \in \mathfrak{I}_{\Lambda'}}$ in $\Lambda^\mathfrak{p}\text{-}\mathbf{SP}(X, \Lambda')$.

Proposition 5.9. *Let X be a separated scheme in $\mathbf{Sch}_\mathbb{F}$ and let $M^\bullet$ be a complex in $\Lambda^\mathfrak{p}\text{-}\mathbf{SP}(\Lambda')$. The natural morphisms*

$$\begin{aligned} \Psi_M \mathrm{R} \Gamma_c(X, \mathcal{F}^\bullet) &\rightarrow \mathrm{R} \Gamma_c(X, \Psi_M \mathcal{F}^\bullet) \\ \Psi_M \mathrm{R} \Gamma_c(\overline{X}, \mathcal{F}^\bullet) &\rightarrow \mathrm{R} \Gamma_c(\overline{X}, \Psi_M \mathcal{F}^\bullet) \end{aligned}$$

are quasi-isomorphisms.

Proof. This is straightforward. See [Wit08, Proposition 5.5.7]. $\square$

6. ADIC SHEAVES INDUCED BY COVERINGS

As before, we let $\mathbb{F}$ be a finite field and X a $\mathbb{F}$ -scheme of finite type. Recall that for any finite étale map $h: Y \rightarrow X$ and any abelian étale sheaf $\mathcal{F}$ on X , $h_!h^*\mathcal{F}$ is the sheaf associated to the presheaf

$$U \mapsto \bigoplus_{\varphi \in \text{Hom}_X(U, Y)} \mathcal{F}(U)$$

with the transition maps

$$\begin{aligned} \bigoplus_{\psi \in \text{Hom}_X(V, Y)} \mathcal{F}(V) &\rightarrow \bigoplus_{\varphi \in \text{Hom}_X(U, Y)} \mathcal{F}(U), \\ (x_\psi) &\mapsto \left(\sum_{\varphi = \psi \circ \alpha} \mathcal{F}(\alpha)(x_\psi) \right) \end{aligned}$$

for $\alpha: U \rightarrow V$. If h is a finite principal covering with Galois group G , then the right action of G on Y induces a right action on $\text{Hom}_X(U, Y)$ and hence, a left action on $h_!h^*\mathcal{F}$ by permutation of the components. The stalk at a geometric point ξ of X is given by

$$(h_!h^*\mathcal{F})_\xi = \bigoplus_{\varphi \in \text{Hom}_X(\xi, Y)} \mathcal{F}_\xi.$$

Since Y is finite over X , the set $\text{Hom}_X(\xi, Y)$ is nonempty. The choice of any element in $\text{Hom}_X(\xi, Y)$ induces an isomorphism of G -sets

$$\text{Hom}_X(\xi, Y) \cong \text{Hom}_Y(\xi, Y \times_X Y) \cong \text{Hom}_Y(\xi, \bigsqcup_{g \in G} Y) \cong G$$

and hence, a $\mathbb{Z}[G]$ -isomorphism

$$(h_!h^*\mathcal{F})_\xi \cong \mathbb{Z}[G] \otimes_{\mathbb{Z}} \mathcal{F}_\xi.$$

Consider an adic $\mathbb{Z}_\ell$ -algebra Λ and let $(f: Y \rightarrow X, G)$ be a virtually pro- ℓ principal covering of X .

Definition 6.1. For $\mathcal{F}^\bullet \in \mathbf{PDG}^{\text{cont}}(X, \Lambda)$ we set

$$f_!f^*\mathcal{F}^\bullet = \left(\varprojlim_{I \in \mathfrak{I}_\Lambda} \varprojlim_{U \in \mathbf{NS}(G)} \Lambda[[G]]/J \otimes_{\Lambda[[G]]} f_{U!}f_U^*\mathcal{F}_I^\bullet \right)_{J \in \mathfrak{I}_{\Lambda[[G]]}}$$

Again, we note that for each $J \in \mathfrak{I}_{\Lambda[[G]]}$, there exists an $I_0 \in \mathfrak{I}_\Lambda$ and an $U_0 \in \mathbf{NS}(G)$ such that $\Lambda[[G]]/J$ is a right $\Lambda/I_0[G/U_0]$ -module and such that

$$(f_!f^*\mathcal{F})_J^\bullet \cong \Lambda[[G]]/J \otimes_{\Lambda/I_0[G/U_0]} f_{U_0!}f_{U_0}^*\mathcal{F}_{I_0}^\bullet.$$

Proposition 6.2. For any complex $\mathcal{F}^\bullet$ in $\mathbf{PDG}^{\text{cont}}(X, \Lambda)$, $f_!f^*\mathcal{F}^\bullet$ is a complex in $\mathbf{PDG}^{\text{cont}}(X, \Lambda[[G]])$. Moreover, the functor

$$f_!f^*: \mathbf{PDG}^{\text{cont}}(X, \Lambda) \rightarrow \mathbf{PDG}^{\text{cont}}(X, \Lambda[[G]])$$

is Waldhausen exact.

Proof. We note that $f_{U!}f_U^*\mathcal{F}_I^\bullet$ is a perfect DG -flat complex of sheaves of $\Lambda/I[G/U]$ -modules. This follows since for every geometric point ξ of X and every étale sheaf of left Λ/I -modules $\mathcal{P}$ on X , we have

$$(f_{U!}f_U^*\mathcal{P})_\xi \cong \Lambda/I[G/U] \otimes_\Lambda (\mathcal{P}_\xi)$$

Moreover, the functor $f_{U!}f_U^*$ is exact as functor from the abelian category of sheaves of Λ/I -modules to the abelian category of sheaves of $\Lambda/I[G/U]$ -modules and for $V \subset U$, $J \subset I$, we have a natural isomorphism of functors

$$\Lambda/I[G/U] \otimes_{\Lambda/J[G/V]} f_{V!}f_V^* \cong f_{U!}f_U^*.$$

These observations suffice to deduce the assertion. $\square$

Sometimes, the following alternative description of the functor $f_! f^*$ is useful.

Proposition 6.3. *The sheaf $(f_! f^* \Lambda)$ is in $\Lambda^\mathcal{P}\text{-}\mathbf{SP}(X, \Lambda[[G]])$ and for any $\mathcal{F}^\bullet$ in $\mathbf{PDG}^{\text{cont}}(X, \Lambda)$, there exists a natural isomorphism*

$$\Psi_{f_! f^* \Lambda}(\mathcal{F}^\bullet) \cong f_! f^* \mathcal{F}^\bullet$$

in $\mathbf{PDG}^{\text{cont}}(X, \Lambda[[G]])$

Proof. One easily reduces to the case that Λ and G are finite. Then the isomorphism is provided by the well-known projection formula:

$$f_! f^* \Lambda \otimes_\Lambda \mathcal{F}^\bullet \cong f_! (f^* \Lambda \otimes_\Lambda f^* \mathcal{F}^\bullet) \cong f_! f^* \mathcal{F}^\bullet.$$

□

Proposition 6.4. *Let $a: X' \rightarrow X$ be a morphism of separated schemes in $\mathbf{Sch}_\mathbb{F}$ and write $(f': Y' \rightarrow X', G)$ for the principal covering obtained by base change. Then*

- (1) *For any $\mathcal{F}^\bullet$ in $\mathbf{PDG}^{\text{cont}}(X, \Lambda)$ there is a natural isomorphism*

$$f'_! f'^* a^* \mathcal{F}^\bullet \cong a^* f_! f^* \mathcal{F}^\bullet$$

in $\mathbf{PDG}^{\text{cont}}(X', \Lambda[[G]])$.

- (2) *For any $\mathcal{F}^\bullet$ in $\mathbf{PDG}^{\text{cont}}(X', \Lambda[[G]])$ there is a natural quasi-isomorphism*

$$f_! f^* R a_! \mathcal{F}^\bullet \xrightarrow{\sim} R a_! f'_! f'^* \mathcal{F}^\bullet$$

in $\mathbf{PDG}^{\text{cont}}(X, \Lambda[[G]])$.

Proof. The first assertion follows from an application of the proper base change theorem in a very trivial case. For the second assertion, we use the projection formula to see that the natural morphism

$$\Psi_{f_! f^* \Lambda} R a_! \mathcal{F}^\bullet \rightarrow R a_! \Psi_{a^* f_! f^* \Lambda} \mathcal{F}^\bullet$$

is a quasi-isomorphism and then the first assertion to identify $a^* f_! f^* \Lambda$ with $f'_! f'^* \Lambda$. □

Proposition 6.5. *Let $\mathcal{F}^\bullet$ be a complex in $\mathbf{PDG}^{\text{cont}}(X, \Lambda)$.*

- (1) *Let H be a closed normal subgroup of G . Then there exists a natural isomorphism*

$$\Psi_{\Lambda[[G/H]]} f_! f^* \mathcal{F}^\bullet \xrightarrow{\sim} (f_H)_! (f_H)^* \mathcal{F}^\bullet$$

in $\mathbf{PDG}^{\text{cont}}(X, \Lambda[[G/H]])$.

- (2) *Let U be an open subgroup of G , let $f_U: Y_U \rightarrow X$ denote the natural projection map, and view $\Lambda[[G]]$ as a $\Lambda[[U]]$ - $\Lambda[[G]]$ -bimodule. Then there exists a natural quasi-isomorphism*

$$\Psi_{\Lambda[[G]]} f_! f^* \mathcal{F}^\bullet \xrightarrow{\sim} (R(f_U)_!) ((f_U^U)_! (f_U^U)^*) f_U^* \mathcal{F}^\bullet$$

in $\mathbf{PDG}^{\text{cont}}(X, \Lambda[[U]])$.

Proof. One reduces to the case that Λ and G are finite and that $\mathcal{F}^\bullet = \Lambda$. The first morphism is induced by the natural map $f_! f^* \Lambda \rightarrow (f_H)_! (f_H)^* \Lambda$ and is easily checked to be an isomorphism by looking at the stalks. The second morphism is the composition of the isomorphism $f_! f^* \Lambda \cong f_{U,!} f_!^U f^{U*} f_U^* \Lambda$ with the functorial morphism $f_{U,!} \rightarrow R f_{U,!}$, the latter being a quasi-isomorphism since f_U is finite. □

Definition 6.6. Let Λ and Λ' be two adic $\mathbb{Z}_\ell$ -algebras and let $\mathcal{K}^\bullet$ be in $\Lambda[[G]]^\mathcal{P}\text{-}\mathbf{SP}(X, \Lambda')$.

- (1) We will write $\mathcal{K}[[G]]^{\delta^\bullet}$ for the complex $\Psi_{\Lambda'[[G]]} \mathcal{K}^\bullet$ in $\Lambda[[G]]^\mathcal{P}\text{-}\mathbf{SP}(X, \Lambda'[[G]])$ with the right $\Lambda[[G]]$ -structure given by the diagonal right operation of G .

(2) We will write $\tilde{\mathcal{K}}^\bullet$ for the complex $\Psi_{\mathcal{K}} f_! f^* \Lambda$ in $\Lambda^{\mathfrak{p}}\text{-}\mathbf{SP}(X, \Lambda')$

Proposition 6.7. *Let $\mathcal{K}^\bullet$ be in $\Lambda[[G]]^{\mathfrak{p}}\text{-}\mathbf{SP}(X, \Lambda')$. For every $\mathcal{F}^\bullet$ in $\mathbf{PDG}^{\text{cont}}(X, \Lambda)$ there exists a natural isomorphism*

$$\Psi_{\mathcal{K}[[G]]^\delta} f_! f^* \mathcal{F}^\bullet \cong f_! f^* \Psi_{\tilde{\mathcal{K}}} \mathcal{F}^\bullet$$

Proof. One easily reduces to the case that Λ' and Λ are finite rings and that G is a finite group. Moreover, it suffices to consider a sheaf $\mathcal{K}$ of $\Lambda[G]$ - Λ -bimodules viewed as a complex in $\Lambda[G]^{\mathfrak{p}}\text{-}\mathbf{SP}(X, \Lambda')$ which is concentrated in degree 0. In view of Proposition 6.3 we may also assume that $\mathcal{F}^\bullet = \Lambda$. We begin by proving two special cases.

Case 1. Assume that G operates trivially on $\mathcal{K}$. Then

$$\Psi_{\mathcal{K}[[G]]^\delta} f_! f^* \Lambda \cong \mathcal{K} \otimes_{\Lambda} f_! f^* \Lambda \cong f_! f^* \mathcal{K}$$

by the projection formula. On the other hand,

$$\Psi_{\tilde{\mathcal{K}}} \Lambda \cong \mathcal{K} \otimes_{\Lambda[G]} f_! f^* \Lambda \cong \mathcal{K},$$

and therefore, $\Psi_{\mathcal{K}[[G]]^\delta} f_! f^* \Lambda \cong f_! f^* \Psi_{\tilde{\mathcal{K}}} \Lambda$.

Case 2. Assume that $\Lambda' = \Lambda[G]$ and that $\mathcal{K}$ is the constant sheaf $\Lambda[G]$. Let $U \rightarrow X$ be finite étale and consider the homomorphism

$$\begin{aligned} u: \bigoplus_{\psi \in \text{Hom}_X(U, Y)} \Lambda &\rightarrow \bigoplus_{\psi \in \text{Hom}_X(U, Y)} \bigoplus_{\phi \in \text{Hom}_X(U, Y)} \Lambda \\ (a_\psi) &\mapsto (a_\psi \delta_{\psi, \phi}) \end{aligned}$$

with

$$\delta_{\psi, \phi} = \begin{cases} 1 & \text{if } \psi = \phi, \\ 0 & \text{else.} \end{cases}$$

Obviously,

$$u(g(a_\psi)) = (g, g)u(a_\psi)$$

for $g \in G$. Hence, u induces a $\Lambda[G][G]$ -homomorphism

$$\Psi_{\Lambda[G][G]^\delta} (f_! f^* \Lambda) = \Lambda[G][G]^\delta \otimes_{\Lambda} f_! f^* \Lambda \rightarrow f_! f^* f_! f^* \Lambda \cong f_! f^* \Psi_{\widetilde{\Lambda[G]}}(\Lambda)$$

which is easily seen to be an isomorphism by checking on the stalks.

To prove the general case, we let $\mathcal{K}'$ be the sheaf $\mathcal{K}$ considered as a sheaf of $\Lambda' \text{-} \Lambda[G][G]$ -bimodules, where the operation of the second copy of G is the trivial one. Then we have an obvious isomorphism of sheaves of $\Lambda'[G] \text{-} \Lambda[G]$ -bimodules

$$\mathcal{K}[G]^\delta \cong \mathcal{G}'[G]^\delta \otimes_{\Lambda[G][G]} \Lambda[G][G]^\delta$$

and by the two cases that we have already proved we obtain

$$\begin{aligned} f_! f^* \Psi_{\tilde{\mathcal{K}}}(\Lambda) &\cong f_! f^* \Psi_{\tilde{\mathcal{K}'}} f_! f^* \Lambda \cong \Psi_{\mathcal{K}'[G]^\delta} f_! f^* f_! f^* \Lambda \\ &\cong \Psi_{\mathcal{K}'[G]^\delta} \Psi_{\Lambda[G][G]^\delta} f_! f^* \Lambda \cong \Psi_{\mathcal{K}[G]^\delta} f_! f^* \Lambda \end{aligned}$$

as desired. $\square$

The following is a version of the well-known equivalence of finite representations of the fundamental group with locally constant étale sheaves on a connected scheme X .

Proposition 6.8. *Let Λ and Λ' be two adic $\mathbb{Z}_\ell$ -algebras. Assume that X is connected and that x is a geometric point of X . Let $(f: Y \rightarrow X, G)$ be a virtually pro- ℓ subcovering of the universal covering $(\tilde{X} \rightarrow X, \pi_1^\ell(X, x))$. The functor*

$$\Lambda[[G]]^{\mathfrak{p}}\text{-}\mathbf{SP}(\Lambda') \rightarrow \Lambda^{\mathfrak{p}}\text{-}\mathbf{SP}(X, \Lambda'), \quad P^\bullet \mapsto \tilde{P}^\bullet$$

identifies $\Lambda[[G]]^\mathcal{O}\text{-}\mathbf{SP}(\Lambda')$ with a full subcategory $\mathbf{C}$ of $\Lambda^\mathcal{O}\text{-}\mathbf{SP}(X, \Lambda')$. The objects of $\mathbf{C}$ are systems of strictly bounded complexes $(\mathcal{F}_I^\bullet)_{I \in \mathfrak{I}_\Lambda}$ of sheaves of Λ'/I - Λ -bimodules such that for each n , $\mathcal{F}_I^n$ is constructible and flat as sheaf of Λ'/I -modules and there exists an open normal subgroup U of G such that $f_U^* \mathcal{F}_I^n$ is a constant sheaf.

Proof. We may assume that Λ' is finite. Clearly, $\tilde{P}^\bullet$ is an object of $\mathbf{C}$ for every complex $P^\bullet$ in $\Lambda^\mathcal{O}\text{-}\mathbf{SP}(\Lambda')$. If $\mathcal{F}^\bullet$ is in $\mathbf{C}$, there exists an open normal subgroup U of G such that $\mathcal{F}(Y_U)^\bullet$ is equal to the stalk $\mathcal{F}_x^\bullet$ in x . Turn $\mathcal{F}_x^\bullet$ into a complex in $\Lambda[[G]]^\mathcal{O}\text{-}\mathbf{SP}(\Lambda')$ by considering the contragredient of the left action of G on $\mathcal{F}(Y_U)^\bullet$. One checks easily that this is an inverse to the functor $P^\bullet \mapsto \tilde{P}^\bullet$. $\square$

Remark 6.9. Extending Definition 5.2 to arbitrary compact rings, one can also prove a corresponding statement for the full universal covering. On the other hand, if Λ and Λ' are adic rings, it follows as in [Wit08, Theorem 5.6.5] that for every $K^\bullet$ in $\Lambda[[\pi_1^\mathcal{A}(X, x)]]^\mathcal{O}\text{-}\mathbf{SP}(\Lambda')$ there exists a virtually pro- ℓ quotient G of $\pi_1^\mathcal{A}(X, x)$ such that $K^\bullet$ also lies in $\Lambda[[G]]^\mathcal{O}\text{-}\mathbf{SP}(\Lambda')$.

Remark 6.10. Proposition 6.8 implies in particular that for any virtually pro- ℓ subcovering $(f: Y \rightarrow X, G)$ of the universal covering, the sheaf $\mathcal{M}(G)$ of the introduction corresponds to $f_! f^* \mathbb{Z}_\ell$.

7. THE CYCLOTOMIC Γ -COVERING

Let X be a separated scheme in $\mathbf{Sch}_\mathbb{F}$. For any complex $\mathcal{F}^\bullet = (\mathcal{F}_I^\bullet)_{I \in \mathfrak{I}_\Lambda}$ in $\mathbf{PDG}^{\text{cont}}(X, \Lambda)$, we write

$$H_c^i(X, \mathcal{F}) = H^i(\varprojlim_{I \in \mathfrak{I}_\Lambda} R\Gamma_c(X, \mathcal{F}_I^\bullet))$$

for the i -th hypercohomology module of the complex $R\Gamma_c(X, \mathcal{F}^\bullet)$.

Proposition 7.1. *Let $(f: X_{\ell^\infty} \rightarrow X, \Gamma_{\ell^\infty})$ be the cyclotomic Γ_{ℓ^∞} -covering of X . For all $i \in \mathbb{Z}$ and any complex $\mathcal{F}^\bullet = (\mathcal{F}_I^\bullet)_{I \in \mathfrak{I}_\Lambda}$ in $\mathbf{PDG}^{\text{cont}}(X, \Lambda)$, we have*

$$H_c^i(X, f_! f^* \mathcal{F}^\bullet) \cong \varprojlim_n H_c^{i-1}(\overline{X}, \mathcal{F}^\bullet) / (\text{id} - \mathfrak{F}_\mathbb{F}^{\ell^n}) H_c^{i-1}(\overline{X}, \mathcal{F}^\bullet)$$

as Λ -modules.

Proof. Write $f_n = f_{\Gamma_{\ell^\infty}^n}$. Since

$$R^1 \varprojlim_{J \in \mathfrak{I}_{\Lambda[[\Gamma_{\ell^\infty}]]}} M_J = 0$$

for any inverse system $(M_J)_{J \in \mathfrak{I}_{\Lambda[[\Gamma_{\ell^\infty}]]}}$ of $\Lambda[[\Gamma_{\ell^\infty}]]$ -modules with surjective transition maps and since the cohomology groups $H_c^i(X, f_{n!} f_n^* \mathcal{F}_I^\bullet)$ are finite for $I \in \mathfrak{I}_\Lambda$, we conclude that

$$H_c^i(X, f_! f^* \mathcal{F}^\bullet) \cong \varprojlim_{I \in \mathfrak{I}_\Lambda} \varprojlim_n H_c^i(X, f_{n!} f_n^* \mathcal{F}_I^\bullet)$$

(see also [Wit08, Proposition 5.3.2]).

Moreover, for every n , there is a commutative diagram with exact rows

$$\begin{array}{ccccccc} \cdots & \longrightarrow & H_c^i(X, f_{n+1!} f_{n+1}^* \mathcal{F}_I^\bullet) & \longrightarrow & H_c^i(\overline{X}, \mathcal{F}_I^\bullet) & \xrightarrow{\text{id} - \mathfrak{F}_\mathbb{F}^{\ell^{n+1}}} & H_c^i(\overline{X}, \mathcal{F}_I^\bullet) \longrightarrow \cdots \\ & & \downarrow \text{tr} & & \downarrow \sum_{k=0}^{\ell-1} \mathfrak{F}_\mathbb{F}^{k\ell^n} & & \downarrow = \\ \cdots & \longrightarrow & H_c^i(X, f_{n!} f_n^* \mathcal{F}_I^\bullet) & \longrightarrow & H_c^i(\overline{X}, \mathcal{F}_I^\bullet) & \xrightarrow{\text{id} - \mathfrak{F}_\mathbb{F}^{\ell^n}} & H_c^i(\overline{X}, \mathcal{F}_I^\bullet) \longrightarrow \cdots \end{array}$$

where $tr: f_{n+1}!f_{n+1}^*\mathcal{F}_I^\bullet \rightarrow f_n!f_n^*\mathcal{F}_I^\bullet$ denotes the usual trace map. Set

$$K_n = \ker \left(H_c^i(\overline{X}, \mathcal{F}_I^\bullet) \xrightarrow{\text{id} - \mathfrak{F}_{\mathbb{F}}^{\ell^n}} H_c^i(\overline{X}, \mathcal{F}_I^\bullet) \right)$$

Since $H_c^i(\overline{X}, \mathcal{F}_I^\bullet)$ is a finite group, the inclusion chain

$$K_0 \subset K_1 \subset \dots \subset K_n \subset \dots$$

becomes stationary. Hence, for large n , $K_n = K_{n+1}$ and the map

$$\sum_{k=0}^{\ell-1} \mathfrak{F}_{\mathbb{F}}^{k\ell^n} : K_{n+1} \rightarrow K_n$$

is multiplication by ℓ . Since K_n is annihilated by a power of ℓ , we conclude

$$\varprojlim_n K_n = 0.$$

The equality claimed in the proposition is an immediate consequence. $\square$

Proposition 7.2. *Let γ denote the image of $\mathfrak{F}_{\mathbb{F}}$ in $\Gamma = \Gamma_{k\ell^\infty}$ and let $(X_{k\ell^\infty} \rightarrow X, \Gamma)$ be the cyclotomic Γ -covering of X . Let $\mathcal{F}^\bullet$ be a complex in $\mathbf{PDG}^{\text{cont}}(X, \Lambda)$. There exists a quasi-isomorphism*

$$\eta: \Psi_{\Lambda[[\Gamma]]} R\Gamma_c(\overline{X}, \mathcal{F}^\bullet) \rightarrow R\Gamma_c(\overline{X}, f_!f^*\mathcal{F}^\bullet)$$

in $\mathbf{PDG}^{\text{cont}}(\Lambda[[\Gamma]])$ such that the following diagram commutes:

$$\begin{array}{ccc} \Psi_{\Lambda[[\Gamma]]} R\Gamma_c(\overline{X}, \mathcal{F}^\bullet) & \xrightarrow{\gamma^{-1} \otimes \mathfrak{F}_{\mathbb{F}}} & \Psi_{\Lambda[[\Gamma]]} R\Gamma_c(\overline{X}, \mathcal{F}^\bullet) \\ \downarrow \eta & & \downarrow \eta \\ R\Gamma_c(\overline{X}, f_!f^*\mathcal{F}^\bullet) & \xrightarrow{\mathfrak{F}_{\mathbb{F}}} & R\Gamma_c(\overline{X}, f_!f^*\mathcal{F}^\bullet) \end{array}$$

Proof. Using Proposition 6.4 we can reduce to the case $X = \text{Spec } \mathbb{F}$. Moreover, it suffices to consider $\mathcal{F}^\bullet = \Lambda$. By Proposition 6.8, the sheaf $f_!f^*\Lambda$ corresponds to the $\Lambda[[\Gamma]]$ -module $\Lambda[[\Gamma]]$ with the left action of the Frobenius $\mathfrak{F}_{\mathbb{F}}$ given by right multiplication with γ^{-1} . The assertion of the proposition is an immediate consequence. $\square$

8. THE IWASAWA MAIN CONJECTURE

The following theorem is the central piece of our analogue of the noncommutative Iwasawa main conjecture. It corresponds to [CFK⁺05, Conjecture 5.1] in conjunction with the conjectured vanishing of the μ -invariant, the complex $R\Gamma_c(X, f_!f^*\mathcal{F}^\bullet)$ playing the role of the module $X(E/F_\infty)$.

Theorem 8.1. *Let X be a separated scheme of finite type over a finite field $\mathbb{F}$. Fix a prime ℓ and let $(f: Y \rightarrow X, G)$ be an admissible covering of X with $G = H \rtimes \Gamma_{\ell^\infty}$. If Λ is an adic $\mathbb{Z}_\ell$ -algebra and $\mathcal{F}^\bullet$ a complex in $\mathbf{PDG}^{\text{cont}}(X, \Lambda)$, then $R\Gamma_c(X, f_!f^*\mathcal{F}^\bullet)$ is in $\mathbf{PDG}^{\text{cont}, w_H}(\Lambda[[G]])$.*

Proof. Proposition 6.5, Proposition 6.7 (for the Λ/I - Λ -bimodule Λ/I with trivial G -operation), and Proposition 5.9 imply that for any closed normal subgroup K of G and each open two-sided ideal I of Λ , there exists a quasi-isomorphism

$$\begin{aligned} \Psi_{\Lambda/I[[G/K]]} R\Gamma_c(X, f_!f^*\mathcal{F}^\bullet) &\xrightarrow{\sim} R\Gamma_c(X, \Psi_{\Lambda/I[[G/K]]} f_!f^*\mathcal{F}^\bullet) \\ &\xrightarrow{\sim} R\Gamma_c(X, f_{K!}f_K^*\Psi_{\Lambda/I}\mathcal{F}^\bullet). \end{aligned}$$

Thus, by Proposition 4.8, we may assume that Λ is a finite ring and that $G = H \rtimes \Gamma_{\ell^\infty}$ with a finite group H . It then suffices to show that $R\Gamma_c(X, f_!f^*\mathcal{F}^\bullet)$ has finite cohomology groups.

As explained in [CFK⁺05, Proof of Lemma 2.1] there exists an open subgroup $\Gamma' \subset \Gamma_{\ell^\infty}$ which lies in the centre of G . The scheme $Y_{\Gamma'}$ is separated of finite type over a finite extension $\mathbb{F}'$ of $\mathbb{F}$ and by Lemma 2.3 the principal covering $(f^{\Gamma'} : Y \rightarrow Y_{\Gamma'}, \Gamma')$ is isomorphic to the cyclotomic Γ_{ℓ^∞} -covering of $Y_{\Gamma'}$ over $\mathbb{F}'$.

By Proposition 6.5 the cohomology groups of $R\Gamma_c(X, f_! f^* \mathcal{F}^\bullet)$ are isomorphic to those of $R\Gamma_c(Y_{\Gamma'}, f_!^{\Gamma'} f^{\Gamma'^*} f_{\Gamma'}^* \mathcal{F}^\bullet)$ which are finite by Proposition 7.1. $\square$

Corollary 8.2. *Under the same assumptions as above,*

$$\mathrm{id} - \mathfrak{F}_{\mathbb{F}} : R\Gamma_c(\overline{X}, f_! f^* \mathcal{F}^\bullet) \rightarrow R\Gamma_c(\overline{X}, f_! f^* \mathcal{F}^\bullet)$$

is a quasi-isomorphism in $w_H \mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G]])$ and hence, gives rise to an element

$$[\mathrm{id} - \mathfrak{F}_{\mathbb{F}}] \in K_1(w_H \mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G]]))$$

satisfying

$$d[\mathrm{id} - \mathfrak{F}_{\mathbb{F}}] = [R\Gamma_c(X, f_! f^* \mathcal{F}^\bullet)]$$

in $K_0(\mathbf{PDG}^{\mathrm{cont}, w_H}(\Lambda[[G]]))$.

Proof. By Proposition 5.6, the cone of $\mathrm{id} - \mathfrak{F}_{\mathbb{F}}$ is $R\Gamma_c(X, f_! f^* \mathcal{F}^\bullet)$ shifted by one. Hence, $\mathrm{id} - \mathfrak{F}_{\mathbb{F}}$ is a quasi-isomorphism in $w_H \mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G]])$. Theorem A.5 then implies $d[\mathrm{id} - \mathfrak{F}_{\mathbb{F}}] = [R\Gamma_c(X, f_! f^* \mathcal{F}^\bullet)]$. $\square$

Definition 8.3. We write $\mathcal{L}_G(X/\mathbb{F}, \mathcal{F})$ for the inverse of the element $[\mathrm{id} - \mathfrak{F}_{\mathbb{F}}]$.

The element $\mathcal{L}_G(X/\mathbb{F}, \mathcal{F})$ may be thought of as our analogue of the noncommutative ℓ -adic L -function that is conjectured to exist in [CFK⁺05]. Note that the assignment $\mathcal{F}^\bullet \mapsto \mathcal{L}_G(X/\mathbb{F}, \mathcal{F})$ extends to a homomorphism

$$K_0(\mathbf{PDG}^{\mathrm{cont}}(X, \Lambda)) \rightarrow K_1(w_H \mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G]])).$$

Moreover, $\mathcal{L}_G(X/\mathbb{F}, \mathcal{F})$ enjoys the following transformation properties.

Theorem 8.4. *Consider a separated scheme X of finite type over a finite field $\mathbb{F}$. Let Λ be any adic $\mathbb{Z}_\ell$ algebra and let $\mathcal{F}^\bullet$ be a complex in $\mathbf{PDG}^{\mathrm{cont}}(X, \Lambda)$.*

- (1) *Let Λ' be another adic $\mathbb{Z}_\ell$ -algebra. For any complex $M^\bullet$ in $\Lambda[[G]]^{\mathrm{op}}\text{-}\mathbf{SP}(\Lambda')$, we have*

$$\Psi_{M[[G]]^\delta}(\mathcal{L}_G(X/\mathbb{F}, \mathcal{F})) = \mathcal{L}_G(X/\mathbb{F}, \Psi_{\widetilde{M}} \mathcal{F})$$

in $K_1(w_H \mathbf{PDG}^{\mathrm{cont}}(\Lambda'[[G]]))$.

- (2) *Let H' be a closed virtually pro- ℓ -subgroup of H which is normal in G . Then*

$$\Psi_{\Lambda[[G/H']]}(\mathcal{L}_G(X/\mathbb{F}, \mathcal{F})) = \mathcal{L}_{G/H'}(X/\mathbb{F}, \mathcal{F})$$

in $K_1(w_{H'} \mathbf{PDG}^{\mathrm{cont}}(\Lambda[[G/H']]))$.

- (3) *Let U be an open subgroup of G and let $\mathbb{F}'$ be the finite extension corresponding to the image of U in Γ_{ℓ^∞} . Then*

$$\Psi_{\Lambda[[G]]}(\mathcal{L}_G(X/\mathbb{F}, \mathcal{F})) = \mathcal{L}_U(Y_U/\mathbb{F}', f_U^* \mathcal{F})$$

in $K_1(w_{H \cap U} \mathbf{PDG}^{\mathrm{cont}}(\Lambda[[U]]))$.

Proof. Assertions (1) and (2) follow from Proposition 6.7 and Proposition 6.5, respectively, in conjunction with Proposition 4.7 and Proposition 5.9. For Assertion (3) we need the following additional reasoning. Consider the commutative diagram

$$\begin{array}{ccccc} Y_U & \xrightarrow{g} & X \times_{\mathbb{F}} \mathbb{F}' & \xrightarrow{a'} & \mathrm{Spec} \mathbb{F}' \\ & \searrow f_U & \downarrow b' & & \downarrow b \\ & & X & \xrightarrow{a} & \mathrm{Spec} \mathbb{F} \end{array}$$

There exists a chain of quasi-isomorphisms

$$R a_! R f_{U!} f_!^U f^{U*} f_U^* \mathcal{F}^\bullet \sim b_! R(a' \circ g)_! f_!^U f^{U*} f_U^* \mathcal{F}^\bullet$$

in $\mathbf{PDG}^{cont}(\mathrm{Spec} \mathbb{F}, \Lambda[[U]])$. Therefore, it suffices to understand the operation of the Frobenius $\mathfrak{F}_{\mathbb{F}}$ on $\Gamma(\mathrm{Spec} \overline{\mathbb{F}}, b_! \mathcal{F})$ for any étale sheaf $\mathcal{F}$ on $\mathrm{Spec} \mathbb{F}'$. Choosing an embedding of $\mathbb{F}'$ into $\overline{\mathbb{F}}$ we obtain an isomorphism

$$\Gamma(\mathrm{Spec} \overline{\mathbb{F}}, b_! \mathcal{F}) \cong \Gamma(\mathrm{Spec} \overline{\mathbb{F}}, \mathcal{F})^{[\mathbb{F}':\mathbb{F}]}$$

under which the Frobenius $\mathfrak{F}_{\mathbb{F}}$ on the left-hand side corresponds to multiplication with the matrix

$$M = \begin{pmatrix} 0 & \dots\dots\dots & 0 & \mathfrak{F}_{\mathbb{F}'} \\ \mathrm{id} & 0 & \dots\dots\dots & 0 \\ 0 & \mathrm{id} & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \mathrm{id} & 0 \end{pmatrix}$$

on the righthand side. Using only elementary row and column operations one can transform $\mathrm{id} - M$ into the matrix

$$\begin{pmatrix} \mathrm{id} & 0 & \dots\dots\dots & 0 \\ 0 & \mathrm{id} & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \mathrm{id} & 0 \\ 0 & \dots & 0 & 0 & \mathrm{id} - \mathfrak{F}_{\mathbb{F}'} \end{pmatrix}.$$

Since these elementary operations have trivial image in the first K-group, we conclude

$$[\mathrm{id} - \mathfrak{F}_{\mathbb{F}}: R \Gamma_c(Y_U \times_{\mathbb{F}} \overline{\mathbb{F}}, f_!^U f^{U*} f_U^* \mathcal{F}^\bullet)] = [\mathrm{id} - \mathfrak{F}_{\mathbb{F}'}: R \Gamma_c(Y_U \times_{\mathbb{F}'} \overline{\mathbb{F}}, f_!^U f^{U*} f_U^* \mathcal{F}^\bullet)]$$

in $K_1(w_{H \cap U} \mathbf{PDG}^{cont}(\Lambda[[U]]))$, from which the assertion follows. $\square$

Let now Ω be a commutative adic $\mathbb{Z}_\ell$ -algebra. Consider the set

$$P = \{f(T) \in \Omega[T]: f(0) \in \Omega^\times\}$$

in the polynomial ring $\Omega[T]$. Then $\Omega[T]_P$ is a commutative semilocal ring and the natural homomorphism of $K_1(\Omega[T]_P) = \Omega[T]_P^\times$ to $K_1(\Omega[[T]]) = \Omega[[T]]^\times$ is an injection. Furthermore, let k be prime to ℓ and

$$S = \{f \in \Omega[[\Gamma_{k\ell^\infty}]]: [f] \in \Omega/\mathrm{Jac}(\Omega)[[\Gamma_{k\ell^\infty}]] \text{ is a nonzerodivisor}\}$$

be the set in $\Omega[[\Gamma_{k\ell^\infty}]]$ that we considered in Lemma 4.2. We write γ for the image of the Frobenius $\mathfrak{F}_{\mathbb{F}}$ in $\Gamma_{k\ell^\infty}$.

Lemma 8.5. *The homomorphism*

$$\Omega[T] \rightarrow \Omega[[\Gamma_{k\ell^\infty}]], \quad T \mapsto \gamma^{-1}$$

maps P into S .

Proof. We can replace Ω by $\Omega/\mathrm{Jac}(\Omega)$, which is a finite product of finite fields. By considering each component separately, we may assume that Ω is a finite field. Enlarging Ω if necessary, we have an isomorphism

$$\Omega[[\Gamma_{k\ell^\infty}]] \cong \prod_{\chi: \mathbb{Z}/k\mathbb{Z} \rightarrow \Omega^\times} \Omega[[\Gamma_{\ell^\infty}]].$$

Recall that

$$\Omega[[\Gamma_{\ell^\infty}]] \rightarrow \Omega[[T]], \quad \gamma^{-1} \mapsto T + 1$$

is an isomorphism. Now it suffices to remark that for any nonzero polynomial $f(T) \in \Omega[T]$ and any $u \in \Omega^\times$, $f(u(T+1))$ is again a nonzero polynomial. $\square$

Extending the classical definition, we define as in [Wit09] the L -function for any $\mathcal{F}^\bullet$ in $\mathbf{PDG}^{\text{cont}}(X, \Omega)$ as the element

$$L(\mathcal{F}, T) = \prod_{x \in X^0} [\text{id} - T^{\deg(x)} \mathfrak{F}_{k(x)} : \Psi_{\Omega[[T]]}(\mathcal{F}_\xi^\bullet)]^{-1} \in K_1(\Omega[[T]]) \cong \Omega[[T]]^\times.$$

Here, the product extends over the set X^0 of closed points of X ,

$$\mathfrak{F}_{k(x)} = \mathfrak{F}_{\mathbb{F}}^{\deg(x)} \in \text{Gal}(\overline{k(x)}/k(x))$$

denotes the geometric Frobenius of the residue field $k(x)$, and ξ is a geometric point over x .

We are ready to establish the link between the classical L -function and the element $\mathcal{L}_G(X/\mathbb{F}, \mathcal{F})$. For this, it is essential to impose the additional condition that ℓ is different from the characteristic of $\mathbb{F}$.

Theorem 8.6. *Let X be a separated scheme in $\mathbf{Sch}_{\mathbb{F}}$, let ℓ be different from the characteristic of $\mathbb{F}$, and let $(f: Y \rightarrow X, G)$ be an ℓ -admissible principal covering containing the cyclotomic $\Gamma_{k\ell^\infty}$ -covering. Furthermore, let Λ and Ω be adic $\mathbb{Z}_\ell$ -algebras with Ω commutative. For every $\mathcal{F}^\bullet$ in $\mathbf{PDG}^{\text{cont}}(X, \Lambda)$ and every $M^\bullet$ in $\Lambda[[G]]^\varphi\text{-}\mathbf{SP}(\Omega)$, we have $L(\Psi_{\widetilde{M}}(\mathcal{F}), T) \in K_1(\Omega[T]_P)$ and*

$$\Psi_{\Omega[[\Gamma_{k\ell^\infty}]]} \Psi_{M[[G]]^\delta}(\mathcal{L}_G(X/\mathbb{F}, \mathcal{F})) = L(\Psi_{\widetilde{M}}(\mathcal{F}), \gamma^{-1})$$

in $K_1(\Omega[[\Gamma_{k\ell^\infty}]]_S)$.

Proof. By Theorem 8.4 it suffices to consider the case $G = \Gamma_{k\ell^\infty}$, $\Lambda = \Omega$, and $M^\bullet = \Omega$. Let $p\mathbf{SP}(\Omega[T])$ be the Waldhausen category of strictly perfect complexes of $\Omega[T]$ -modules with quasi-isomorphisms being the morphisms which become quasi-isomorphisms in $\mathbf{SP}(\Omega[[T]])$, that means, precisely those whose cone has P -torsion cohomology groups. Then

$$K_n(p\mathbf{SP}(\Omega[T])) = K_n(\Omega[T]_P)$$

for $n \geq 1$ according to [WY92]. It is easy to show that there exists a strictly perfect complex of Ω -modules $Q^\bullet$ with an endomorphism f and a quasi-isomorphism $q: Q^\bullet \rightarrow R\Gamma_c(\overline{X}, \mathcal{F}^\bullet)$ such that the following diagram commutes up to homotopy:

$$\begin{array}{ccc} Q^\bullet & \xrightarrow{q} & R\Gamma_c(\overline{X}, \mathcal{F}^\bullet) \\ \downarrow f & & \downarrow \mathfrak{F}_{\mathbb{F}} \\ Q^\bullet & \xrightarrow{q} & R\Gamma_c(\overline{X}, \mathcal{F}^\bullet) \end{array}$$

(see e.g. [Wit08, Lemma 3.3.2]). By the Grothendieck trace formula [Del77, Fonction $L \bmod \ell^n$, Theorem 2.2] (or also [Wit09, Theorem 7.2]) we know that

$$L(\mathcal{F}, T) = [\text{id} - T \mathfrak{F}_{\mathbb{F}} : \Psi_{\Omega[[T]]} R\Gamma_c(\overline{X}, \mathcal{F}^\bullet)]^{-1}$$

in $K_1(\Omega[[T]])$. On the other hand, the above homotopy-commutative diagram implies

$$L(\mathcal{F}, T) = [\text{id} - T f : \Psi_{\Omega[[T]]} Q^\bullet]^{-1}$$

in $K_1(\Omega[[T]])$ and by Proposition 7.2 also

$$\mathcal{L}_{\Gamma_{k\ell^\infty}}(X/\mathbb{F}, \mathcal{F}) = [\text{id} - \gamma^{-1} f : Q^\bullet]^{-1}$$

in $K_1(\Omega[[\Gamma_{k\ell^\infty}]]_S)$. Hence, $L(\mathcal{F}, T)$ and $\mathcal{L}_{\Gamma_{k\ell^\infty}}(X/\mathbb{F}, \mathcal{F})$ are the images of the element $[\text{id} - T f : \Omega[T] \otimes_\Omega Q^\bullet]^{-1}$ under the homomorphisms

$$K_1(\Omega[T]_P) \rightarrow K_1(\Omega[[T]]), \quad K_1(\Omega[T]_P) \rightarrow K_1(\Omega[[\Gamma_{k\ell^\infty}]]_S),$$

respectively. $\square$

Remark 8.7. The theorem would make sense and the same proof would work for noncommutative Ω if one could show that

$$K_1(p\mathbf{SP}(\Omega[T])) \rightarrow K_1(\Omega[[T]])$$

is always injective.

Theorem 1.1 in the introduction is easily seen to be a special case of Theorem 8.1, Corollary 8.2, and Theorem 8.6 for $\Lambda = \mathbb{Z}_\ell$ and $(f: Y \rightarrow X, G)$ being a subcovering of the universal covering of a connected scheme X .

APPENDIX

Let $\mathbf{wW}$ be a Waldhausen category with weak equivalences $\mathbf{w}$, and let $\mathbf{vW}$ be the same category with the same notion of cofibrations, but with a coarser notion $\mathbf{v} \subset \mathbf{w}$ of weak equivalences. We assume that $\mathbf{wW}$ is *saturated* and *extensional*, i. e.

- (1) if f and g are composable and any two of the morphisms f , g and $g \circ f$ are in $\mathbf{w}$, then so is the third;
- (2) if the two outer components of a morphism of exact sequences in $\mathbf{wW}$ are in $\mathbf{w}$, then so is the middle one.

We denote by $\mathbf{vW}^{\mathbf{w}}$ the full subcategory of $\mathbf{vW}$ consisting of those objects A such that $0 \rightarrow A$ is in $\mathbf{w}$. With the notions of cofibrations and weak equivalences in $\mathbf{vW}$, this subcategory is again a Waldhausen category.

Under the additional assumption that there exists an appropriate notion of cylinder functors in $\mathbf{vW}$ and $\mathbf{wW}$, which we will explain below, Waldhausen's localisation theorem [TT90], Theorem 1.8.2, states that the natural inclusion functors $\mathbf{vW}^{\mathbf{w}} \rightarrow \mathbf{vW} \rightarrow \mathbf{wW}$ induce a homotopy fibre sequence of the associated K-theory spaces and hence, a long exact sequence

$$\begin{aligned} \dots \rightarrow K_n(\mathbf{vW}^{\mathbf{w}}) \rightarrow K_n(\mathbf{vW}) \rightarrow K_n(\mathbf{wW}) \xrightarrow{d} K_{n-1}(\mathbf{vW}^{\mathbf{w}}) \rightarrow \dots \\ \dots \rightarrow K_1(\mathbf{vW}) \rightarrow K_1(\mathbf{wW}) \xrightarrow{d} K_0(\mathbf{vW}^{\mathbf{w}}) \rightarrow K_0(\mathbf{vW}) \rightarrow K_0(\mathbf{wW}) \rightarrow 0. \end{aligned}$$

In this appendix, we will give an explicit description of the connecting homomorphism $d: K_1(\mathbf{wW}) \rightarrow K_0(\mathbf{vW}^{\mathbf{w}})$ in terms of the 1-types of the Waldhausen categories, as defined in [MT07]. A similar description has also been derived in [Sta09, Theorem 4.1] (up to some obvious sign errors) using more sophisticated arguments.

We begin by recalling the definition of a cylinder functor. For any Waldhausen category $\mathbf{W}$, the *category of morphisms* $\mathbf{Mor}(\mathbf{W})$ is again a Waldhausen category with the following cofibrations and weak equivalences. A morphism $\alpha \rightarrow \beta$ in $\mathbf{Mor}(\mathbf{W})$, i. e. a commutative square

$$\begin{array}{ccc} A & \xrightarrow{\epsilon} & B \\ \alpha \downarrow & & \downarrow \beta \\ A' & \xrightarrow{\epsilon'} & B', \end{array}$$

is a cofibration if both ϵ and ϵ' are cofibrations. It is a weak equivalence if both ϵ and ϵ' are weak equivalences. One checks easily that the functors

$$\begin{aligned} s: \mathbf{Mor}(\mathbf{W}) &\rightarrow \mathbf{W}, & (A \xrightarrow{\alpha} A') &\mapsto A, \\ t: \mathbf{Mor}(\mathbf{W}) &\rightarrow \mathbf{W}, & (A \xrightarrow{\alpha} A') &\mapsto A', \\ S: \mathbf{W} &\rightarrow \mathbf{Mor}(\mathbf{W}), & A &\mapsto (A \rightarrow 0), \\ T: \mathbf{W} &\rightarrow \mathbf{Mor}(\mathbf{W}), & A &\mapsto (0 \rightarrow A) \end{aligned}$$

are Waldhausen exact. Let $ar: s \rightarrow t$ denote the natural transformation given by $ar(\alpha) = \alpha$.

For two Waldhausen categories $\mathbf{W}_1$ and $\mathbf{W}_2$ we let

$$\mathbf{Fun}(\mathbf{W}_1, \mathbf{W}_2)$$

denote the *Waldhausen category of exact functors* with natural transformations as morphisms. A natural transformation $\alpha: F \rightarrow G$ is a cofibration if

- (1) for each object C in $\mathbf{W}_1$, the morphism $\alpha(C): F(C) \rightarrow G(C)$ is a cofibration,
- (2) for each cofibration $C \rightarrowtail C'$ in $\mathbf{W}_1$, $G(C) \cup_{F(C)} F(C') \rightarrow G(C')$ is a cofibration.

A natural transformation $\alpha: F \rightarrow G$ is a weak equivalence if for each object C in $\mathbf{W}_1$, the morphism $\alpha(C): F(C) \rightarrow G(C)$ is a weak equivalence.

Definition A.1. A *cylinder functor* for $\mathbf{W}$ is an exact functor

$$\text{Cyl}: \mathbf{Mor}(\mathbf{W}) \rightarrow \mathbf{W}$$

together with natural transformations $j_1: s \rightarrow \text{Cyl}$, $j_2: t \rightarrow \text{Cyl}$, $p: \text{Cyl} \rightarrow t$ such that

- (1) $p \circ j_1 = ar$, $p \circ j_2 = \text{id}$,
- (2) $j_1 \oplus j_2: s \oplus t \rightarrow \text{Cyl}$ is a cofibration in $\mathbf{Fun}(\mathbf{Mor}(\mathbf{W}), \mathbf{W})$,
- (3) $\text{Cyl} \circ T = \text{id}$ and the compositions of j_2 with T and p with T are the identity transformation on id .

A cylinder functor satisfies the *cylinder axiom* if

- (4) $p: \text{Cyl} \xrightarrow{\sim} t$ is a weak equivalence in $\mathbf{Fun}(\mathbf{Mor}(\mathbf{W}), \mathbf{W})$.

Remark A.2. The above definition of a cylinder functor is clearly equivalent to the one given in [Wal85], Definition 1.6. Thomason claims that it is also equivalent to the one given in [TT90], Definition 1.3.1. However, it seems at least not to be completely evident from the axioms stated there that Cyl preserves pushouts along cofibrations.

We further set

$$\begin{aligned} \text{Cone} &= \text{Cyl}/s: \mathbf{Mor}(\mathbf{W}) \rightarrow \mathbf{W}, & (A \xrightarrow{\alpha} A') &\mapsto \text{Cyl}(\alpha)/A, \\ \Sigma &= \text{Cone} \circ S: \mathbf{W} \rightarrow \mathbf{W}, & A &\mapsto \text{Cone}(A \rightarrow 0). \end{aligned}$$

Note that $t \rightarrow \text{Cone} \rightarrow \Sigma \circ s$ is an exact sequence in $\mathbf{Fun}(\mathbf{Mor}(\mathbf{W}), \mathbf{W})$ for any cylinder functor Cyl .

Definition A.3. A *stable quadratic module* M_* is a homomorphism of groups $\partial_M: M_1 \rightarrow M_0$ together with a pairing

$$\langle -, - \rangle: M_0 \times M_0 \rightarrow M_1$$

satisfying the following identities for any $a, b \in M_1$ and $X, Y, Z \in M_0$:

- (1) $\langle \partial_M a, \partial_M b \rangle = [b, a]$,
- (2) $\partial_M \langle X, Y \rangle = [Y, X]$,
- (3) $\langle X, Y \rangle \langle Y, X \rangle = 1$,
- (4) $\langle X, YZ \rangle = \langle X, Y \rangle \langle X, Z \rangle$.

We set $a^X = a \langle X, \partial a \rangle$ for $a \in M_1$, $X \in M_0$. Note that this defines a right action of M_0 on M_1 . Furthermore, we let

$$\pi_1(M_*) = \ker \partial_M, \quad \pi_0(M_*) = \text{coker } \partial_M$$

denote the *homotopy groups* of M_* .

Assume that $f: M_* \rightarrow N_*$ is any morphism of stable quadratic modules such that f_0 is injective. Then $f_1(N_1)$ is a normal subgroup of $\partial_N^{-1}(f_0(N_0))$. We set

$$\pi_0(M_*, N_*) = \partial_N^{-1}(f_0(N_0))/f_1(N_1)$$

and obtain an exact sequence

$$\pi_1(M_*) \rightarrow \pi_1(N_*) \rightarrow \pi_0(M_*, N_*) \rightarrow \pi_0(M_*) \rightarrow \pi_0(N_*).$$

Muro and Tonks give the following definition of the 1-type of a Waldhausen category [MT07, Definition 1.2].

Definition A.4. Let $\mathbf{W}$ be a Waldhausen category. The *algebraic 1-type* $\mathcal{D}_*\mathbf{W}$ of $\mathbf{W}$ is the stable quadratic module generated by

- (G0) the symbols $[X]$ for each object X in $\mathbf{W}$ in degree 0,
- (G1) the symbols $[w]$ and $[\Delta]$ for each weak equivalence w and each exact sequence Δ in $\mathbf{W}$,

with ∂ given by

- (R1) $\partial[\alpha] = [B]^{-1}[A]$ for $\alpha: A \xrightarrow{\sim} B$,
- (R2) $\partial[\Delta] = [B]^{-1}[C][A]$ for $\Delta: A \rightarrowtail B \twoheadrightarrow C$.

and

- (R3) $\langle [A], [B] \rangle = [B \rightarrowtail A \oplus B \twoheadrightarrow A]^{-1}[A \rightarrowtail A \oplus B \twoheadrightarrow B]$ for any pair of objects A, B .

Moreover, we impose the following relations:

- (R4) $[0 \rightarrowtail 0 \twoheadrightarrow 0] = 1_{\mathcal{D}_1}$,
- (R5) $[\beta\alpha] = [\beta][\alpha]$ for $\alpha: A \xrightarrow{\sim} B$, $\beta: B \xrightarrow{\sim} C$,
- (R6) $[\Delta'][\alpha][\gamma]^{[A]} = [\beta][\Delta]$ for any commutative diagram

$$\begin{array}{ccccc} \Delta: & A & \twoheadrightarrow & B & \twoheadrightarrow & C \\ & \alpha \downarrow \sim & & \beta \downarrow \sim & & \gamma \downarrow \sim \\ \Delta': & A' & \twoheadrightarrow & B' & \twoheadrightarrow & C' \end{array}$$

- (R7) $[\Gamma_1][\Delta_1] = [\Delta_2][\Gamma_2]^{[A]}$ for any commutative diagram

$$\begin{array}{ccccc} & & \Gamma_1: & & \Gamma_2: \\ \Delta_1: & A & \twoheadrightarrow & B & \twoheadrightarrow & C \\ & \parallel & & \downarrow & & \downarrow \\ \Delta_2: & A & \twoheadrightarrow & D & \twoheadrightarrow & E \\ & \downarrow & & \downarrow & & \downarrow \\ & 0 & \twoheadrightarrow & F & = & F \end{array}$$

Muro and Tonks then prove that

$$K_1(\mathbf{W}) = \pi_1(\mathcal{D}_*(\mathbf{W})), \quad K_0(\mathbf{W}) = \pi_0(\mathcal{D}_*(\mathbf{W})).$$

The following theorem gives our explicit description of the connecting homomorphism.

Theorem A.5. *Let $\mathbf{wW}$ be a Waldhausen category and $\mathbf{vW}$ the same category with a coarser notion of weak equivalences. Assume that $\mathbf{wW}$ is saturated and extensional and let Cyl be a cylinder functor for both $\mathbf{wW}$ and $\mathbf{vW}$ which satisfies the cylinder axiom for $\mathbf{wW}$. Then the assignment*

$$\begin{aligned} d(\Delta) &= 1 && \text{for every exact sequence } \Delta \text{ in } \mathbf{wW}, \\ d(\alpha) &= [\text{Cone}(\alpha)]^{-1}[\text{Cone}(\text{id}_A)] && \text{for every weak equivalence } \alpha: A \rightarrow A' \text{ in } \mathbf{wW} \end{aligned}$$

defines a homomorphism $d: \mathcal{D}_1(\mathbf{wW}) \rightarrow K_0(\mathbf{vW}^{\mathbf{w}})$ and the sequence

$$K_1(\mathbf{vW}) \rightarrow K_1(\mathbf{wW}) \xrightarrow{d} K_0(\mathbf{vW}^{\mathbf{w}}) \rightarrow K_0(\mathbf{vW}) \rightarrow K_0(\mathbf{wW}) \rightarrow 0$$

is exact.

Proof. We may view $K_0(\mathbf{vW}^{\mathbf{w}})$ as a stable quadratic module with trivial group in degree zero and $K_0(\mathbf{vW}^{\mathbf{w}})$ in degree one. By the universal property of $\mathcal{D}_*(\mathbf{wW})$ it suffices to verify the following two assertions in order to show that the homomorphism $d: \mathcal{D}_1(\mathbf{wW}) \rightarrow K_0(\mathbf{vW}^{\mathbf{w}})$ is well-defined.

- (1) For commutative diagrams

$$\begin{array}{ccccc} A & \twoheadrightarrow & B & \twoheadrightarrow & C \\ \alpha \downarrow \sim & & \beta \downarrow \sim & & \gamma \downarrow \sim \\ A' & \twoheadrightarrow & B' & \twoheadrightarrow & C' \end{array}$$

in $\mathbf{wW}$ we have $d(\beta) = d(\alpha)d(\gamma)$.

- (2) For weak equivalences $\alpha: A \xrightarrow{\sim} B$, $\beta: B \xrightarrow{\sim} C$ in $\mathbf{wW}$ we have $d(\beta \circ \alpha) = d(\beta)d(\alpha)$.

Assertion (1) follows easily by applying the exact functors Cone and $\alpha \mapsto \text{Cone}(\text{id}_s(\alpha))$ to the exact sequence $\alpha \mapsto \beta \mapsto \gamma$ in $\mathbf{Mor}(\mathbf{wW})$. We prove Assertion (2).

First, we consider a weak equivalence $\alpha: A \xrightarrow{\sim} B$ in $\mathbf{wW}$ between objects A and B in $\mathbf{vW}^{\mathbf{w}}$. The exact sequences

$$\begin{aligned} B &\mapsto \text{Cone}(\alpha) \twoheadrightarrow \Sigma A, \\ B &\mapsto \text{Cone}(\text{id}_B) \twoheadrightarrow \Sigma B \end{aligned}$$

in $\mathbf{vW}^{\mathbf{w}}$ imply

$$\begin{aligned} d(\alpha)d(B \xrightarrow{\sim} 0) &= [\text{Cone}(\alpha)]^{-1}[\text{Cone}(\text{id}_A)][\Sigma B]^{-1}[\text{Cone}(\text{id}_B)] \\ &= [\Sigma A]^{-1}[\text{Cone}(\text{id}_A)] \\ &= d(A \xrightarrow{\sim} 0). \end{aligned}$$

We obtain

$$d(\beta \circ \alpha) = d(C \xrightarrow{\sim} 0)^{-1}d(A \xrightarrow{\sim} 0) = d(\beta)d(\alpha)$$

in the special case that $\alpha: A \xrightarrow{\sim} B$ and $\beta: B \xrightarrow{\sim} C$ are weak equivalences in $\mathbf{wW}$ between objects A , B , and C in $\mathbf{vW}^{\mathbf{w}}$.

Let now $\alpha: A \xrightarrow{\sim} B$ and $\beta: B \xrightarrow{\sim} C$ be arbitrary weak equivalences in $\mathbf{wW}$. Viewing the vertical morphisms in the commutative diagram

$$\begin{array}{ccccc} A & \xlongequal{\quad} & A & \xlongequal{\quad} & A \\ \parallel & & \alpha \downarrow \sim & & \beta \circ \alpha \downarrow \sim \\ A & \xrightarrow[\sim]{\alpha} & B & \xrightarrow[\sim]{\beta} & C \end{array}$$

as morphisms $\text{id}_A \xrightarrow{\sim} \alpha \xrightarrow{\sim} \beta \circ \alpha$ in $\mathbf{Mor}(\mathbf{wW})$ and applying the exact sequence $t \mapsto \text{Cone} \mapsto \Sigma s$ we obtain the following commutative diagram with exact rows.

$$\begin{array}{ccccc}
A & \xrightarrow{\quad} & \text{Cone}(\text{id}_A) & \twoheadrightarrow & \Sigma A \\
\alpha \downarrow \sim & & \alpha_* \downarrow \sim & & \parallel \\
B & \xrightarrow{\quad} & \text{Cone}(\alpha) & \twoheadrightarrow & \Sigma A \\
\beta \downarrow \sim & & \beta_* \downarrow \sim & & \parallel \\
C & \xrightarrow{\quad} & \text{Cone}(\beta \circ \alpha) & \twoheadrightarrow & \Sigma A
\end{array}$$

Assertion (1) and the previously proved special case of Assertion (2) imply

$$d(\beta \circ \alpha) = d(\beta_* \circ \alpha_*) = d(\beta_*)d(\alpha_*) = d(\beta)d(\alpha).$$

This completes the proof of Assertion (2) in general. Hence, we have also proved the existence of the homomorphism $d: \mathcal{D}_1(\mathbf{wW}) \rightarrow K_0(\mathbf{vW}^{\mathbf{w}})$.

We will now prove the exactness of the sequence in the statement of the theorem. Note that $\mathcal{D}_0(\mathbf{vW}^{\mathbf{w}})$ injects into $\mathcal{D}_0(\mathbf{vW})$ and that $\mathcal{D}_0(\mathbf{vW}) = \mathcal{D}_0(\mathbf{wW})$. Write $K = \pi_0(\mathcal{D}_*(\mathbf{vW}), \mathcal{D}_*(\mathbf{wW}))$, i.e. K is the cokernel of the natural homomorphism $\mathcal{D}_1(\mathbf{vW}) \rightarrow \mathcal{D}_1(\mathbf{wW})$. As explained above, the sequence of abelian groups

$$K_1(\mathbf{vW}) \rightarrow K_1(\mathbf{wW}) \rightarrow K \rightarrow K_0(\mathbf{vW}) \rightarrow K_0(\mathbf{wW}) \rightarrow 0$$

is exact.

Let $\alpha: A \xrightarrow{\sim} B$ be a weak equivalence in $\mathbf{vW}$. Since $\text{Cone}: \mathbf{Mor}(\mathbf{vW}) \rightarrow \mathbf{vW}$ is an exact functor, we see that the induced morphism $\alpha_*: \text{Cone}(\text{id}_A) \xrightarrow{\sim} \text{Cone}(\alpha)$ is a weak equivalence in $\mathbf{vW}^{\mathbf{w}}$. Hence,

$$d(\alpha) = d(\alpha_*) = 1,$$

i.e. the homomorphism d factors through K . It remains to show that $d: K \rightarrow K_0(\mathbf{vW}^{\mathbf{w}})$ is an isomorphism.

Consider the homomorphism $h: \mathcal{D}_0(\mathbf{vW}^{\mathbf{w}}) \rightarrow \mathcal{D}_1(\mathbf{wW})$ induced by sending an object X of $\mathbf{vW}^{\mathbf{w}}$ to $[X \xrightarrow{\sim} 0]$ in $\mathcal{D}_1(\mathbf{wW})$. One checks easily that $h \circ \partial_{\mathcal{D}_*(\mathbf{vW}^{\mathbf{w}})}$ agrees with the natural homomorphism $\mathcal{D}_1(\mathbf{vW}^{\mathbf{w}}) \rightarrow \mathcal{D}_1(\mathbf{wW})$; hence, h induces a homomorphism $H: K_0(\mathbf{vW}^{\mathbf{w}}) \rightarrow K$.

For any weak equivalence $\alpha: A \xrightarrow{\sim} B$ in $\mathbf{wW}$ we have

$$\begin{aligned}
H(d(\alpha)) &= [\text{Cone}(\alpha) \xrightarrow{\sim} 0]^{-1} [\text{Cone}(\text{id}_A) \xrightarrow{\sim} 0] \\
&= [\text{Cone}(\text{id}_A) \xrightarrow{\alpha_*} \text{Cone}(\alpha)] \\
&= [\alpha];
\end{aligned}$$

for any object X in $\mathbf{vW}^{\mathbf{w}}$ we have

$$d(H(X)) = [\Sigma X]^{-1} [\text{Cone}(\text{id}_X)] = [X].$$

Therefore, $d: K \rightarrow K_0(\mathbf{vW}^{\mathbf{w}})$ is indeed an isomorphism with inverse H . $\square$

Note that if Cyl also satisfies the cylinder axiom in $\mathbf{vW}$, then $[\text{Cone}(\text{id}_A)] = 1$ in $K_0(\mathbf{vW}^{\mathbf{w}})$ for every object A in $\mathbf{wW}$. This will be the case in the most common situations.

The above theorem can also be applied to derive the description of Weiss' generalised Whitehead torsion given in [Mur08, Remark 6.3].

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