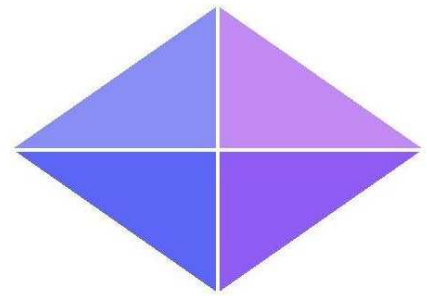


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Noncommutative L -functions for
varieties over finite fields

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NONCOMMUTATIVE L -FUNCTIONS FOR VARIETIES OVER FINITE FIELDS

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ABSTRACT. In this article we prove a Grothendieck trace formula for L -functions of not necessarily commutative adic sheaves.

1. INTRODUCTION

Let $\mathcal{F}$ be an ℓ -adic sheaf on a separated scheme X over a finite field $\mathbb{F}$ of characteristic different from ℓ . The L -function of $\mathcal{F}$ is defined as the product over all closed points x of X of the characteristic polynomials of the geometric Frobenius automorphism $\mathfrak{F}_x$ at x acting on the stalk $\mathcal{F}_x$:

$$L(X, \mathcal{F}, T) = \prod_x \det(1 - \mathfrak{F}_x T^{\deg x} : \mathcal{F}_x)^{-1}.$$

The Grothendieck trace formula relates the L -function to the action of the geometric Frobenius $\mathfrak{F}_{\mathbb{F}}$ on the ℓ -adic cohomology groups with proper support over the base change $\overline{X}$ of X to the algebraic closure:

$$L(X, \mathcal{F}, T) = \prod_{i \in \mathbb{Z}} \det(1 - \mathfrak{F}_{\mathbb{F}} T : H_c^i(\overline{X}, \mathcal{F}))^{(-1)^{i+1}}.$$

It was used by Grothendieck to establish the rationality and the functional equation of the zeta function of X , both of which are parts of the Weil conjectures.

The Grothendieck trace formula may also be viewed as an equality between two elements of the first K-group of the power series ring $\mathbb{Z}_{\ell}[[T]]$. Since the ring $\mathbb{Z}_{\ell}[[T]]$ is a semilocal commutative ring, $K_1(\mathbb{Z}_{\ell}[[T]])$ may be identified with the group of units $\mathbb{Z}_{\ell}[[T]]^{\times}$ via the map induced by the determinant. For each closed point x of X , the $\mathbb{Z}_{\ell}[[T]]$ -automorphism $1 - \mathfrak{F}_x T$ on $\mathbb{Z}_{\ell}[[T]] \otimes_{\mathbb{Z}_{\ell}} \mathcal{F}_x$ defines a class in $K_1(\mathbb{Z}_{\ell}[[T]])$. The product of all these classes converges in the profinite topology induced on $K_1(\mathbb{Z}_{\ell}[[T]])$ by the isomorphism

$$K_1(\mathbb{Z}_{\ell}[[T]]) \cong \varprojlim_n K_1(\mathbb{Z}_{\ell}[[T]]/(\ell^n, T^n)).$$

The image of the limit under the determinant map agrees with the inverse of the L -function of $\mathcal{F}$. On the other hand, the $\mathbb{Z}_{\ell}[[T]]$ -automorphisms

$$\mathbb{Z}_{\ell}[[T]] \otimes_{\mathbb{Z}_{\ell}} H_c^i(\overline{X}, \mathcal{F}) \xrightarrow{1 - \mathfrak{F}_{\mathbb{F}} T} \mathbb{Z}_{\ell}[[T]] \otimes_{\mathbb{Z}_{\ell}} H_c^i(\overline{X}, \mathcal{F})$$

also give rise to elements in the group $K_1(\mathbb{Z}_{\ell}[[T]])$. The Grothendieck trace formula may thus be translated into an equality between the alternating product of those elements and the class corresponding to the L -function.

In this article, we will show that in the above formulation of the Grothendieck trace formula, one may replace $\mathbb{Z}_{\ell}$ by any adic $\mathbb{Z}_{\ell}$ -algebra, i. e. a compact, semilocal $\mathbb{Z}_{\ell}$ -algebra Λ whose Jacobson radical is finitely generated. These rings play an important role in noncommutative Iwasawa theory.

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For any such Λ , we introduced in [Wit08] the notion of a perfect complex of adic sheaves of Λ -modules on X . Furthermore, we presented an explicit functorial construction of a perfect complex of Λ -modules $R\Gamma_c(\overline{X}, \mathcal{F}^\bullet)$ that computes the cohomology with proper support of $\mathcal{F}^\bullet$. By the same pattern as above, we define the L -function of such a complex $\mathcal{F}^\bullet$ as an element $L(\mathcal{F}^\bullet, T)$ of $K_1(\Lambda[[T]])$. The automorphism $1 - \mathfrak{F}_\mathbb{F}T$ on $\Lambda[[T]] \otimes_\Lambda R\Gamma_c(\overline{X}, \mathcal{F}^\bullet)$ gives rise to another class in $K_1(\Lambda[[T]])$. Below, we shall prove the following theorem.

Theorem 1.1. *Let $\mathcal{F}^\bullet$ be a perfect complex of adic sheaves of Λ -modules on X . Then*

$$L(\mathcal{F}^\bullet, T) = [\Lambda[[T]] \otimes_\Lambda R\Gamma_c(\overline{X}, \mathcal{F}^\bullet) \xrightarrow{1 - \mathfrak{F}_\mathbb{F}T} \Lambda[[T]] \otimes_\Lambda R\Gamma_c(\overline{X}, \mathcal{F}^\bullet)]^{-1}$$

in $K_1(\Lambda[[T]])$.

As in the proof of the classical Grothendieck trace formula, one may reduce everything to the case of X being a smooth geometrically connected curve over the finite field $\mathbb{F}$. Moreover, one can replace Λ by $\mathbb{Z}_\ell[G]$, where G is the Galois group of a Galois covering of X . The above theorem is then deduced from the classical Grothendieck trace formula and the fact that

$$\varprojlim_L SK_1(\mathbb{Z}_\ell[\text{Gal}(L/K)]) = 0,$$

where L runs through the finite Galois extensions of the function field K of X .

The paper is structured as follows. Section 2 recalls briefly Waldhausen's construction of algebraic K-theory. In Section 3 we introduce a special Waldhausen category that computes the K-theory of an adic ring. A similar construction is then used in Section 4 to define the categories of perfect complexes of adic sheaves. In Section 5 we study the first K-group of $\mathbb{Z}_\ell[G][[T]]$ and prove that $SK_1(\mathbb{Z}_\ell[\text{Gal}(L/K)])$ vanishes in the limit. In Section 6 we define the L -function of a perfect complex of adic sheaves. Section 7 contains the proof of the Grothendieck trace formula for these L -functions.

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2. WALDHAUSEN CATEGORIES

Waldhausen [Wal85] introduced a construction of algebraic K-theory that is both more transparent and more flexible than Quillen's original approach. He associates K-groups to any category of the following kind.

Definition 2.1. A *Waldhausen category* $\mathbf{W}$ is a category with a zero object $*$, together with two subcategories $\text{co}(\mathbf{W})$ (*cofibrations*) and $\text{w}(\mathbf{W})$ (*weak equivalences*) subject to the following set of axioms.

- (1) Any isomorphism in $\mathbf{W}$ is a morphism in $\text{co}(\mathbf{W})$ and $\text{w}(\mathbf{W})$.
- (2) For every object A in $\mathbf{W}$, the unique map $* \rightarrow A$ is in $\text{co}(\mathbf{W})$.
- (3) If $A \rightarrow B$ is a map in $\text{co}(\mathbf{W})$ and $A \rightarrow C$ is a map in $\mathbf{W}$, then the pushout $B \cup_A C$ exists and the canonical map $C \rightarrow B \cup_A C$ is in $\text{co}(\mathbf{W})$.
- (4) If in the commutative diagram

$$\begin{array}{ccccc} B & \longleftarrow & A & \xrightarrow{f} & C \\ \downarrow & & \downarrow & & \downarrow \\ B' & \longleftarrow & A' & \xrightarrow{g} & C' \end{array}$$

the morphisms f and g are cofibrations and the downwards pointing arrows are weak equivalences, then the natural map $B \cup_A C \rightarrow B' \cup_{A'} C'$ is a weak equivalence.

We denote maps from A to B in $\text{co}(\mathbf{W})$ by $A \twoheadrightarrow B$, those in $\text{w}(\mathbf{W})$ by $A \xrightarrow{\sim} B$. If $C = B \cup_A *$ is a cokernel of the cofibration $A \twoheadrightarrow B$, we denote the natural quotient map from B to C by $B \twoheadrightarrow C$. The sequence

$$A \twoheadrightarrow B \twoheadrightarrow C$$

is called *exact sequence* or *cofibre sequence*.

Definition 2.2. A functor between Waldhausen categories is called (*Waldhausen*) *exact* if it preserves cofibrations, weak equivalences, and pushouts along cofibrations.

If $\mathbf{W}$ is a Waldhausen category, then Waldhausen's S -construction yields a topological space $\mathbb{K}(\mathbf{W})$ and Waldhausen exact functors $F: \mathbf{W} \rightarrow \mathbf{W}'$ yield continuous maps $\mathbb{K}(F): \mathbb{K}(\mathbf{W}) \rightarrow \mathbb{K}(\mathbf{W}')$ [Wal85].

Definition 2.3. The n -th K-group of $\mathbf{W}$ is defined to be the n -th homotopy group of $\mathbb{K}(\mathbf{W})$:

$$K_n(\mathbf{W}) = \pi_n(\mathbb{K}(\mathbf{W})).$$

Example 2.4.

- (1) Any exact category $\mathbf{E}$ may be viewed as a Waldhausen category by taking the admissible monomorphisms as cofibrations and isomorphisms as weak equivalences. Then the Waldhausen K-groups of $\mathbf{E}$ agree with the Quillen K-groups of $\mathbf{E}$ [TT90, Theorem 1.11.2].
- (2) Let $\mathbf{Kom}^b(\mathbf{E})$ be the category of bounded complexes over the exact category $\mathbf{E}$ with degreewise admissible monomorphisms as cofibrations and quasi-isomorphisms (in the category of complexes of an ambient abelian category $\mathbf{A}$) as weak equivalences. By the Gillet-Waldhausen theorem [TT90, Theorem 1.11.7], the Waldhausen K-groups of $\mathbf{Kom}^b(\mathbf{E})$ also agree with the K-groups of $\mathbf{E}$.
- (3) In fact, Thomason showed that if $\mathbf{W}$ is any sufficiently nice Waldhausen category of complexes and $F: \mathbf{W} \rightarrow \mathbf{Kom}^b(\mathbf{E})$ a Waldhausen exact functor that induces an equivalence of the derived categories of $\mathbf{W}$ and $\mathbf{Kom}^b(\mathbf{E})$, then F induces an isomorphism of the corresponding K-groups [TT90, Theorem 1.9.8].

Remark 2.5. In the view of Example 2.4.(3) one might wonder whether it is possible to define a reasonable K-theory for triangulated categories. However, [Sch02] shows that such a construction fails to exist.

The zeroth K-group of a Waldhausen category can be described fairly explicitly as follows.

Proposition 2.6. *Let $\mathbf{W}$ be a Waldhausen category. The group $K_0(\mathbf{W})$ is the abelian group generated by the objects of $\mathbf{W}$ modulo the relations*

- (1) $[A] = [B]$ if there exists a weak equivalence $A \xrightarrow{\sim} B$,
- (2) $[B] = [A][C]$ if there exists a cofibre sequence $A \twoheadrightarrow B \twoheadrightarrow C$.

Proof. See [TT90, §1.5.6]. □

There also exists a description of $K_1(\mathbf{W})$ for general $\mathbf{W}$ as the kernel of a certain group homomorphism [MT07]. We shall come back to this description later in a more specific situation.

3. THE K -THEORY OF ADIC RINGS

All rings will be associative with unity, but not necessarily commutative. For any ring R , we let

$$\text{Jac}(R) = \{x \in R \mid 1 - rx \text{ is invertible for any } r \in R\}$$

denote the *Jacobson radical* of R , i. e. the intersection of all maximal left ideals. It is the largest two-sided ideal I of R such that $1 + I \subset R^\times$ [Lam91, Chapter 2, §4]. The ring R is called *semilocal* if $R/\text{Jac}(R)$ is artinian.

Definition 3.1. A ring Λ is called an *adic ring* if it satisfies any of the following equivalent conditions:

- (1) Λ is compact, semilocal and the Jacobson radical is finitely generated.
- (2) For each integer $n \geq 1$, the ideal $\text{Jac}(\Lambda)^n$ is of finite index in Λ and

$$\Lambda = \varprojlim_n \Lambda / \text{Jac}(\Lambda)^n.$$

- (3) There exists a two-sided ideal I such that for each integer $n \geq 1$, the ideal I^n is of finite index in Λ and

$$\Lambda = \varprojlim_n \Lambda / I^n.$$

Example 3.2. The following rings are adic rings:

- (1) any finite ring,
- (2) $\mathbb{Z}_\ell$,
- (3) the group ring $\Lambda[G]$ for any finite group G and any adic ring Λ ,
- (4) the power series ring $\Lambda[[T]]$ for any adic ring Λ and an indeterminate T that commutes with all elements of Λ ,
- (5) the profinite group ring $\Lambda[[G]]$, when Λ is a adic $\mathbb{Z}_\ell$ -algebra and G is a profinite group whose ℓ -Sylow subgroup has finite index in G .

Note that adic rings are not noetherian in general, the power series over $\mathbb{Z}_\ell$ in two noncommuting indeterminates being a counterexample.

We will now examine the K -theory of Λ .

Definition 3.3. Let R be any ring. A complex $M^\bullet$ of left R -modules is called *strictly perfect* if it is strictly bounded and for every n , the module M^n is finitely generated and projective. We let $\mathbf{SP}(R)$ denote the Waldhausen category of strictly perfect complexes, with quasi-isomorphisms as weak equivalences and injective complex morphisms as cofibrations.

Definition 3.4. Let R and S be two rings. We denote by $R^{\mathcal{P}}\text{-}\mathbf{SP}(S)$ the Waldhausen category of complexes of S - R -bimodules (with S acting from the left, R acting from the right) which are strictly perfect as complexes of S -modules. The weak equivalences are given by quasi-isomorphisms, the cofibrations are the injective complex morphisms.

By Example 2.4 we know that the Waldhausen K -theory of $\mathbf{SP}(R)$ coincides with the Quillen K -theory of R :

$$K_n(\mathbf{SP}(R)) = K_n(R).$$

For complexes $M^\bullet$ and $N^\bullet$ of right and left R -modules, respectively, we let

$$(M \otimes_R N)^\bullet$$

denote the total complex of the bicomplex $M^\bullet \otimes_R N^\bullet$. Any complex $M^\bullet$ in $R^{\mathcal{P}}\text{-}\mathbf{SP}(S)$ clearly gives rise to a Waldhausen exact functor

$$(M \otimes_R (-))^\bullet: \mathbf{SP}(R) \rightarrow \mathbf{SP}(S).$$

and hence, to homomorphisms $K_n(R) \rightarrow K_n(S)$.

Let now Λ be an adic ring. The first algebraic K-group of Λ has the following useful property.

Proposition 3.5 ([FK06], Prop. 1.5.3). *Let Λ be an adic ring. Then*

$$K_1(\Lambda) = \varprojlim_{I \in \mathfrak{I}_\Lambda} K_1(\Lambda/I)$$

In particular, $K_1(\Lambda)$ is a profinite group.

It will be convenient to introduce another Waldhausen category that computes the K-theory of Λ .

Definition 3.6. Let R be any ring. A complex $M^\bullet$ of left R -modules is called *DG-flat* if every module M^n is flat and for every acyclic complex $N^\bullet$ of right R -modules, the complex $(N \otimes_R M)^\bullet$ is acyclic.

We shall denote the lattice of open ideals of an adic ring Λ by $\mathfrak{I}_\Lambda$.

Definition 3.7. Let Λ be an adic ring. We denote by $\mathbf{PDG}^{\text{cont}}(\Lambda)$ the following Waldhausen category. The objects of $\mathbf{PDG}^{\text{cont}}(\Lambda)$ are inverse system $(P_I^\bullet)_{I \in \mathfrak{I}_\Lambda}$ satisfying the following conditions:

- (1) for each $I \in \mathfrak{I}_\Lambda$, $P_I^\bullet$ is a DG-flat complex of left Λ/I -modules and *perfect*, i. e. quasi-isomorphic to a complex in $\mathbf{SP}(\Lambda)$,
- (2) for each $I \subset J \in \mathfrak{I}_\Lambda$, the transition morphism of the system

$$\varphi_{IJ} : P_I^\bullet \rightarrow P_J^\bullet$$

induces an isomorphism

$$\Lambda/J \otimes_{\Lambda/I} P_I^\bullet \cong P_J^\bullet.$$

A morphism of inverse systems $(f_I : P_I^\bullet \rightarrow Q_I^\bullet)_{I \in \mathfrak{I}_\Lambda}$ in $\mathbf{PDG}^{\text{cont}}(\Lambda)$ is a weak equivalence if every f_I is a quasi-isomorphism. It is a cofibration if every f_I is injective.

The following proposition is an easy consequence of Waldhausen's approximation theorem.

Proposition 3.8. *The Waldhausen exact functor*

$$\mathbf{SP}(\Lambda) \rightarrow \mathbf{PDG}^{\text{cont}}(\Lambda), \quad P^\bullet \rightarrow (\Lambda/I \otimes_\Lambda P^\bullet)_{I \in \mathfrak{I}_\Lambda}$$

identifies $\mathbf{SP}(\Lambda)$ with a full Waldhausen subcategory of $\mathbf{PDG}^{\text{cont}}(\Lambda)$. Moreover, it induces isomorphisms

$$K_n(\mathbf{SP}(\Lambda)) \cong K_n(\mathbf{PDG}^{\text{cont}}(\Lambda)).$$

Proof. See [Wit08, Proposition 5.2.5]. □

We will now extend the definition of the tensor product to $\mathbf{PDG}^{\text{cont}}(\Lambda)$.

Definition 3.9. For $(P_I^\bullet)_{I \in \mathfrak{I}_\Lambda} \in \mathbf{PDG}^{\text{cont}}(\Lambda)$ and $M^\bullet \in \Lambda^\varphi\text{-}\mathbf{SP}(\Lambda')$ we set

$$\Psi_M((P_I^\bullet)_{I \in \mathfrak{I}_\Lambda}) = (\varprojlim_{J \in \mathfrak{I}_\Lambda} \Lambda'/I \otimes_{\Lambda'} (M \otimes_\Lambda P_J)^\bullet)_{I \in \mathfrak{I}_{\Lambda'}}$$

and obtain a Waldhausen exact functor

$$\Psi_M : \mathbf{PDG}^{\text{cont}}(\Lambda) \rightarrow \mathbf{PDG}^{\text{cont}}(\Lambda').$$

Proposition 3.10. *Let $M^\bullet$ be a complex in $\Lambda^{\mathcal{P}}\text{-}\mathbf{SP}(\Lambda')$. Then the following diagram commutes.*

$$\begin{array}{ccc} K_n(\mathbf{SP}(\Lambda)) & \xrightarrow{\cong} & K_n(\mathbf{PDG}^{\text{cont}}(\Lambda)) \\ \downarrow K_n(M^\bullet \otimes_\Lambda (-)) & & \downarrow K_n(\Psi_M) \\ K_n(\mathbf{SP}(\Lambda')) & \xrightarrow{\cong} & K_n(\mathbf{PDG}^{\text{cont}}(\Lambda')) \end{array}$$

Proof. Let $P^\bullet$ be a strictly perfect complex in $\mathbf{SP}(\Lambda)$. There exists a canonical isomorphism

$$(\Lambda'/I \otimes_{\Lambda'} (M \otimes_\Lambda P)^\bullet)_{I \in \mathfrak{I}_{\Lambda'}} \cong (\varprojlim_{J \in \mathfrak{I}_\Lambda} \Lambda'/I \otimes_{\Lambda'} (M \otimes_\Lambda \Lambda/J \otimes_\Lambda P)^\bullet)_{I \in \mathfrak{I}_{\Lambda'}}.$$

□

From [MT07] we deduce the following properties of the group $K_1(\Lambda)$.

Proposition 3.11. *The group $K_1(\Lambda)$ is generated by the weak autoequivalences $(f_I: P_I^\bullet \xrightarrow{\sim} P_I^\bullet)_{I \in \mathfrak{I}_\Lambda}$ in $\mathbf{PDG}^{\text{cont}}(\Lambda)$. Moreover, we have the following relations:*

- (1) $[(f_I: P_I^\bullet \xrightarrow{\sim} P_I^\bullet)_{I \in \mathfrak{I}_\Lambda}] = [(g_I: P_I^\bullet \xrightarrow{\sim} P_I^\bullet)_{I \in \mathfrak{I}_\Lambda}][h_I: P_I^\bullet \xrightarrow{\sim} P_I^\bullet]_{I \in \mathfrak{I}_\Lambda}$ if for each $I \in \mathfrak{I}_\Lambda$, one has $f_I = g_I \circ h_I$,
- (2) $[(f_I: P_I^\bullet \xrightarrow{\sim} P_I^\bullet)_{I \in \mathfrak{I}_\Lambda}] = [(g_I: Q_I^\bullet \xrightarrow{\sim} Q_I^\bullet)_{I \in \mathfrak{I}_\Lambda}]$ if for each $I \in \mathfrak{I}_\Lambda$, there exists a quasi-isomorphism $a_I: P_I^\bullet \xrightarrow{\sim} Q_I^\bullet$ such that the square

$$\begin{array}{ccc} P_I^\bullet & \xrightarrow{f_I} & P_I^\bullet \\ \downarrow a_I & & \downarrow a_I \\ Q_I^\bullet & \xrightarrow{g_I} & Q_I^\bullet \end{array}$$

commutes up to homotopy,

- (3) $[(g_I: P_I^\bullet \xrightarrow{\sim} P_I^\bullet)_{I \in \mathfrak{I}_\Lambda}] = [(f_I: P_I^\bullet \xrightarrow{\sim} P_I^\bullet)_{I \in \mathfrak{I}_\Lambda}][h_I: P_I^{\bullet\bullet} \xrightarrow{\sim} P_I^{\bullet\bullet}]_{I \in \mathfrak{I}_\Lambda}$ if for each $I \in \mathfrak{I}_\Lambda$, there exists an exact sequence $P \rightarrow P' \rightarrow P''$ such that the diagram

$$\begin{array}{ccccc} P_I^\bullet & \longrightarrow & P_I^{\bullet\bullet} & \twoheadrightarrow & P_I^{\bullet\bullet\bullet} \\ \downarrow f_I & & \downarrow g_I & & \downarrow h_I \\ P_I^\bullet & \longrightarrow & P_I^{\bullet\bullet} & \twoheadrightarrow & P_I^{\bullet\bullet\bullet} \end{array}$$

commutes in the strict sense.

Proof. The description of $K_1(\mathbf{PDG}^{\text{cont}}(\Lambda))$ as the kernel of

$$\mathcal{D}_1 \mathbf{PDG}^{\text{cont}}(\Lambda) \xrightarrow{\partial} \mathcal{D}_0 \mathbf{PDG}^{\text{cont}}(\Lambda)$$

given in [MT07] shows that the weak autoequivalences are indeed elements of $K_1(\mathbf{PDG}^{\text{cont}}(\Lambda))$. Together with Proposition 3.5 this description also implies that relations (1) and (3) are satisfied. For relation (2), one can use [Wit08, Lemma 3.1.6]. Finally, the classical description of $K_1(\Lambda)$ implies that $K_1(\mathbf{PDG}^{\text{cont}}(\Lambda))$ is already generated by isomorphisms of finitely generated, projective modules viewed as strictly perfect complexes concentrated in degree 0. □

Remark 3.12. Despite the relatively explicit description of $K_1(\mathbf{W})$ for a Waldhausen category $\mathbf{W}$ in [MT07] it is not an easy task to deduce from it a presentation of $K_1(\mathbf{W})$ as an abelian group. We refer to [MT06] for a partial result in this direction.

In particular, one should not expect that the relations (1)–(3) describe the group $K_1(\mathbf{PDG}^{\text{cont}}(\Lambda))$ completely. However, they will suffice for the purpose of this paper.

4. PERFECT COMPLEXES OF ADIC SHEAVES

We let $\mathbb{F}$ denote a finite field of characteristic p , with $q = p^\nu$ elements. Furthermore, we fix an algebraic closure $\overline{\mathbb{F}}$ of $\mathbb{F}$.

For any scheme X in the category $\mathbf{Sch}_{\mathbb{F}}^{sep}$ of separated $\mathbb{F}$ -schemes of finite type and any adic ring Λ we introduced in [Wit08] a Waldhausen category $\mathbf{PDG}^{cont}(X, \Lambda)$ of perfect complexes of adic sheaves on X . Below, we will recall the definition.

Definition 4.1. Let R be a finite ring and X be a scheme in $\mathbf{Sch}_{\mathbb{F}}^{sep}$. A complex $\mathcal{F}^\bullet$ of étale sheaves of left R -modules on X is called *strictly perfect* if it is strictly bounded and each $\mathcal{F}^n$ is constructible and flat. A complex is called *perfect* if it is quasi-isomorphic to a strictly perfect complex. It is *DG-flat* if for each geometric point of X , the complex of stalks is DG-flat.

Definition 4.2. We will denote by $\mathbf{PDG}(X, R)$ the category of DG-flat perfect complexes of R -modules on X . It is a Waldhausen category with quasi-isomorphisms as weak equivalences and injective complex morphisms as cofibrations.

Definition 4.3. Let X be a scheme in $\mathbf{Sch}_{\mathbb{F}}$ and let Λ be an adic ring. The category of perfect complexes of adic sheaves $\mathbf{PDG}^{cont}(X, \Lambda)$ is the following Waldhausen category. The objects of $\mathbf{PDG}^{cont}(X, \Lambda)$ are inverse system $(\mathcal{F}_I^\bullet)_{I \in \mathfrak{I}_\Lambda}$ such that:

- (1) for each $I \in \mathfrak{I}_\Lambda$, $\mathcal{F}_I^\bullet$ is in $\mathbf{PDG}(X, \Lambda/I)$,
- (2) for each $I \subset J \in \mathfrak{I}_\Lambda$, the transition morphism

$$\varphi_{IJ} : \mathcal{F}_I^\bullet \rightarrow \mathcal{F}_J^\bullet$$

of the system induces an isomorphism

$$\Lambda/J \otimes_{\Lambda/I} \mathcal{F}_I^\bullet \xrightarrow{\sim} \mathcal{F}_J^\bullet.$$

Weak equivalences and cofibrations are those morphisms of inverse systems that are weak equivalences or cofibrations for each $I \in \mathfrak{I}_\Lambda$, respectively.

Remark 4.4. If Λ is a finite ring, the zero ideal is open and hence, an element in $\mathfrak{I}_\Lambda$. In particular, the following Waldhausen exact functors are mutually inverse equivalences for finite rings Λ :

$$\begin{aligned} \mathbf{PDG}^{cont}(X, \Lambda) &\rightarrow \mathbf{PDG}(X, \Lambda), & (\mathcal{F}_I^\bullet)_{I \in \mathfrak{I}_\Lambda} &\mapsto \mathcal{F}_{(0)}^\bullet, \\ \mathbf{PDG}(X, \Lambda) &\rightarrow \mathbf{PDG}^{cont}(X, \Lambda), & \mathcal{F}^\bullet &\mapsto (\Lambda/I \otimes_\Lambda \mathcal{F}^\bullet)_{I \in \mathfrak{I}_\Lambda}. \end{aligned}$$

We use these functors to identify the two categories.

If $\Lambda = \mathbb{Z}_\ell$, then the subcategory of complexes concentrated in degree 0 of $\mathbf{PDG}^{cont}(X, \mathbb{Z}_\ell)$ corresponds precisely to the exact category of flat constructible ℓ -adic sheaves on X in the sense of [Gro77, Exposé VI, Definition 1.1.1]. In this sense, we recover the classical theory.

If $f: Y \rightarrow X$ is a morphism of schemes, we define a Waldhausen exact functor

$$f^*: \mathbf{PDG}^{cont}(X, \Lambda) \rightarrow \mathbf{PDG}^{cont}(Y, \Lambda), \quad (\mathcal{F}_I^\bullet)_{I \in \mathfrak{I}_\Lambda} \mapsto (f^* \mathcal{F}_I^\bullet)_{I \in \mathfrak{I}_\Lambda}.$$

We will also need a Waldhausen exact functor that computes higher direct images with proper support. For the purposes of this article it suffices to use the following construction.

Definition 4.5. Let $f: X \rightarrow Y$ be a morphism in $\mathbf{Sch}_{\mathbb{F}}^{sep}$. Then there exists a factorisation $f = p \circ j$ with $j: X \hookrightarrow X'$ an open immersion and $p: X' \rightarrow Y$ a proper morphism. Let $G_{X'}^\bullet$ denote the Godement resolution of a complex $\mathcal{G}^\bullet$ of abelian étale sheaves on X' . Define

$$\begin{aligned} Rf_!: \mathbf{PDG}^{cont}(X, \Lambda) &\rightarrow \mathbf{PDG}^{cont}(Y, \Lambda) \\ (\mathcal{F}_I^\bullet)_{I \in \mathfrak{I}_\Lambda} &\mapsto (f_* G_{X'}^\bullet j_! \mathcal{F}_I^\bullet)_{I \in \mathfrak{I}_\Lambda} \end{aligned}$$

Obviously, this definition depends on the particular choice of the compactification $f = p \circ j$. However, all possible choices will induce the same homomorphisms

$$K_n(R f_!): K_n(\mathbf{PDG}^{cont}(X, \Lambda)) \rightarrow K_n(\mathbf{PDG}^{cont}(Y, \Lambda))$$

and this is all we need.

Remark 4.6. In [Wit08, Section 4.5] we present a way to make the construction of $R f_!$ independent of the choice of a particular compactification.

Proposition 4.7. *Let $f: X \rightarrow Y$ be a morphism in $\mathbf{Sch}_{\mathbb{F}}^{sep}$.*

- (1) *$K_n(R f_!)$ is independent of the choice of a compactification $f = p \circ j$.*
- (2) *Let $\mathbb{F}'$ be a subfield of $\mathbb{F}$ and consider f as a morphism in $\mathbf{Sch}_{\mathbb{F}'}^{sep}$. Then $K_n(R f_!)$ remains the same.*
- (3) *If $g: Y \rightarrow Z$ is another morphism in $\mathbf{Sch}_{\mathbb{F}}^{sep}$, then*

$$K_n(R(g \circ f)_!) = K_n(R g_!) \circ K_n(R f_!)$$

- (4) *For any cartesian square*

$$\begin{array}{ccc} Y \times_X Z & \xrightarrow{f'} & Z \\ g' \downarrow & & \downarrow g \\ Y & \xrightarrow{f} & X \end{array}$$

in $\mathbf{Sch}_{\mathbb{F}}^{sep}$ we have

$$K_n(f^* R g_!) = K_n(R g'_! f'^*)$$

Proof. All of this follows easily from [AGV72, Exposé XXVII]. See also [Wit08, Section 4.5]. $\square$

Definition 4.8. Let X be a scheme in $\mathbf{Sch}_{\mathbb{F}}^{sep}$ and write $h: X \rightarrow \mathrm{Spec} \mathbb{F}$ for the structure map, $s: \mathrm{Spec} \bar{\mathbb{F}} \rightarrow \mathrm{Spec} \mathbb{F}$ for the map induced by the embedding into the algebraic closure. We define the Waldhausen exact functor

$$R_{\mathbb{F}} \Gamma_c(\bar{X}, -): \mathbf{PDG}^{cont}(X, \Lambda) \rightarrow \mathbf{PDG}^{cont}(\Lambda)$$

to be the composition of

$$R h_!: \mathbf{PDG}^{cont}(X, \Lambda) \rightarrow \mathbf{PDG}^{cont}(\mathrm{Spec} \mathbb{F}, \Lambda)$$

with the section functor

$$\mathbf{PDG}^{cont}(\mathrm{Spec} \mathbb{F}, \Lambda) \rightarrow \mathbf{PDG}^{cont}(\Lambda), \quad (\mathcal{F}_I^\bullet)_{I \in \mathcal{I}_\Lambda} \rightarrow (\Gamma(\mathrm{Spec} \bar{\mathbb{F}}, s^* \mathcal{F}_I^\bullet))_{I \in \mathcal{I}_\Lambda}.$$

Remark 4.9. If $\mathbb{F}'$ is a subfield of $\mathbb{F}$, then $R_{\mathbb{F}} \Gamma_c(\bar{X}, -)$ and $R_{\mathbb{F}'} \Gamma_c(\bar{X}, -)$ are in fact quasi-isomorphic and hence, they induce the same homomorphism of K-groups. Nevertheless, it will be convenient to distinguish between the two functors. We will omit the index if the base field is clear from the context.

The definition of Ψ_M extends to $\mathbf{PDG}^{cont}(X, \Lambda)$.

Definition 4.10. For $(\mathcal{F}_I^\bullet)_{I \in \mathcal{I}_\Lambda} \in \mathbf{PDG}^{cont}(X, \Lambda)$ and $M^\bullet \in \Lambda^{\varphi\text{-}\mathbf{SP}}(\Lambda')$ we set

$$\Psi_M((\mathcal{F}_I^\bullet)_{I \in \mathcal{I}_\Lambda}) = (\varprojlim_{J \in \mathcal{I}_\Lambda} \Lambda'/I \otimes_{\Lambda'} (M \otimes_{\Lambda} \mathcal{F}_J^\bullet))_{I \in \mathcal{I}_{\Lambda'}}$$

and obtain a Waldhausen exact functor

$$\Psi_M: \mathbf{PDG}^{cont}(X, \Lambda) \rightarrow \mathbf{PDG}^{cont}(X, \Lambda').$$

Proposition 4.11. *Let $M^\bullet$ be a complex in $\Lambda^{\text{op}}\text{-}\mathbf{SP}(\Lambda')$. Then the following diagram commutes.*

$$\begin{array}{ccc} K_n(\mathbf{PDG}^{\text{cont}}(X, \Lambda)) & \xrightarrow{K_n \text{R}\Gamma_c(X, -)} & K_n(\mathbf{PDG}^{\text{cont}}(\Lambda)) \\ \downarrow K_n(\Psi_M) & & \downarrow K_n(\Psi_M) \\ K_n(\mathbf{PDG}^{\text{cont}}(X, \Lambda')) & \xrightarrow{K_n \text{R}\Gamma_c(X, -)} & K_n(\mathbf{PDG}^{\text{cont}}(\Lambda')) \end{array}$$

Proof. See [Wit08, Proposition 5.5.7]. $\square$

Finally, we need the following result. Let X be a connected scheme and $f: Y \rightarrow X$ a finite Galois covering of X with Galois group G , i.e. f is finite étale and the degree of f is equal to the order of $G = \text{Aut}_X(Y)$. We set

$$\mathbb{Z}_\ell[G]_X^\sharp = f_* f^* \mathbb{Z}_\ell.$$

Then $\mathbb{Z}_\ell[G]_X^\sharp$ is a locally constant constructible flat sheaf of $\mathbb{Z}_\ell[G]$ -modules. In fact, it corresponds to the continuous Galois module $\mathbb{Z}_\ell[G]$ on which the fundamental group of X acts contragrediently.

Lemma 4.12. *Let X be a connected scheme in $\mathbf{Sch}_{\mathbb{F}}^{\text{sep}}$. Let R be a finite $\mathbb{Z}_\ell$ -algebra and let $\mathcal{F}^\bullet$ be a bounded complex of flat, locally constant, and constructible sheaves in $\mathbf{PDG}(X, R)$. Then there exists a finite Galois covering Y of X with Galois group G and a complex $M^\bullet$ in $\mathbb{Z}_\ell[G]^{\text{op}}\text{-}\mathbf{SP}(R)$ such that*

$$\mathcal{F}^\bullet \cong \Psi_M(\mathbb{Z}_\ell[G]_X^\sharp).$$

Proof. Choose a large enough Galois covering $f: Y \rightarrow X$ such that $f^* \mathcal{F}^\bullet$ is a complex of constant sheaves and set $M^\bullet = \Gamma(Y, f^* \mathcal{F}^\bullet)$. This is in a natural way a complex in $\mathbb{Z}_\ell[G]^{\text{op}}\text{-}\mathbf{SP}(R)$ and

$$M^\bullet \otimes_{\mathbb{Z}_\ell[G]} \mathbb{Z}_\ell[G]_X^\sharp \cong \mathcal{F}^\bullet$$

See [Wit08, Section 5.6] for further details. $\square$

5. ON $K_1(\mathbb{Z}_\ell[G][[T]])$

Let G be a finite group and let T denote an indeterminate that commutes with every element of $\mathbb{Z}_\ell[G]$. We need some results on the structure of $K_1(\mathbb{Z}_\ell[G][[T]])$. Recall that there exists a finite extension F of $\mathbb{Q}_\ell$ such that $F[G]$ is *split semisimple*:

$$F[G] \cong \prod_{k=1}^r \text{End}_F(F^{s_k})$$

for some integers $r, s_1, \dots, s_r$. Write $\mathcal{O}_F$ for the valuation ring of F and let M be a *maximal $\mathbb{Z}_\ell$ -order* in $F[G]$, i.e. an $\mathbb{Z}_\ell$ -lattice in $F[G]$ which is a subring and which is maximal with respect to this property. Then

$$M \cong \prod_{k=1}^r \text{End}_{\mathcal{O}_F}(\mathcal{O}_F^{s_k})$$

according to [Oli88, Theorem 1.9]. In particular, the determinant map induces an isomorphism

$$K_1(M) \cong \bigoplus_{k=1}^r \mathcal{O}_F^\times.$$

Theorem 2.5 of *loc. cit.* then implies

$$\text{SK}_1(\mathbb{Z}_\ell[G]) = \ker K_1(\mathbb{Z}_\ell[G]) \rightarrow K_1(\mathbb{Q}_\ell[G]) = \ker K_1(\mathbb{Z}_\ell[G]) \rightarrow K_1(M).$$

Analogously, we define a subgroup in $K_1(\mathbb{Z}_\ell[G][[T]])$.

Definition 5.1. Let G be a finite group and choose a finite extension F of $\mathbb{Q}_\ell$ such that $F[G]$ is split semisimple. We set

$$\mathrm{SK}_1(\mathbb{Z}_\ell[G][[T]]) = \ker \mathrm{K}_1(\mathbb{Z}_\ell[G][[T]]) \rightarrow \mathrm{K}_1(M[[T]])$$

where M denotes the maximal $\mathbb{Z}_\ell$ -order in $F[G]$.

Lemma 5.2. For any finite group G ,

$$\mathrm{SK}_1(\mathbb{Z}_\ell[G][[T]]) \cong \varprojlim_n \mathrm{SK}_1(\mathbb{Z}_\ell[G \times \mathbb{Z}/(\ell^n)]).$$

Proof. Let F and F' be splitting fields for $\mathbb{Z}_\ell[G]$ and $\mathbb{Z}_\ell[G \times \mathbb{Z}/(\ell^n)]$, respectively and denote the corresponding maximal orders by M and M' . The commutativity of the diagram

$$\begin{array}{ccc} \mathrm{K}_1(M[\mathbb{Z}/(\ell^n)]) & \longrightarrow & \mathrm{K}_1(M') \\ \cong \downarrow \det & & \cong \downarrow \det \\ \bigoplus_{k=1}^r \mathcal{O}_F[\mathbb{Z}/(\ell^n)]^\times & \xrightarrow{\subseteq} & \bigoplus_{k=1}^{r'} \mathcal{O}_{F'}^\times \end{array}$$

implies that

$$\mathrm{SK}_1(\mathbb{Z}_\ell[G \times \mathbb{Z}/(\ell^n)]) = \ker \mathrm{K}_1(\mathbb{Z}_\ell[G \times \mathbb{Z}/(\ell^n)]) \rightarrow \mathrm{K}_1(M[\mathbb{Z}/(\ell^n)]).$$

By [NSW00, Theorem 5.3.5] the choice of a topological generator $\gamma \in \mathbb{Z}_\ell$ induces an isomorphism

$$\mathbb{Z}_\ell[[T]] \cong \varprojlim_n \mathbb{Z}_\ell[\mathbb{Z}/(\ell^n)], \quad T \mapsto \gamma - 1.$$

In particular, we have compatible isomorphisms

$$\mathrm{K}_1(\mathbb{Z}_\ell[G][[T]]) \cong \varprojlim_n \mathrm{K}_1(\mathbb{Z}_\ell[G \times \mathbb{Z}/(\ell^n)]), \quad \mathrm{K}_1(M[[T]]) \cong \varprojlim_n \mathrm{K}_1(M[\mathbb{Z}/(\ell^n)]).$$

by Proposition 3.5. Hence, we obtain an isomorphism

$$\mathrm{SK}_1(\mathbb{Z}_\ell[G][[T]]) \cong \varprojlim_n \mathrm{SK}_1(\mathbb{Z}_\ell[G \times \mathbb{Z}/(\ell^n)]),$$

as claimed. $\square$

Proposition 5.3. For any finite group G ,

$$\mathrm{SK}_1(\mathbb{Z}_\ell[G][[T]]) \cong \mathrm{SK}_1(\mathbb{Z}_\ell[G]).$$

Proof. By Lemma 5.2 it suffices to prove that the projection $\mathbb{Z}_\ell[G \times \mathbb{Z}/(\ell^n)] \rightarrow \mathbb{Z}_\ell[G]$ induces an isomorphism

$$\mathrm{SK}_1(\mathbb{Z}_\ell[G \times \mathbb{Z}/(\ell^n)]) \cong \mathrm{SK}_1(\mathbb{Z}_\ell[G]).$$

Let $g_1, \dots, g_k$ be a system of representatives for the $\mathbb{Q}_\ell$ -conjugacy classes of elements of order prime to ℓ in G . (Two elements g, h of order r prime to ℓ are called $\mathbb{Q}_\ell$ -conjugated if $g^a = xhx^{-1}$ for some $x \in G$, $a \in \mathrm{Gal}(\mathbb{Q}_\ell(\zeta_r)/\mathbb{Q}_\ell) \subset (\mathbb{Z}/(r))^\times$.) Let r_i denote the order of g_i and set

$$\begin{aligned} N_i(G) &= \{x \in G \mid xg_ix^{-1} = g_i^a \text{ for some } a \in \mathrm{Gal}(\mathbb{Q}_\ell(\zeta_{r_i})/\mathbb{Q}_\ell)\}, \\ Z_i(G) &= \{x \in G \mid xg_ix^{-1} = g_i\}. \end{aligned}$$

Furthermore, let

$$\mathrm{H}_2^{ab}(Z_i(G), \mathbb{Z}) = \mathrm{im} \bigoplus_{\substack{H \subset Z_i(G) \\ \text{abelian}}} \mathrm{H}_2(H, \mathbb{Z}) \rightarrow \mathrm{H}_2(Z_i(G), \mathbb{Z})$$

denote the subgroup of the second homology group generated by elements induced up from abelian subgroups of $Z_i(G)$. According to [Oli88, Theorem 12.5] there exists an isomorphism

$$\mathrm{SK}_1(\mathbb{Z}_\ell[G]) \cong \bigoplus_{i=0}^k \mathrm{H}_0(N_i(G)/Z_i(G), \mathrm{H}_2(Z_i(G), \mathbb{Z})/\mathrm{H}_2^{ab}(Z_i(G), \mathbb{Z}))_{(\ell)}.$$

Now, $(g_1, 0), \dots, (g_k, 0)$ is a system of representatives for the $\mathbb{Q}_\ell$ -conjugacy classes of elements of order prime to ℓ in $G \times \mathbb{Z}/(\ell^n)$ and

$$N_i(G \times \mathbb{Z}/(\ell^n)) = N_i(G) \times \mathbb{Z}/(\ell^n), \quad Z_i(G \times \mathbb{Z}/(\ell^n)) = Z_i(G) \times \mathbb{Z}/(\ell^n).$$

By *loc. cit.*, Proposition 8.12, we have

$$\begin{aligned} \mathrm{H}_2(Z_i(G) \times \mathbb{Z}/(\ell^n), \mathbb{Z})/\mathrm{H}_2^{ab}(Z_i(G) \times \mathbb{Z}/(\ell^n), \mathbb{Z}) = \\ \mathrm{H}_2(Z_i(G), \mathbb{Z})/\mathrm{H}_2^{ab}(Z_i(G), \mathbb{Z}) \times \mathrm{H}_2(\mathbb{Z}/(\ell^n), \mathbb{Z})/\mathrm{H}_2^{ab}(\mathbb{Z}/(\ell^n), \mathbb{Z}) \end{aligned}$$

and clearly,

$$\mathrm{H}_2(\mathbb{Z}/(\ell^n), \mathbb{Z}) = \mathrm{H}_2^{ab}(\mathbb{Z}/(\ell^n), \mathbb{Z}).$$

From this, the claim of the lemma follows easily. $\square$

The following proposition was proved in [FK06, Proposition 2.3.7] in the case of number fields.

Proposition 5.4. *Let Q be a function field of transcendence degree 1 over a finite field $\mathbb{F}$ and let ℓ be any prime. Then*

$$\varprojlim_L \mathrm{SK}_1(\mathbb{Z}_\ell[\mathrm{Gal}(L/Q)]) = 0.$$

where L runs through the finite Galois extensions of Q in a fixed separable closure $\overline{Q}$.

Proof. By the same argument as in the proof of Proposition 2.3.7 in [FK06] it suffices to prove that

$$\mathrm{H}^2(\mathrm{Gal}(\overline{Q}/L), \mathbb{Q}_\ell/\mathbb{Z}_\ell) = 0$$

for any finite extension L of Q .

If ℓ is different from the characteristic of $\mathbb{F}$, the vanishing of this group can be deduced via the same argument as the analogous statement for number fields given in [Sch79, § 4, Satz 1]: Let

$$L_\infty = \bigcup_n L(\zeta_{\ell^n}).$$

Then

$$\mathrm{H}^2(\mathrm{Gal}(\overline{Q}/L), \mathbb{Q}_\ell/\mathbb{Z}_\ell) = \mathrm{H}^1(\mathrm{Gal}(L_\infty/L), \mathrm{H}^1(\mathrm{Gal}(\overline{Q}/L_\infty), \mathbb{Q}_\ell/\mathbb{Z}_\ell))$$

and by Kummer theory,

$$\mathrm{H}^1(\mathrm{Gal}(\overline{Q}/L_\infty), \mathbb{Q}_\ell/\mathbb{Z}_\ell) = L_\infty^\times \otimes_{\mathbb{Z}} \mathbb{Q}_\ell/\mathbb{Z}_\ell(-1).$$

Now

$$\mathrm{H}^1(\mathrm{Gal}(L_\infty/L), L_\infty^\times \otimes_{\mathbb{Z}} \mathbb{Q}_\ell/\mathbb{Z}_\ell(-1)) = \varinjlim_n \mathrm{H}^1(\mathrm{Gal}(L_\infty/L), L(\zeta_{\ell^n})^\times \otimes_{\mathbb{Z}} \mathbb{Q}_\ell/\mathbb{Z}_\ell(-1))$$

by [NSW00, Proposition 1.5.1] and

$$\mathrm{H}^1(\mathrm{Gal}(L_\infty/L), L(\zeta_{\ell^n})^\times \otimes_{\mathbb{Z}} \mathbb{Q}_\ell/\mathbb{Z}_\ell(-1)) = (L(\zeta_{\ell^n})^\times \otimes_{\mathbb{Z}} \mathbb{Q}_\ell/\mathbb{Z}_\ell(-1))_{\mathrm{Gal}(L_\infty/L)}$$

by *loc. cit.*, Proposition 1.6.13. Since the latter group is a factor group of

$$(L(\zeta_{\ell^n})^\times \otimes_{\mathbb{Z}} \mathbb{Q}_\ell/\mathbb{Z}_\ell(-1))_{\mathrm{Gal}(L_\infty/L(\zeta_{\ell^n}))} = 0,$$

the claim is proved.

If ℓ is equal to the characteristic of $\mathbb{F}$, then the cohomological ℓ -dimension of $\text{Gal}(\overline{Q}/L)$ is known to be 1 and hence, the second cohomology group of $\mathbb{Q}_\ell/\mathbb{Z}_\ell$ vanishes for trivial reasons. $\square$

6. L -FUNCTIONS

Consider an adic ring Λ and let $\Lambda[[T]]$ denote the ring of power series in the indeterminate T (where T is assumed to commute with every element of Λ). The ring $\Lambda[[T]]$ is still an adic ring whose Jacobson radical $\text{Jac}(\Lambda[[T]])$ is generated by $\text{Jac}(\Lambda)$ and T . In particular, we conclude from Proposition 3.5 that

$$K_1(\Lambda[[T]]) = \varprojlim_n K_1(\Lambda[[T]]/\text{Jac}(\Lambda[[T]])^n)$$

is a profinite group.

Let $\mathbb{F}$ be a finite field. We write X^0 for the set of closed points of a scheme X in $\mathbf{Sch}_{\mathbb{F}}$. If $x \in X^0$ is a closed point, then

$$\overline{x} = x \times_{\text{Spec } \mathbb{F}} \text{Spec } \overline{\mathbb{F}}$$

consists of finitely many points, whose number is given by the degree $\deg(x)$ of x , i. e. the degree of the residue field $k(x)$ of x as a field extension of $\mathbb{F}$. Let

$$s_x: \overline{x} \rightarrow X$$

denote the structure map. For any complex

$$\mathcal{F}^\bullet = (\mathcal{F}_I^\bullet)_{I \in \mathfrak{I}_\Lambda}$$

in $\mathbf{PDG}^{\text{cont}}(X, \Lambda)$, we write

$$\mathcal{F}_x^\bullet = (\Gamma(\overline{x}, s_x^* \mathcal{F}_I^\bullet))_{I \in \mathfrak{I}_\Lambda}.$$

This defines a Waldhausen exact functor

$$\mathbf{PDG}^{\text{cont}}(X, \Lambda) \rightarrow \mathbf{PDG}^{\text{cont}}(\Lambda), \quad \mathcal{F}^\bullet \mapsto \mathcal{F}_x^\bullet.$$

Note that $\mathcal{F}_x^\bullet$ can also be written as the product over the stalks of $\mathcal{F}$ in the points of $\overline{x}$:

$$\mathcal{F}_x^\bullet \cong \prod_{\xi \in \overline{x}} ((\mathcal{F}_I^\bullet)_\xi)_{I \in \mathfrak{I}_\Lambda}.$$

The geometric Frobenius automorphism

$$\mathfrak{F}_{\mathbb{F}} \in \text{Gal}(\overline{\mathbb{F}}/\mathbb{F})$$

operates on $\mathcal{F}_x^\bullet$ through its action on $\overline{x}$. Hence, it also operates on $\Psi_{\Lambda[[T]]}(\mathcal{F}_x^\bullet)$. Here,

$$\Psi_{\Lambda[[T]]}: \mathbf{PDG}^{\text{cont}}(\Lambda) \rightarrow \mathbf{PDG}^{\text{cont}}(\Lambda[[T]])$$

denotes the change of ring functor with respect to the $\Lambda[[T]]$ - Λ -bimodule $\Lambda[[T]]$, as constructed in Definition 3.9. The morphism

$$\text{id} - \mathfrak{F}_{\mathbb{F}} T: \Psi_{\Lambda[[T]]}(\mathcal{F}_x^\bullet) \rightarrow \Psi_{\Lambda[[T]]}(\mathcal{F}_x^\bullet).$$

is a natural isomorphism whose inverse is given by

$$\sum_{n=0}^{\infty} \mathfrak{F}_{\mathbb{F}}^n T^n.$$

Definition 6.1. The class

$$E_x(\mathcal{F}^\bullet, T) = [\Psi_{\Lambda[[T]]}(\mathcal{F}_x^\bullet) \xrightarrow{\text{id} - \mathfrak{F}_{\mathbb{F}} T} \Psi_{\Lambda[[T]]}(\mathcal{F}_x^\bullet)]^{-1}$$

in $K_1(\Lambda[[T]])$ is called the *Euler factor* of $\mathcal{F}^\bullet$ at x .

One can easily verify that the Euler factor is multiplicative on exact sequences and that

$$E_x(\mathcal{F}^\bullet, T) = E_x(\mathcal{G}^\bullet, T)$$

if the complexes $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$ are quasi-isomorphic. Hence, the above assignment extends to a homomorphism

$$E_x(-, T): K_0(\mathbf{PDG}^{\text{cont}}(X, \Lambda)) \rightarrow K_1(\Lambda[[T]]).$$

Lemma 6.2. *Let $\xi \in \bar{x}$ be a geometric point. Then*

$$E_x(\mathcal{F}^\bullet, T) = [\Psi_{\Lambda[[T]]}(\mathcal{F}_\xi^\bullet) \xrightarrow{\text{id} - \mathfrak{F}_{k(x)} T^{\deg(x)}} \Psi_{\Lambda[[T]]}(\mathcal{F}_\xi^\bullet)]^{-1}.$$

Proof. The Frobenius automorphism $\mathfrak{F}_{\mathbb{F}}$ induces isomorphisms $\mathcal{F}_{\mathfrak{F}^k \xi}^\bullet \cong \mathcal{F}_\xi^\bullet$ for $k = 1, \dots, \deg(x)$. For $k = \deg(x)$ we have $\mathfrak{F}_{\mathbb{F}}^k \xi = \xi$ and the isomorphism is given by the operation of $\mathfrak{F}_{k(x)}$ on $\mathcal{F}_\xi^\bullet$. Hence, we may identify $\mathcal{F}_x^\bullet$ with the complex $(\mathcal{F}_\xi^\bullet)^{\deg(x)}$, on which the Frobenius $\mathfrak{F}_{\mathbb{F}}$ acts through the matrix

$$\begin{pmatrix} 0 & \dots & 0 & \mathfrak{F}_{k(x)} \\ \text{id} & 0 & \dots & 0 \\ 0 & \text{id} & 0 & \dots \\ \vdots & \ddots & \ddots & \ddots \\ 0 & \dots & 0 & \text{id} \end{pmatrix}.$$

Let A be the automorphism of $\Psi_{\Lambda[[T]]}((\mathcal{F}_\xi^\bullet)^{\deg(x)})$ given by the matrix

$$\begin{pmatrix} \text{id} & 0 & \dots & 0 \\ \text{id}T & \text{id} & 0 & \dots \\ \text{id}T^2 & \text{id}T & \text{id} & 0 \\ \vdots & \ddots & \ddots & \ddots \\ \text{id}T^{\deg(x)-1} & \dots & \text{id}T^2 & \text{id}T & \text{id} \end{pmatrix}$$

Then $A(\text{id} - \mathfrak{F}_{\mathbb{F}}T)$ corresponds to the matrix

$$\begin{pmatrix} \text{id} & 0 & \dots & 0 & -\mathfrak{F}_{k(x)}T \\ 0 & \text{id} & 0 & \dots & -\mathfrak{F}_{k(x)}T^2 \\ \vdots & & \ddots & & \vdots \\ 0 & \dots & 0 & \text{id} & -\mathfrak{F}_{k(x)}T^{\deg(x)-1} \\ 0 & \dots & 0 & (\text{id} - \mathfrak{F}_{k(x)}T^{\deg(x)}) \end{pmatrix}$$

Moreover, we have $[A] = 1$ in $K_1(\Lambda[[T]])$. Hence,

$$[\Psi_{\Lambda[[T]]}(\mathcal{F}_x^\bullet) \xrightarrow{\text{id} - \mathfrak{F}_{\mathbb{F}}T} \Psi_{\Lambda[[T]]}(\mathcal{F}_x^\bullet)] = [\Psi_{\Lambda[[T]]}(\mathcal{F}_\xi^\bullet) \xrightarrow{\text{id} - \mathfrak{F}_{k(x)}T^{\deg(x)}} \Psi_{\Lambda[[T]]}(\mathcal{F}_\xi^\bullet)]$$

as claimed. $\square$

Proposition 6.3. *The infinite product*

$$\prod_{x \in X^0} E_x(\mathcal{F}^\bullet, T)$$

converges in the profinite topology of $K_1(\Lambda[[T]])$.

Proof. For each integer m , there exist only finitely many closed points $x \in X^0$ with $\deg(x) < m$. If $\deg(x) \geq m$, then we conclude from Lemma 6.2 that the image of $E_x(\mathcal{F}^\bullet, T)$ in $K_1(\Lambda[[T]]/(T^m))$ is 1. $\square$

Definition 6.4. The L -function of the complex $\mathcal{F}^\bullet$ in $\mathbf{PDG}^{\text{cont}}(X, \Lambda)$ is given by

$$L_{\mathbb{F}}(\mathcal{F}^\bullet, T) = \prod_{x \in X^0} E_x(\mathcal{F}^\bullet, T) \in K_1(\Lambda[[T]])$$

Remark 6.5. If $\mathbb{F}'$ is a subfield of $\mathbb{F}$, then Lemma 6.2 implies that

$$L_{\mathbb{F}'}(\mathcal{F}^\bullet, T) = L_{\mathbb{F}}(\mathcal{F}^\bullet, T^{[\mathbb{F}:\mathbb{F}']}) \in K_1(\Lambda[[T]]).$$

Remark 6.6. If Λ is commutative, the determinant induces an isomorphism

$$\det: K_1(\Lambda[[T]]) \rightarrow \Lambda[[T]]^\times.$$

In particular, we see that the L -function agrees with the one defined in [Del77, Fonction $L \bmod \ell^n$] in the case of commutative adic rings.

7. THE GROTHENDIECK TRACE FORMULA

In this section, we will prove the Grothendieck trace formula for our L -functions.

Definition 7.1. For a scheme X in $\mathbf{Sch}_{\mathbb{F}}^{\text{sep}}$ and a complex $\mathcal{F}^\bullet$ in $\mathbf{PDG}^{\text{cont}}(X, \Lambda)$ we let $\mathcal{L}_{\mathbb{F}}(\mathcal{F}^\bullet, T)$ denote the element

$$[\Psi_{\Lambda[[T]]}(\mathrm{R}_{\mathbb{F}} \Gamma_c(\overline{X}, \mathcal{F}^\bullet)) \xrightarrow{\mathrm{id} - \mathfrak{F}_{\mathbb{F}} T} \Psi_{\Lambda[[T]]}(\mathrm{R}_{\mathbb{F}} \Gamma_c(\overline{X}, \mathcal{F}^\bullet))]^{-1}$$

in $K_1(\Lambda[[T]])$.

Theorem 7.2 (Grothendieck trace formula). *Let $\mathbb{F}$ be a finite field of characteristic p and let Λ be an adic ring such that p is invertible in Λ . Then*

$$L_{\mathbb{F}}(\mathcal{F}^\bullet, T) = \mathcal{L}_{\mathbb{F}}(\mathcal{F}^\bullet, T)$$

for every scheme X in $\mathbf{Sch}_{\mathbb{F}}^{\text{sep}}$ and every complex $\mathcal{F}^\bullet$ in $\mathbf{PDG}^{\text{cont}}(X, \Lambda)$.

We proceed by a series of lemmas, following closely along the lines of [Mil80, Chapter VI, §13].

Lemma 7.3. *Let U be an open subscheme of X with closed complement Z . Theorem 7.2 is true for X if it is true for U and Z .*

Proof. Write $j: U \hookrightarrow X$ and $i: Z \hookrightarrow X$ for the corresponding immersions,

$$u: U \rightarrow \mathrm{Spec} \mathbb{F}, \quad x: X \rightarrow \mathrm{Spec} \mathbb{F}, \quad z: Z \rightarrow \mathrm{Spec} \mathbb{F}$$

for the structure morphisms. Clearly,

$$L(X, \mathcal{F}^\bullet) = L(U, j^* \mathcal{F}^\bullet) L(Z, i^* \mathcal{F}^\bullet)$$

On the other hand, we have an exact sequence

$$\mathrm{R} x_! j_! j^* \mathcal{F}^\bullet \rightarrow \mathrm{R} x_! \mathcal{F}^\bullet \rightarrow \mathrm{R} x_! i_* i^* \mathcal{F}^\bullet$$

and (chains of) quasi-isomorphisms

$$\mathrm{R} u_! j^* \mathcal{F}^\bullet \simeq \mathrm{R} x_! j_! j^* \mathcal{F}^\bullet \quad \mathrm{R} z_! i^* \mathcal{F}^\bullet \simeq \mathrm{R} x_! i_* i^* \mathcal{F}^\bullet.$$

Hence,

$$[\mathrm{R} x_! \mathcal{F}^\bullet] = [\mathrm{R} u_! j^* \mathcal{F}^\bullet] [\mathrm{R} z_! i^* \mathcal{F}^\bullet]$$

in $K_0 \mathbf{PDG}^{\text{cont}}(\mathrm{Spec} \mathbb{F}, \Lambda)$. The homomorphism

$$K_0 \mathbf{PDG}^{\text{cont}}(\mathrm{Spec} \mathbb{F}, \Lambda) \rightarrow K_1(\Lambda[[T]]),$$

$$[\mathcal{F}^\bullet] \mapsto [\Psi_{\Lambda[[T]]}(\Gamma(\mathrm{Spec} \overline{\mathbb{F}}, s^* \mathcal{F}^\bullet)) \xrightarrow{1 - \mathfrak{F}_{\mathbb{F}} T} \Psi_{\Lambda[[T]]}(\Gamma(\mathrm{Spec} \overline{\mathbb{F}}, s^* \mathcal{F}^\bullet))]^{-1}$$

preserves this relation. $\square$

Next, we prove that the formula is compatible with changes of the base field.

Lemma 7.4. *Let $\mathbb{F}'$ be a subfield of $\mathbb{F}$ and X a scheme in $\mathbf{Sch}_{\mathbb{F}}^{\text{sep}}$. Then*

$$\mathcal{L}_{\mathbb{F}'}(\mathcal{F}^\bullet, T) = \mathcal{L}_{\mathbb{F}}(\mathcal{F}^\bullet, T^{[\mathbb{F}:\mathbb{F}']}).$$

Proof. Let $r: \text{Spec } \mathbb{F} \rightarrow \text{Spec } \mathbb{F}'$ be the morphism induced by the inclusion $\mathbb{F}' \subset \mathbb{F}$ and write

$$\begin{aligned} h: X \times_{\text{Spec } \mathbb{F}} \text{Spec } \overline{\mathbb{F}} &\rightarrow X, & h': X \times_{\text{Spec } \mathbb{F}'} \text{Spec } \overline{\mathbb{F}} &\rightarrow X \\ s: \text{Spec } \overline{\mathbb{F}} &\rightarrow \text{Spec } \mathbb{F}, & s': \text{Spec } \overline{\mathbb{F}} &\rightarrow \text{Spec } \mathbb{F}' \end{aligned}$$

for the corresponding structure morphisms. For any $\mathcal{F}^\bullet$ in $\mathbf{PDG}^{\text{cont}}(X, \Lambda)$, the complexes $r_* R h_! \mathcal{F}^\bullet$, $R r_! R h_! \mathcal{F}^\bullet$, and $R h_! \mathcal{F}^\bullet$ in $\mathbf{PDG}^{\text{cont}}(\text{Spec } \mathbb{F}', \Lambda)$ are quasi-isomorphic. Moreover, for any complex $\mathcal{G}^\bullet$ in $\mathbf{PDG}^{\text{cont}}(\text{Spec } \mathbb{F}, \Lambda)$, the following diagram is commutative:

$$\begin{array}{ccc} \Gamma(\text{Spec } \overline{\mathbb{F}}, s^* r^* r_* \mathcal{G}^\bullet) & \xrightarrow{\cong} & \bigoplus_{k=1}^{[\mathbb{F}:\mathbb{F}']} \Gamma(\text{Spec } \overline{\mathbb{F}}, s^* \mathcal{G}^\bullet) \\ \downarrow \mathfrak{F}_{\mathbb{F}'} & & \downarrow \begin{pmatrix} 0 & \dots & 0 & \mathfrak{F}_{\mathbb{F}} \\ \text{id} & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & \text{id} & 0 \end{pmatrix} \\ \Gamma(\text{Spec } \overline{\mathbb{F}}, s^* r^* r_* \mathcal{G}^\bullet) & \xrightarrow{\cong} & \bigoplus_{k=1}^{[\mathbb{F}:\mathbb{F}']} \Gamma(\text{Spec } \overline{\mathbb{F}}, s^* \mathcal{G}^\bullet) \end{array}$$

As in the proof of Lemma 6.2 one concludes

$$\begin{aligned} & [\Psi_{\Lambda[[T]]}(\mathbf{R}_{\mathbb{F}'} \Gamma_c(\overline{X}, \mathcal{F}^\bullet)) \xrightarrow{\text{id} - \mathfrak{F}_{\mathbb{F}'} T} \Psi_{\Lambda[[T]]}(\mathbf{R}_{\mathbb{F}'} \Gamma_c(\overline{X}, \mathcal{F}^\bullet))] = \\ & [\Psi_{\Lambda[[T]]}(\Gamma(\text{Spec } \overline{\mathbb{F}}, s^* r^* r_* R h_! \mathcal{F}^\bullet)) \xrightarrow{\text{id} - \mathfrak{F}_{\mathbb{F}'} T} \Psi_{\Lambda[[T]]}(\Gamma(\text{Spec } \overline{\mathbb{F}}, s^* r^* r_* R h_! \mathcal{F}^\bullet))] = \\ & [\Psi_{\Lambda[[T]]}(\Gamma(\text{Spec } \overline{\mathbb{F}}, s^* R h_! \mathcal{F}^\bullet)) \xrightarrow{\text{id} - \mathfrak{F}_{\mathbb{F}} T^{[\mathbb{F}:\mathbb{F}']}} \Psi_{\Lambda[[T]]}(\Gamma(\text{Spec } \overline{\mathbb{F}}, s^* R h_! \mathcal{F}^\bullet))] = \\ & [\Psi_{\Lambda[[T]]}(\mathbf{R}_{\mathbb{F}} \Gamma_c(\overline{X}, \mathcal{F}^\bullet)) \xrightarrow{\text{id} - \mathfrak{F}_{\mathbb{F}} T^{[\mathbb{F}:\mathbb{F}']}} \Psi_{\Lambda[[T]]}(\mathbf{R}_{\mathbb{F}} \Gamma_c(\overline{X}, \mathcal{F}^\bullet))]. \end{aligned}$$

□

Clearly, Theorem 7.2 is true for schemes of dimension 0. Next, we consider the case that X is a curve.

Lemma 7.5. *The formula in Theorem 7.2 is true for any smooth and geometrically connected curve X , $\Lambda = \mathbb{Z}_\ell[G]$, and $\mathcal{F}^\bullet = \mathbb{Z}_\ell[G]_X^\sharp$, where ℓ is a prime different from the characteristic of $\mathbb{F}$ and G is the Galois group of a finite Galois covering of X .*

Proof. Let Q be the function field of X and let F the function field of a finite Galois covering of X . Let d_F denote the element

$$d_F = L(\mathbb{Z}_\ell[\text{Gal}(F/Q)]_X^\sharp, T) \mathcal{L}(\mathbb{Z}_\ell[\text{Gal}(F/Q)]_X^\sharp, T)^{-1}$$

in $K_1(\mathbb{Z}_\ell[\text{Gal}(F/Q)][[T]])$.

Note that d_F does not change if we replace X by an open subscheme of X . Hence, we may define d_F for any finite Galois extension F of Q such that, if F'/F is Galois, $d_{F'}$ is mapped onto d_F under the canonical homomorphism

$$K_1(\mathbb{Z}_\ell[\text{Gal}(F'/Q)][[T]]) \rightarrow K_1(\mathbb{Z}_\ell[\text{Gal}(F/Q)][[T]]).$$

Let L be a splitting field for $\mathbb{Q}_\ell[\text{Gal}(F/Q)]$ and $M \subset L[\text{Gal}(F/Q)]$ a maximal $\mathbb{Z}_\ell$ -order. By the classical Grothendieck trace formula [Del77, Fonction $L \bmod \ell^n$,

Theorem 2.2.(a)], the image of d_F under the homomorphism

$$K_1(\mathbb{Z}_\ell[\text{Gal}(F/Q)][[T]]) \rightarrow K_1(M[[T]]) \cong \bigoplus_{r=1}^s \mathcal{O}_L[[T]]^\times$$

is trivial; hence $d_F \in \text{SK}_1(\mathbb{Z}_\ell[\text{Gal}(F/Q)][[T]]) = \text{SK}_1(\mathbb{Z}_\ell[\text{Gal}(F/Q)])$. From Proposition 5.4 we conclude $d_F = 0$. $\square$

Lemma 7.6. *The formula in Theorem 7.2 is true for any scheme X in $\mathbf{Sch}_{\mathbb{F}}^{\text{sep}}$ of dimension less or equal 1, any adic ring Λ with $p \in \Lambda^\times$ and any complex $\mathcal{F}^\bullet$ in $\mathbf{PDG}^{\text{cont}}(X, \Lambda)$.*

Proof. By Proposition 3.5 it suffices to consider finite rings Λ . The ℓ -Sylow subgroups of Λ are subrings of Λ and Λ is equal to their direct product. Since p is invertible, the p -Sylow subgroup is trivial. Hence, we may further assume that Λ is a $\mathbb{Z}_\ell$ -algebra for $\ell \neq p$.

Shrinking X if necessary we may assume that X is smooth, irreducible curve and that $\mathcal{F}^\bullet$ is a strictly perfect complex of locally constant sheaves. By replacing $\mathbb{F}$ with its algebraic closure in the function field of X and using Lemma 7.4, we may assume that X is geometrically connected. By Lemma 4.12 and Proposition 4.11 we have

$$\mathcal{L}(\mathcal{F}^\bullet, T) = K_1(\Psi_{M \otimes_{\mathbb{Z}_\ell[G]} \mathbb{Z}_\ell[G][[T]]})(\mathcal{L}(\mathbb{Z}_\ell[G]_X^\sharp, T))$$

for a suitable Galois group G and a complex $M^\bullet$ in $\mathbb{Z}_\ell[G]^\mathbf{q}\text{-}\mathbf{SP}(\Lambda)$. Likewise,

$$L(\mathcal{F}^\bullet, T) = K_1(\Psi_{M \otimes_{\mathbb{Z}_\ell[G]} \mathbb{Z}_\ell[G][[T]]})(L(\mathbb{Z}_\ell[G]_X^\sharp, T)).$$

Now the assertion follows from Lemma 7.5. $\square$

We complete the proof of Theorem 7.2 by induction on the dimension d of X . By shrinking X if necessary we may assume that there exists a morphism $f: X \rightarrow Y$ such that Y and all fibres of f have dimension less than d . Then Proposition 4.7.(3) and the induction hypothesis imply

$$\mathcal{L}(\mathcal{F}^\bullet, T) = \mathcal{L}(\mathbf{R} f_* \mathcal{F}^\bullet, T) = L(\mathbf{R} f_* \mathcal{F}^\bullet, T).$$

Let now y be a closed point of Y . Write $f_y: X_y \rightarrow X$ for the fibre over y . Then

$$\begin{aligned} E_y(\mathbf{R} f_* \mathcal{F}^\bullet, T) &= [\Psi_{\Lambda[[T]]}(\mathbf{R} \Gamma_c(\overline{X}_y, f_y^* \mathcal{F}^\bullet)) \xrightarrow{\text{id} - \mathfrak{F}_T} \Psi_{\Lambda[[T]]}(\mathbf{R} \Gamma_c(\overline{X}_y, f_y^* \mathcal{F}^\bullet))]^{-1} \\ &= L(f_y^* \mathcal{F}^\bullet, T) \end{aligned}$$

by Proposition 4.7.(4) and the induction hypothesis. Since clearly

$$L(\mathcal{F}^\bullet, T) = \prod_{y \in Y^0} L(f_y^* \mathcal{F}^\bullet, T),$$

Theorem 7.2 follows.

Remark 7.7. The formula in Theorem 7.2 is also valid if Λ is a finite field of characteristic p , see [Del77, Fonction $L \bmod \ell^n$, Theorem 2.2.(b)]. However, it does not extend to general adic $\mathbb{Z}_p$ -algebras. We refer to *loc. cit.*, §4.5 for a counterexample.

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