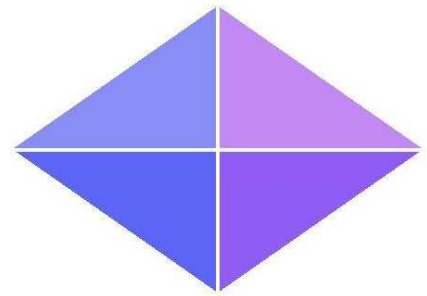


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Extensions of profinite duality groups

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Let G be a profinite group and let p be a prime number. By $Mod_p(G)$ we denote the category of discrete p -primary G -modules. For $A \in Mod_p(G)$ and $i \geq 0$, let

$$D_i(G, A) = \varinjlim_U H^i(U, A)^*,$$

where $*$ is $\text{Hom}(-, \mathbb{Q}_p/\mathbb{Z}_p)$, the direct limit is taken over all open subgroups U of G and the transition maps are the duals of the corestriction maps. $D_i(G, A)$ is a discrete G -module in a natural way. Assume that $n = cd_p G$ is finite. Then the G -module

$$I(G) = \varinjlim_{\nu \in \mathbb{N}} D_n(G, \mathbb{Z}/p^\nu \mathbb{Z})$$

is called the **dualizing module** of G at p . Its importance lies in the functorial isomorphism

$$H^n(G, A)^* \cong \text{Hom}_G(A, I(G))$$

for all $A \in Mod_p(G)$. This isomorphism is induced by the cup-products ($V \subseteq U$)

$$H^n(G, A)^* \times_{p^\nu} A^U \longrightarrow H^n(V, \mathbb{Z}/p^\nu \mathbb{Z})^*, (\phi, a) \longmapsto \left(\alpha \mapsto \phi(\text{cor}_G^V(\alpha \cup a)) \right)$$

by passing to the limit over ν and V , and then over U . The identity-map of $I(G)$ gives rise to the homomorphism

$$\text{tr} : H^n(G, I(G)) \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p,$$

called the **trace map**.

The profinite group G is called a **duality group at p of dimension n** if for all $i \in \mathbb{Z}$ and all finite G -modules $A \in Mod_p(G)$, the cup-product and the trace map

$$H^i(G, \operatorname{Hom}(A, I(G))) \times H^{n-i}(G, A) \xrightarrow{\cup} H^n(G, I(G)) \xrightarrow{tr} \mathbb{Q}_p/\mathbb{Z}_p$$

yield an isomorphism

$$H^i(G, \operatorname{Hom}(A, I(G))) \cong H^{n-i}(G, A)^*.$$

Remark: In [Ve], J.-L. Verdier used the name **strict Cohen-Macaulay at p** for what we call a profinite duality group at p here. In [Pl], A. Pletch defined D_p^n -groups (and called them duality groups at p of dimension n). The D_p^n -groups of Pletch are exactly the duality groups at p (in our sense) which, in addition, satisfy the following finiteness condition:

$FC(G, p)$: $H^i(G, A)$ is finite for all finite $A \in \operatorname{Mod}_p(G)$ and for all $i \geq 0$.

Since any finite, discrete G -module is trivialized by an open subgroup U of G , condition $FC(G, p)$ can also be rephrased in the form:

$FC(G, p)$: $H^i(U, \mathbb{Z}/p\mathbb{Z})$ is finite for all open subgroups U of G and all $i \geq 0$.

By a duality theorem due to J. Tate, see [Ta] Thm. 3 or [Ve] Prop. 4.3 or [NSW] (3.4.6), a profinite group G of cohomological p -dimension n is a duality group at p if and only if

$$D_i(G, \mathbb{Z}/p\mathbb{Z}) = 0 \quad \text{for } 0 \leq i < n.$$

As a consequence we see that every open subgroup of a duality group at p is a duality group at p as well (of the same cohomological dimension), and if an open subgroup of G is a duality group at p and $cd_p G < \infty$, then G is a duality group at p of the same cohomological dimension (use [NSW] (3.3.5)(ii)). Furthermore, any profinite group of cohomological p -dimension 1 is a duality group at p .

We call a profinite group G **virtually a duality group at p of (virtual) dimension $gcd_p G = n$** if an open subgroup U of G is a duality group at p of dimension n .

The objective of this paper is to give a proof of Theorem 1 below, which states that the class of duality groups is closed under group extensions $1 \rightarrow H \rightarrow G \rightarrow G/H \rightarrow 1$ if the kernel satisfies $FC(H, p)$. Weaker forms of Theorem 1 were first proved by A. Pletch (for D_p^n -groups, see [Pl]¹) and by the second author (for Poincaré groups, see [Wi]).

¹The proof given by Pletch in [Pl] is only correct for pro- p -groups as the author assumes that finitely generated projective modules over the complete group ring $\mathbb{Z}_p[[G]]$ are free.

Theorem 1. *Let*

$$1 \longrightarrow H \longrightarrow G \longrightarrow G/H \longrightarrow 1$$

be an exact sequence of profinite groups such that condition $FC(H, p)$ is satisfied. Then the following assertions hold:

- (i) *If G is a duality group at p , then H is a duality group at p and G/H is virtually a duality group at p .*
- (ii) *If H and G/H are duality groups at p , then G is a duality group at p .*

Moreover, in both cases we have:

$$cd_p G = cd_p H + vcd_p G/H,$$

and there is a canonical G -isomorphism

$$I(G)^\vee \cong I(H)^\vee \hat{\otimes}_{\mathbb{Z}_p} I(G/H)^\vee,$$

where $^\vee$ is the Pontryagin dual and $\hat{\otimes}_{\mathbb{Z}_p}$ is the tensor-product in the category of compact $\mathbb{Z}_p$ -modules.

Remark: The assumption $FC(H, p)$ is necessary, as the following examples show:

1. Let G be the free pro- p -group on two generators x, y and let $H \subset G$ be the normal subgroup generated by x . Then H is free of infinite rank, G/H is free of rank one and $1 \rightarrow H \rightarrow G \rightarrow G/H \rightarrow 1$ is an exact sequence in which all three groups are duality groups of dimension one.
2. Let D be a duality group at p of dimension 2, F a duality group at p of dimension 1 and $G = F * D$ their free product. The kernel of the projection $G \twoheadrightarrow D$ has cohomological p -dimension 1, hence is a duality group at p of dimension 1. The group G has cohomological p -dimension 2 but is not a duality group at p .

In the proof of Theorem 1, we make use of the following

Proposition 2. *Let*

$$1 \longrightarrow H \longrightarrow G \longrightarrow G/H \longrightarrow 1$$

be an exact sequence of profinite groups. Assume that $FC(H, p)$ holds. Then there is a spectral sequence of homological type

$$E_{ij}^2 = D_i(G/H, \mathbb{Z}/p\mathbb{Z}) \otimes D_j(H, \mathbb{Z}/p\mathbb{Z}) \implies D_{i+j}(G, \mathbb{Z}/p\mathbb{Z}).$$

Proof. Let g run through the open normal subgroups of G . Then $gH/H \cong g/g \cap H$ runs through the open normal subgroups of G/H . For a G -module $A \in \text{Mod}_p(G)$, we consider the Hochschild-Serre spectral sequence

$$E(g, g \cap H, A) : E_2^{ij}(g, g \cap H, A) = H^i(g/g \cap H, H^j(g \cap H, A)) \implies H^{i+j}(g, A).$$

If $g' \subseteq g$ is another open normal subgroup of G , then the corestriction yields a morphism

$$\text{cor} : E(g', g' \cap H, A) \longrightarrow E(g, g \cap H, A)$$

of spectral sequences. The map

$$E_2^{ij}(g', g' \cap H, A) \longrightarrow E_2^{ij}(g, g \cap H, A)$$

is the composite of the maps

$$\begin{aligned} H^i(g'/g' \cap H, H^j(g' \cap H, A)) &\xrightarrow{\text{cor}_{g' \cap H}^{g' \cap H}} H^i(g'/g' \cap H, H^j(g \cap H, A)) \\ &\xrightarrow{\text{cor}_{g/g \cap H}^{g'/g' \cap H}} H^i(g/g \cap H, H^j(g \cap H, A)) \end{aligned}$$

and the map between the limit terms is the corestriction

$$\text{cor}_g^{g'} : H^{i+j}(g', A) \longrightarrow H^{i+j}(g, A).$$

For $2 \leq r \leq \infty$ we set

$$E_{ij}^2 = D_{ij}^r(G, H, A) := \varinjlim_g E_r^{ij}(g, g \cap H, A)^*.$$

As taking duals and direct limits are exact operations, the terms $D_{ij}^r(G, H, A)$, $2 \leq r \leq \infty$, establish a homological spectral sequence which converges to $D_n(G, A)$. If h runs through the open subgroups of H which are normal in G , then the cohomology groups $H^j(h, A)$ are G -modules in a natural way. If g is open in G with $g \cap H \subseteq h$, then these groups are $g/g \cap H$ -modules. We see that

$$D_{ij}^2(G, H, A) = \varinjlim_{\substack{h \subseteq H \\ h \trianglelefteq G}} \varinjlim_{\substack{g \subseteq G \\ g \cap H \subseteq h}} H^i(g/g \cap H, H^j(h, A))^*,$$

where for both limits the transition maps are (induced by) cor^* . In order to conclude the proof of the proposition, it remains to construct isomorphisms

$$D_{ij}^2(G, H, \mathbb{Z}/p\mathbb{Z}) \cong D_i(G/H, \mathbb{Z}/p\mathbb{Z}) \otimes D_j(H, \mathbb{Z}/p\mathbb{Z})$$

for all i and j . To this end note that all occurring abelian groups are $\mathbb{F}_p$ -vector spaces, so that $*$ is $\text{Hom}(-, \mathbb{F}_p)$. Further note that for vector spaces V, W over a field k the homomorphism

$$V^* \otimes W^* \longrightarrow (V \otimes W)^*, \quad \phi \otimes \psi \longmapsto (v \otimes w \mapsto \phi(v)\psi(w))$$

is an isomorphism provided that V or W is finite-dimensional. Let h be an open subgroup of H which is normal in G and let $g' \subseteq g$ be open subgroups of G such that g acts trivially on the finite group $H^j(h, \mathbb{Z}/p\mathbb{Z})$. Then, by [NSW] (1.5.3)(iv), the diagram

$$\begin{array}{ccc} H^i(g'/g' \cap H, \mathbb{Z}/p\mathbb{Z}) \otimes H^j(h, \mathbb{Z}/p\mathbb{Z}) & \xrightarrow{\cup} & H^i(g'/g' \cap H, H^j(h, \mathbb{Z}/p\mathbb{Z})) \\ \downarrow \text{cor} \otimes \text{id} & & \downarrow \text{cor} \\ H^i(g/g \cap H, \mathbb{Z}/p\mathbb{Z}) \otimes H^j(h, \mathbb{Z}/p\mathbb{Z}) & \xrightarrow{\cup} & H^i(g/g \cap H, H^j(h, \mathbb{Z}/p\mathbb{Z})) \end{array}$$

commutes. For fixed h , we therefore obtain isomorphisms

$$\begin{aligned} D_i(G/H, \mathbb{Z}/p\mathbb{Z}) \otimes H^j(h, \mathbb{Z}/p\mathbb{Z})^* & \\ & \cong \left(\varinjlim_g H^i(g/g \cap H, \mathbb{Z}/p\mathbb{Z})^* \right) \otimes H^j(h, \mathbb{Z}/p\mathbb{Z})^* \\ & \cong \varinjlim_g H^i(g/g \cap H, \mathbb{Z}/p\mathbb{Z})^* \otimes H^j(h, \mathbb{Z}/p\mathbb{Z})^* \\ & \cong \varinjlim_g \left(H^i(g/g \cap H, \mathbb{Z}/p\mathbb{Z}) \otimes H^j(h, \mathbb{Z}/p\mathbb{Z}) \right)^* \\ & \cong \varinjlim_g H^i(g/g \cap H, H^j(h, \mathbb{Z}/p\mathbb{Z}))^*. \end{aligned}$$

Passing to the limit over h , we obtain the required isomorphism

$$D_i(G/H, \mathbb{Z}/p\mathbb{Z}) \otimes D_j(H, \mathbb{Z}/p\mathbb{Z}) \cong D_{ij}^2(G, H, \mathbb{Z}/p\mathbb{Z}).$$

□

Corollary 3. *Under the assumptions of Proposition 2, let i_0 and j_0 be the smallest integers such that $D_{i_0}(G/H, \mathbb{Z}/p\mathbb{Z}) \neq 0$ and $D_{j_0}(H, \mathbb{Z}/p\mathbb{Z}) \neq 0$, respectively. Then $D_{i_0+j_0}(G, \mathbb{Z}/p\mathbb{Z}) \neq 0$.*

Proof. The spectral sequence constructed in Proposition 2 induces an isomorphism

$$D_{i_0+j_0}(G, \mathbb{Z}/p\mathbb{Z}) \cong D_{i_0}(G/H, \mathbb{Z}/p\mathbb{Z}) \otimes D_{j_0}(H, \mathbb{Z}/p\mathbb{Z}) \neq 0.$$

□

Proof of Theorem 1. Assume that G is a duality group at p of dimension d . Let $cd_p H = m$ and $n = d - m$. Then there exists an open subgroup H_1 of H such that $H^m(H_1, \mathbb{Z}/p\mathbb{Z}) \neq 0$. Let G_1 be an open subgroup of G such that $H_1 = G_1 \cap H$. Then G_1 is a duality group at p of dimension d , $cd_p H_1 = m$ and G_1/H_1 is an open subgroup of G/H . We consider the exact sequence

$$1 \longrightarrow H_1 \longrightarrow G_1 \longrightarrow G_1/H_1 \longrightarrow 1.$$

As $H^m(H_1, \mathbb{Z}/p\mathbb{Z})$ is finite and nonzero, we have $vcd_p G_1/H_1 = n$, see [NSW] (3.3.9). Furthermore, $D_i(G_1, \mathbb{Z}/p\mathbb{Z}) = 0$, $i < n + m$. Using Corollary 3, we see that $D_i(G_1/H_1, \mathbb{Z}/p\mathbb{Z}) = 0$ for all $i < n$ and $D_j(H_1, \mathbb{Z}/p\mathbb{Z}) = 0$ for all $j < m$. Thus G_1/H_1 , hence G/H , is virtually a duality group at p of dimension n , and H_1 , and so H , is a duality group at p of dimension m . This shows (i).

Assume now that H and G/H are duality groups at p of dimension m and n . Then, $cd_p G = n + m$ by [NSW] (3.3.8), and in the spectral sequence of Proposition 2 we have $E_{ij}^2 = 0$ for $(i, j) \neq (n, m)$. Hence $D_r(G, \mathbb{Z}/p\mathbb{Z}) = 0$ for $r \neq n + m$ showing that G is a duality group at p of dimension $n + m$.

In order to prove the assertion about the dualizing modules, let h run through all open subgroups of H which are normal in G and g runs through the open subgroups of G . Since $m = cd_p H$, the Hochschild-Serre spectral sequence induces isomorphisms

$$H^{m+n}(g, \mathbb{Z}/p^\nu\mathbb{Z}) \cong H^n(g/g \cap H, H^m(g \cap H, \mathbb{Z}/p^\nu\mathbb{Z})),$$

and we obtain

$$\begin{aligned} I(G) &\cong \varinjlim_\nu \varinjlim_g H^{m+n}(g, \mathbb{Z}/p^\nu\mathbb{Z})^* \\ &\cong \varinjlim_\nu \varinjlim_h \varinjlim_g H^n(g/g \cap H, H^m(h, \mathbb{Z}/p^\nu\mathbb{Z}))^* \\ &\cong \varinjlim_\nu \varinjlim_h \varinjlim_{g, \text{res}} H^0(g/g \cap H, \text{Hom}(H^m(h, \mathbb{Z}/p^\nu\mathbb{Z}), I(G/H))) \\ &\cong \varinjlim_\nu \varinjlim_h \text{Hom}(H^m(h, \mathbb{Z}/p^\nu\mathbb{Z}), I(G/H)) \\ &\cong \text{Hom}_{cts}(\varprojlim_\nu \varprojlim_h H^m(h, \mathbb{Z}/p^\nu\mathbb{Z}), I(G/H)) \\ &\cong \text{Hom}_{cts}((\varinjlim_\nu \varinjlim_h H^m(h, \mathbb{Z}/p^\nu\mathbb{Z})^*)^\vee, I(G/H)) \\ &\cong \text{Hom}_{cts}(I(H)^\vee, I(G/H)) \cong (I(H)^\vee \hat{\otimes}_{\mathbb{Z}_p} I(G/H)^\vee)^\vee \end{aligned}$$

(see [NSW] (5.2.9) for the last isomorphism). This completes the proof of the theorem. $\square$

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