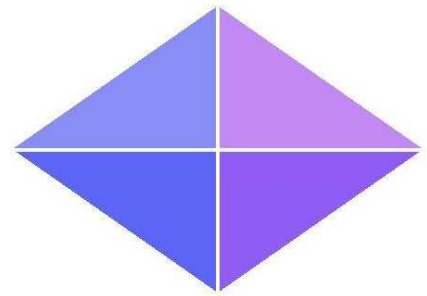


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Periods for flat algebraic connections

Marco Hien

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PERIODS FOR FLAT ALGEBRAIC CONNECTIONS

by

Marco Hien

Abstract. — In previous work [9], we established a duality between the algebraic de Rham cohomology of a flat algebraic connection on a smooth quasi-projective surface over the complex numbers and the rapid decay homology of the dual connection relying on a conjecture by C. Sabbah. In the present article, we generalize this result to arbitrary dimensions unconditionally using recent results of T. Mochizuki.

1. Introduction

Let U be a quasi-projective smooth algebraic variety over the complex numbers and E a vector bundle on U together with a flat algebraic connection $\nabla : E \rightarrow E \otimes_{\mathcal{O}_U} \Omega_{U/\mathbb{C}}^1$. The *algebraic de Rham cohomology* $H_{dR}^\bullet(U, E, \nabla)$ of (E, ∇) is the hypercohomology of the associated de Rham complex on U . The complex points of U give an analytic manifold U^{an} over $\mathbb{C}$ which carries the associated flat analytic connection $\nabla : E^{\text{an}} \rightarrow E^{\text{an}} \otimes_{\mathcal{O}_{U^{\text{an}}}} \Omega_{U^{\text{an}}/\mathbb{C}}^1$ on the holomorphic vector bundle E^{an} and gives rise to the *analytic de Rham cohomology* $H_{dR}^\bullet(U^{\text{an}}, E^{\text{an}}, \nabla)$, the hypercohomology of the analogous analytic de Rham complex. There is a canonical morphism

$$(1.1) \quad H_{dR}^\bullet(U, E, \nabla) \longrightarrow H_{dR}^\bullet(U^{\text{an}}, E^{\text{an}}, \nabla) .$$

between these cohomologies. A. Grothendieck proved in [8] that in the case of the trivial connection $(\mathcal{O}_U, d)$, this morphism is an isomorphism. P. Deligne examined the general case in [7], where he introduced the notion of (E, ∇) being *regular singular at infinity*, and proves the Comparison Theorem: the morphism (1.1) is an isomorphism provided that (E, ∇) is regular singular at infinity ([7], Théorème II.6.2). If $\mathcal{E}^\vee$ denotes the local system $\mathcal{E}^\vee := \ker(\nabla^\vee)$ associated to the dual analytic connection $\nabla^\vee$ on $E^{\text{an}, \vee}$, the holomorphic Poincaré-Lemma yields that the analytic de Rham

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cohomology $H_{dR}^\bullet(U^{\text{an}}, E^{\text{an}}, \nabla)$ is naturally isomorphic to the cohomology of U^{an} with values in this local system and hence by Deligne's Comparison Theorem the same is true for the algebraic de Rham cohomology

$$(1.2) \quad H_{dR}^\bullet(U, E, \nabla) \cong H^\bullet(U^{\text{an}}, \mathcal{E}^\vee)$$

still under the hypotheses of regular singularities at infinity.

In the irregular singular case, the morphism is not an isomorphism in general. On curves, the defect between algebraic and analytic de Rham cohomology can be expressed in terms of the irregularity numbers ([7], II.6.21). In [4], S. Bloch and H. Esnault provide a way to generalize (1.2) **on curves** in the following sense. They construct modified homology groups $H_{dR}^{rd}(U^{\text{an}}, E^\vee, \nabla^\vee)$ on the affine curve U^{an} as well as a bilinear pairing

$$H_{dR}^n(U, E, \nabla) \times H_n^{rd}(U^{\text{an}}, E^\vee, \nabla^\vee) \longrightarrow \mathbb{C}$$

given by integration along chains, which they prove to be a perfect pairing.

In [9], we investigated the two-dimensional situation, i.e. U being a smooth surface. In loc. cit., we construct a period pairing and prove its perfectness assuming a conjecture of C. Sabbah on the existence of the so-called good formal structure after birational pull-back of meromorphic connections on surfaces. Lately, a proof of Sabbah's Conjecture in the case of algebraic connections (which lie in the main focus of our work) has been achieved by T. Mochizuki (see [16]). Moreover, in another outstanding work [17], T. Mochizuki is able to prove a far reaching higher-dimensional generalization of Sabbah's conjecture, namely the existence of a good lattice again after a suitable birational pull-back (loc. cit., Theorem 19.5). As in the one- and two-dimensional case, the canonical next step into structural examinations of flat meromorphic connections consists in the question of lifting the good formal properties to asymptotic analogues. This step is also carried out in Mochizuki's paper [17] and results in the construction of Stokes structures for flat meromorphic connections in arbitrary dimensions.

In the present article, we now construct the rapid decay homology in all dimensions as well as the period pairing between the algebraic de Rham cohomology of a flat algebraic connection and the rapid decay homology associated to the dual connection. We then prove perfectness of this period pairing unconditionally in all dimensions for arbitrary flat algebraic connections (Theorem 5.3, section 5):

Theorem. — *There is a perfect pairing*

$$H_p^{rd}(U^{\text{an}}, E^\vee, \nabla^\vee) \otimes_{\mathbb{C}} H_{dR}^p(U; E, \nabla) \longrightarrow \mathbb{C}$$

between the algebraic de Rham cohomology of (E, ∇) and the rapid decay homology of the dual connection given by period integrals.

The result relies on a local duality between the algebraic de Rham complex and an analytic de Rham complex of the dual bundle with asymptotically flat coefficients which is proved in Theorem 4.4, section 4. The link between this local duality result and the rapid decay homology – and hence the period pairing – is carried out in section 5 and leads via Proposition 5.1 to the main result stated above.

As a consequence, assuming rational structures of the given geometric data over a subfield $k \subset \mathbb{C}$ as well as of the local system associated to the flat connection over another subfield $F \subset \mathbb{C}$, we deduce a well-defined notion of the **period determinant** arising from the comparison of these rational structures by means of the perfect period pairing. This determinant will be an element in the quotient $\mathbb{C}^\times / k^\times F^\times$. One of the motivations to study this determinant lies in the mysterious analogies between flat connections over the complex numbers and ℓ -adic sheaves on varieties over finite fields, according to which the period determinant is expected to behave as the ε -factor in the latter theory. If one restricts to the subcategory of regular singular connections, corresponding to the tamely ramified sheaves over finite fields, the period determinant and its analogies to the ε -factor has been extensively studied by T. Saito and T. Terasoma in [22]. With the present work, we hope to contribute to a generalization of these lines of thought to the irregular singular case.

2. Formal completion and real oriented blow-up

Let U be a smooth quasi-projective variety over $\mathbb{C}$ and X a smooth projective variety such that $D := X \setminus U$ is a divisor with normal crossings. Introducing polar coordinates around each irreducible component of D leads to the real oriented blow-up $\pi : \tilde{X} \rightarrow X^{\text{an}}$. Locally at $p \in D$ with complex coordinates $x_1, \dots, x_d$ such that $p = 0$ and $D = \{x_1 \cdots x_k = 0\}$, π reads as

$$\pi : ([0, \varepsilon) \times S^1)^k \times Y \rightarrow X^{\text{an}}, \quad ((r_\nu, e^{i\vartheta_\nu})_{\nu=1}^k, y) \mapsto ((r_\nu \cdot e^{i\vartheta_\nu})_{\nu=1}^k, y),$$

where Y is a small analytic neighbourhood of $0 \in \mathbb{C}^{d-k}$ and $y = (x_{k+1}, \dots, x_d)$, $\varepsilon > 0$ a small positive number. Restricted to $\pi^{-1}(U^{\text{an}})$ the real oriented blow-up is a homeomorphism by which we read U^{an} as a subspace of $\tilde{X}$. We write $\tilde{j} : U^{\text{an}} \hookrightarrow \tilde{X}$ for the inclusion.

On $\tilde{X}$, we consider the following sheaves of functions (cp. [18], II.1.). Firstly, $\tilde{X}$ carries the structure of a real manifold with boundary. Additionally, the logarithmic differential operators $\bar{x}_\nu \bar{\partial}_{x_\nu}$ act on the sheaf C_X^∞ of C^∞ functions. Let $\mathcal{A}_{\tilde{X}}$ be the sheaf of functions which in the local situation as above is given by

$$\mathcal{A}_{\tilde{X}} := \bigcap_{\nu=1}^k \ker(\bar{x}_\nu \bar{\partial}_{x_\nu}) \cap \bigcap_{\nu=k+1}^d \ker \bar{\partial}_{x_\nu} \subset C_X^\infty.$$

Local sections in $\mathcal{A}_{\tilde{X}}$ for some open $\Omega \subset \tilde{X}$ are differentiable functions on Ω which are holomorphic on $\Omega \cap U^{\text{an}}$ and admit an asymptotic development in the higher-dimensional analogue of Poincaré's asymptotic developments due to Majima (cp. Proposition B.2.1 in [18] and [11]).

Next, if $\mathcal{P}_{\tilde{X}}^{\leq D}$ denotes the sheaf of C^∞ -functions on $\tilde{X}$ which are flat on $\pi^{-1}(D)$, i.e. all of whose derivations vanish on $\pi^{-1}(D)$ (cp. [13]), we consider $\mathcal{A}_{\tilde{X}}^{\leq D} := \mathcal{A}_{\tilde{X}} \cap \mathcal{P}_{\tilde{X}}^{\leq D}$. Similarly, the local sections of $\mathcal{A}_{\tilde{X}}^{\leq D}$ are holomorphic restricted to U^{an} and with vanishing asymptotic development (cp. Proposition II.1.1.11 in [18]). More precisely, if $u \in \mathcal{A}_{\tilde{X}}^{\leq D}(\Omega)$ then for any compact $K \subset \Omega$ and any $N \in \mathbb{N}^k$, the function

u satisfies

$$(2.1) \quad |u(x)| \leq C_{K,N} \cdot |x_1|^{N_1} \cdots |x_k|^{N_k} \text{ for all } x \in K \setminus \pi^{-1}(D),$$

in terms of local coordinates as above such that locally $D = \{x_1 \cdots x_k = 0\}$.

Due to Lemme II.1.1.18 in [18], we have the following fine resolution of $\mathcal{A}_{\tilde{X}}^{\leq D}$

$$(2.2) \quad \mathcal{A}_{\tilde{X}}^{\leq D} \hookrightarrow (\mathcal{P}_{\tilde{X}}^{\leq D} \otimes_{\pi^{-1}C_X^\infty} \pi^{-1}\Omega_{X^{\text{an}}}^{\infty,(0,\cdot)}, \bar{\partial}),$$

where $\Omega_{X^{\text{an}}}^{\infty,(p,q)}$ denotes the sheaf of C^∞ -forms of degree (p,q) on X .

Analogously, if $\mathcal{A}_{\tilde{X}}^{\text{mod}D}$ denotes the sheaf of functions on $\tilde{X}$ which are holomorphic on U^{an} and of moderate growth along $\tilde{D}$, we have a fine resolution

$$(2.3) \quad \mathcal{A}_{\tilde{X}}^{\text{mod}D} \hookrightarrow (\mathcal{P}_{\tilde{X}}^{\text{mod}D} \otimes_{\pi^{-1}C_X^\infty} \pi^{-1}\Omega_{X^{\text{an}}}^{\infty,(0,\cdot)}, \bar{\partial}),$$

with $\mathcal{P}_{\tilde{X}}^{\text{mod}D}$ being the sheaf of C^∞ -functions on $\tilde{X}$ with moderate growth at $\pi^{-1}(D)$.

Finally, let $\mathfrak{D}_{\tilde{X}}^{\text{rd},-s}$ denote the sheaf of *rapid decay distributions* on $\tilde{X}$, i.e. the sheaf whose local section on small open $V \subset \tilde{X}$ are distributions

$$\varphi \in \mathfrak{D}_{\tilde{X}}^{-s}(V) := \text{Hom}_{\text{cont}}(\Gamma_c(V, \Omega_{\tilde{X}}^{\infty,s}), \mathbb{C})$$

on the space $\Omega_{\tilde{X}}^{\infty,s}$ of C^∞ differential forms on $\tilde{X}$ of degree s with compact support in $\tilde{X}$ satisfying the following condition: choosing coordinates $x_1, \dots, x_n$ on X such that $D \cap V = \{x_1 \cdots x_k = 0\}$, we require that for any compact $K \subset V$ and any element $N \in \mathbb{N}^k$ there are $m \in \mathbb{N}$ and $C_{K,N} > 0$ such that for any test form η with compact support in K the estimate

$$(2.4) \quad |\varphi(\eta)| \leq C_{K,N} \sum_i \sup_{|\alpha| \leq m} \sup_K \{|x|^{N_i} |\partial^\alpha f_i|\}$$

holds, where α runs over all multi-indices of degree less than or equal to m and ∂^α denotes the α -fold partial derivative of the coefficient functions f_i of η in the chosen coordinates.

3. Good formal lattices and decompositions (after T. Mochizuki)

3.1. Deligne-Malgrange lattices

Let there be given a flat meromorphic connection ∇ on the locally free $\mathcal{O}_{X^{\text{an}}}(*D)$ -module E of rank r , where $D \subset X$ is a divisor with normal crossings. After choice of a local trivialization $E \cong (\mathcal{O}_{X^{\text{an}}}(*D))^r$, the connection reads as $\nabla = d + A$ with the connection matrix $A \in M(r \times r, \Omega_{X^{\text{an}}}^1(*D))$. A change $T \in \text{GL}_r(\mathcal{O}_{X^{\text{an}}}(*D))$ of the trivialization transforms the connection matrix due to the formula

$$A' := T^{-1} dT + T^{-1} A T.$$

The local classification of these connections, i.e. of the connection matrices up to this transformation, is a difficult task in general.

The first major distinction is the one between regular singular connections and irregular singular ones. Let us recall this notion in the given geometric situation, i.e.

with D being a normal crossing divisor. Then, a flat meromorphic connection (E, ∇) is **regular singular**, if there is a trivialization $E \cong (\mathcal{O}_X(*D))^r$ such that the resulting connection matrix has logarithmic poles along D at most, i.e. A can be written as $A = \sum_{i=1}^k A_i(x) \operatorname{dlog} x_i + \sum_{j=k+1}^d A_j(x) dx_j$ with holomorphic matrices $A_i(x) \in M(r \times r, \mathcal{O}_{X^{\text{an}}})$. The structure of regular singular connections is well understood (see [7]), in particular we know that each regular singular connection is a successive extension of rank one connections. For the latter, one finds a basis vector e over $\mathcal{O}_{X^{\text{an}}}$ such that

$$\nabla e = e \otimes (\lambda_1 \operatorname{dlog} x_1 + \dots + \lambda_k \operatorname{dlog} x_k)$$

for some $\lambda \in \mathbb{C}^k$. Such a connection will be denoted by x^λ .

Next, the most elementary irregular singular connections are rank one connections which in a suitable basis vector e read as $\nabla e = e \otimes d\mathbf{a}$ for some $\mathbf{a} \in \mathcal{O}_{X^{\text{an}}}(*D)$, which up to isomorphism depends on $\mathbf{a} \bmod \mathcal{O}_{X^{\text{an}}}$ only. Such a connection will be denoted by $e^{\mathbf{a}}$.

Now, let $Y \subset D$ be a stratum in the natural stratification of the normal crossing divisor D . Passing from $\mathcal{O}_{X^{\text{an}}}$ to the formal completion $\widehat{\mathcal{O}_{X|Y}}$ of $\mathcal{O}_{X^{\text{an}}}$ along Y leads to the problem of formal classification, to which a extensive answer is given in the case $\dim X = 1$ by the Levelt-Turrittin Theorem (see [12], Chapter III). In [18], C. Sabbah investigated the two-dimensional situation leading to a precise conjecture as well as partial results in this direction. Recently, T. Mochizuki was able to give a proof Sabbah's Conjecture and a higher-dimensional generalization using a different approach by examining the so-called good lattices which go back to Malgrange (see [14]). We will now explain Mochizuki's result about these formal properties of meromorphic connections. For more details, we refer to [17], chapter 5. Let us adopt the notation

$$E|_{\widehat{Y}} := E \otimes_{\mathcal{O}_{X^{\text{an}}}} \widehat{\mathcal{O}_{X|Y}}$$

for any $\mathcal{O}_{X^{\text{an}}}$ -module E .

Let us choose local coordinates such that $D = \{x_1 \cdots x_k = 0\}$ with the irreducible components $D_i = \{x_i = 0\}$ and consider the local situation, i.e. $X^{\text{an}} = \Delta^k \times Y$ where Δ^k is a small poly-disc in $\mathbb{C}^k$ centered at the origin and Y is a small neighbourhood of the origin in the remaining variables. Accordingly, we will denote the first k variables by $z_1, \dots, z_k$ and the others by $y_1, \dots, y_{d-k}$. Let $\leq$ be the partial order on $\mathbb{Z}^k$ given by $m \leq n$ if $m_i \leq n_i$ for all i . Now, any meromorphic function $f \in \mathcal{O}_{X^{\text{an}}}(*D)$ admits a Laurent expansion $f = \sum_{m \in \mathbb{Z}^k} f_m(y) z^m$ with holomorphic functions $f_m \in \mathcal{O}_Y$. The *order of f* will be the minimum

$$\operatorname{ord}(f) = \min\{m \in \mathbb{Z}^k \mid f_m \neq 0\},$$

assuming that this minimum with respect to $\leq$ exists. T.Mochizuki defines the notion of a **good set of irregular values on (X, D)** to be a finite set $S \subset \mathcal{O}_{X^{\text{an}}}(*D)/\mathcal{O}_{X^{\text{an}}}$, such that

- i) $\operatorname{ord}(\mathbf{a})$ and $\operatorname{ord}(\mathbf{a} - \mathbf{b})$ exist for all $\mathbf{a} \neq \mathbf{b}$ in S ,
- ii) the set $\{\operatorname{ord}(\mathbf{a} - \mathbf{b}) \mid \mathbf{a}, \mathbf{b} \in S\}$ is totally ordered with respect to $\leq$ on $\mathbb{Z}^k$,
- iii) the leading terms $\mathbf{a}_{\operatorname{ord}(\mathbf{a})}(p)$ and $(\mathbf{a} - \mathbf{b})_{\operatorname{ord}(\mathbf{a} - \mathbf{b})}(p)$ of the Laurent expansions are non-vanishing for all $p \in Y$.

For any subset $I \subset \{1, \dots, k\}$, let I^c be its complement and furthermore $D_I := \bigcap_{i \in I} D_i$ and $D(I) := \bigcup_{i \in I} D_i$. The completion of X along D_I resp. $D(I)$ will be denoted by $\widehat{D}_I$ resp. $\widehat{D}(I)$. For a given subset $S \subset \mathcal{O}_{X^{\text{an}}}(*D)/\mathcal{O}_{X^{\text{an}}}$ let $S(I) := \{\mathfrak{a} \bmod \mathcal{O}_{X^{\text{an}}}(*D(I^c)) \mid \mathfrak{a} \in S\}$.

Let $\mathbb{E}$ be a lattice in E , i.e. a locally free $\mathcal{O}_{X^{\text{an}}}$ -module such that $\mathbb{E} \otimes_{\mathcal{O}_{X^{\text{an}}}} (*D) = E$. Then $\mathbb{E}$ is called an **unramified good lattice** at $p \in D$ if there is a good set of irregular values S as above such that for each I we have a formal decomposition

$$(3.1) \quad (\mathbb{E}, \nabla)|_{\widehat{D}_I} = \bigoplus_{\mathfrak{a} \in S(I)} ({}^I\widehat{\mathbb{E}}_{\mathfrak{a}}, {}^I\widehat{\nabla}_{\mathfrak{a}}),$$

such that

$$(3.2) \quad (\widehat{\nabla}_{\mathfrak{a}} - d\mathfrak{a})({}^I\widehat{\mathbb{E}}_{\mathfrak{a}}) \subset {}^I\widehat{\mathbb{E}}_{\mathfrak{a}} \otimes (\Omega_{X^{\text{an}}}^1(\log D(I)) + \Omega_{X^{\text{an}}}^1(*D(I^c))).$$

The set S of irregular values is uniquely determined by the given connection and hence it is denoted by $S := \text{Irr}(\nabla)$.

The given connection (E, ∇) is said to admit a **good lattice** at $p \in D$, if for some coordinate neighbourhood $U = \Delta^d$ of p as above, there exists a ramification map

$$\rho_e : \Delta^d \rightarrow U, (t_1, \dots, t_d) \mapsto (t_1^e, \dots, t_k^e, t_{k+1}, \dots, t_d)$$

for some $e \in \mathbb{N}_0$ such that $\rho_e^*(E, \nabla)$ admits an unramified good lattice $\widetilde{\mathbb{E}}$. With these notions, T. Mochizuki proves the following far reaching result

Theorem 3.1 (T. Mochizuki, [17], Theorem 19.5). — *Let (E, ∇) be a flat meromorphic connection on (X, D) . Then there exists a regular birational map $\varphi : X' \rightarrow X$ such that $D' := \varphi^{-1}(D)$ is again normal crossing, $\varphi|_{X' \setminus D'}$ is an isomorphism onto U and the pull-back $\varphi^*(E, \nabla)$ admits a good lattice at any $p \in D'$.*

Actually, the theorem in [17] is more precise in the sense that due to Malgrange's work ([14]) one knows that generically there is a "canonical lattice", which Mochizuki calls the Deligne-Malgrange lattice. He then proves that its extension to the whole X is again locally free over $\mathcal{O}_{X^{\text{an}}}$ and a good lattice in the above sense.

3.2. Good decomposition in multisectors

If we apply Mochizuki's Theorem 3.1 to the given flat meromorphic connection (E, ∇) on (X, D) , we know that the pull-back with respect to some birational map and some finite ramification admits an unramified good lattice $\mathbb{E}$. It will be important for our purposes to see that the resulting formal decomposition can be lifted to an asymptotic one on small multisectors. In dimension two, this is the ingredient of [18], II.2, in higher dimensions it can be derived from Mochizuki's approach of good lattices and can be found in [17] also, see the remark after the theorem below.

Let $\widehat{\mathcal{A}}_{\widehat{X}|Z}$ denote the formal completion of $\mathcal{A}_{\widehat{X}}$ along $\pi^{-1}(Z)$ for a closed subset $Z \subset X$ and T_Z the natural morphism $\mathcal{A}_{\widehat{X}} \rightarrow \widehat{\mathcal{A}}_{\widehat{X}|Z}$. We will consider mainly the case where $Z \subset D$ is a union of local irreducible components of D . Note that $\widehat{\mathcal{A}}_{\widehat{X}|Z}|_{\pi^{-1}(Z)} =$

$\pi^{-1}\mathcal{O}_{\widehat{X|Z}}$ in such a case. The following generalization of the one-dimensional Borel-Ritt theorem tells us that the sequence $0 \rightarrow \mathcal{A}_{\tilde{X}}^{\leq D} \rightarrow \mathcal{A}_{\tilde{X}} \xrightarrow{T_D} \mathcal{A}_{\widehat{X|D}} \rightarrow 0$ is exact (see [18], II.1.1.16). Let us again use the notation

$$E|_Y := (\pi^*E \otimes_{\pi^{-1}\mathcal{O}_{X^{\text{an}}}} \mathcal{A}_{\tilde{X}})|_Y$$

for some $Y \subset \tilde{X}$ and $\mathcal{O}_{X^{\text{an}}}$ -module E .

The formal decompositions (3.1) given by the existence of an unramified good lattice can indeed be lifted in the following sense:

Theorem 3.2 (T. Mochizuki, [17]). — *Let (E, ∇) be a flat meromorphic connection and assume that it admits an unramified good lattice $\mathbb{E}$. Let $\vartheta \in \pi^{-1}(D)$ be any multi-direction. Then there exists a small multisector $S \subset X \setminus D$ around ϑ with closure $\bar{S} \subset \tilde{X}$ such that we have a ∇ -flat decomposition*

$$(3.3) \quad \mathbb{E}|_{\bar{S}} = \bigoplus_{\alpha \in \text{Irr}(\nabla)} \mathcal{G}_{\alpha, S} ,$$

where the connection on $\mathcal{G}_{\alpha, S}$ induced by $\nabla - d\alpha$ is logarithmic, hence regular singular.

The proof of Theorem 3.2 is completely contained in [17]. However, since Mochizuki develops the theory of Stokes structures in a more general setting, adapted to the examination of wild twistor $\mathcal{D}$ -modules, and in much more detail than needed here, the proof of the above theorem is not so easily found in loc. cit. It occupies several steps and culminates in the statement to be found in Remark 7.73, [17]. For this reason and the reader's convenience, we give an overview over the proof of Theorem 3.2 in the appendix.

4. The local duality pairing

Consider the same situation as above, $U \hookrightarrow X$ a smooth projective compactification of U with $D := X \setminus U$ a normal crossing divisor. Let (E, ∇) be a flat algebraic connection U . It induces a flat meromorphic connection on X^{an} with poles along D^{an} . Our aim now is to study the meromorphic de Rham complex $\text{DR}_{X^{\text{an}}}(\nabla)$ associated to this flat meromorphic connection by lifting it to the real oriented blow-up $\pi : \tilde{X} \rightarrow X^{\text{an}}$. To this end, we consider the following de Rham complexes on $\tilde{X}$:

Definition 4.1. — *The asymptotically flat de Rham complex is defined to be the complex*

$$\text{DR}_{\tilde{X}}^{\leq D}(\nabla^\vee) := \mathcal{A}_{\tilde{X}}^{\leq D} \otimes_{\pi^{-1}(\mathcal{O}_{X^{\text{an}}})} \pi^{-1}(\text{DR}_{X^{\text{an}}}(\nabla^\vee)) \in D^b(\mathbb{C}_{\tilde{X}})$$

and the **moderate de Rham complex** to be the complex

$$\text{DR}_{\tilde{X}}^{\text{mod}D}(\nabla) := \mathcal{A}_{\tilde{X}}^{\text{mod}D} \otimes_{\pi^{-1}(\mathcal{O}_{X^{\text{an}}})} \pi^{-1}(\text{DR}_{X^{\text{an}}}(\nabla)) \in D^b(\mathbb{C}_{\tilde{X}}) .$$

The moderate de Rham complex computes the meromorphic (and hence also the algebraic) de Rham cohomology of (E, ∇) (cf. [18], Corollaire 1.1.8):

$$\mathbb{H}^k(\tilde{X}, \text{DR}_{\tilde{X}}^{\text{mod}D}(\nabla)) = H_{dR}^k(U, E, \nabla) .$$

Since multiplication gives a map $\mathcal{A}_{\tilde{X}}^{\text{mod}D} \otimes_{\mathbb{C}} \mathcal{A}_{\tilde{X}}^{<D} \rightarrow \mathcal{A}_{\tilde{X}}^{<D}$, we deduce that the usual wedge product of a differential form with moderate growth and an asymptotically flat differential form is again asymptotically flat. This leads to the following

Definition 4.2. — *The local duality pairing is the natural pairing*

$$(4.1) \quad \text{DR}_{\tilde{X}}^{\text{mod}D}(\nabla^{\vee}) \otimes_{\mathbb{C}} \text{DR}_{\tilde{X}}^{<D}(\nabla) \rightarrow \text{DR}_{\tilde{X}}^{<D}(\mathcal{O}_X, d)$$

induced by the wedge product of forms and the natural contraction $E^{\vee} \otimes E \rightarrow \mathcal{O}_X$ of the vector bundle E and its dual to the trivial line bundle.

Before stating the local duality result, we want to prove that the complexes involved are concentrated in one degree whenever (E, ∇) admits a good lattice on (X, D) .

Proposition 4.3. — *If (E, ∇) admits a good lattice on (X, D) , the complexes $\text{DR}_{\tilde{X}}^{\text{mod}D}(\nabla)$ and $\text{DR}_{\tilde{X}}^{<D}(\nabla)$ have cohomology in degree zero only.*

Proof. — Due to Mochizuki's results, Theorem 3.1 and Theorem 3.2 above, there exists a cyclic ramification $\rho : Y \rightarrow X$ such that locally on $\tilde{Y}$ the pull-back connection $\rho^{-1}(\nabla)$ admits a decomposition (3.3) for small enough multisectors S . Let $\pi_X : \tilde{X} \rightarrow X^{\text{an}}$ and $\pi_Y : \tilde{Y} \rightarrow Y^{\text{an}}$ denote the oriented real blow-up of $Z := \rho^{-1}(D)$ and D respectively. The projection formula for the lift $\tilde{\rho} : \tilde{Y} \rightarrow \tilde{X}$ gives

$$(4.2) \quad R\tilde{\rho}_* \text{DR}_{\tilde{Y}}^{\text{mod}Z}(\rho^{-1}\nabla) = R\tilde{\rho}_* \mathcal{A}_{\tilde{Y}}^{\text{mod}Z} \otimes_{\pi^{-1}(\mathcal{O}_X)}^{\mathbb{L}} \pi_X^{-1} \text{DR}_{X^{\text{an}}}(\nabla).$$

Now, $\tilde{\rho}$ being a finite map and since obviously $R\tilde{\rho}_* \mathcal{A}_{\tilde{Y}}^{\text{mod}Z} = \mathcal{A}_{\tilde{X}}^{\text{mod}D}$ (using the resolution (2.3)), it follows that it suffices to prove the claim on $\tilde{Y}$. Hence, we can assume that we have the decompositions (3.3) for the connection (E, ∇) locally on $\tilde{X}$. Since the decomposition is ∇ -flat, we can further restrict to the case of one summand $\mathcal{G}_{\mathbf{a}, S}$, i.e. we only have to consider the case where a priori (E, ∇) is of the form $(E, \nabla) = e^{\mathbf{a}} \otimes R_{\mathbf{a}}$ for some regular singular connection $R_{\mathbf{a}}$. Since every regular singular connection is a successive extension of regular singular line bundles, we can further reduce to the case $R_{\mathbf{a}} = x^{\lambda}$ with a $\lambda \in \mathbb{C}^d$.

The proof will now follow the same arguments in the asymptotically flat as well as in the moderate case. Since it is a local statement, we restrict to the local situation at some point $x_0 \in D = \{x_1 \cdots x_k = 0\}$. Let $\vartheta \in \pi^{-1}(x_0) \simeq (S^1)^k$ be a direction in $\tilde{D}$ over x_0 . Then the complex of stalks at ϑ which we have to consider is given as

$$(4.3) \quad \dots \rightarrow \left(\mathcal{A}_{\tilde{X}}^{?D_t} \otimes \pi^{-1} \Omega_{X^{\text{an}}}^p \right)_{\vartheta} \xrightarrow{\nabla} \left(\mathcal{A}_{\tilde{X}}^{?D} \otimes \pi^{-1} \Omega_{X^{\text{an}}}^{p+1} \right)_{\vartheta} \rightarrow \dots$$

where ? stands for either $<$ or mod. In degree p and with the usual basis dx_I for $I = \{1 \leq i_1 < \dots < i_p \leq d\}$ of $\Omega_{X^{\text{an}}}^p$, the connection map ∇ in (4.3) reads as

$$\sum_{\#I=p} w_I dx_I \mapsto \sum_{\#J=p+1} \left(\sum_{j \in J} \text{sgn}_J(j) (Q_j w_{J \setminus \{j\}}) \right) dx_J$$

with

$$Q_j u := \frac{\partial}{\partial x_j} u + \frac{\partial \mathbf{a}}{\partial x_j} \cdot u + x_j^{-1} \lambda_j \cdot u,$$

and where we define $\text{sgn}_J(j) := (-1)^{\nu}$ for $J = \{j_1 < \dots < j_{p+1}\}$ and $j_{\nu} = j$.

Consider a germ of a section ω of $\mathcal{A}_{\tilde{X}}^{?D} \otimes \pi^{-1}\Omega_{X^{\text{an}}}^{p+1}$,

$$\omega = \sum_{\#J=p+1} w_J dx_J ,$$

such that $\nabla\omega = 0$. We have to find a p -form η with appropriate growth condition such that $\nabla\eta = \omega$. To achieve this, let $s \in \mathbb{N}$ be an integer such that $w_J = 0$ for all J with $J \cap \{1, \dots, s-1\} \neq \emptyset$ (which is an empty condition for $s = 1$). We prove that we can find a p -form η with coefficients in $\mathcal{A}_{\tilde{X}}^{?D}$ such that

$$(4.4) \quad (\omega - \nabla\eta) \in \sum_{J \cap \{1, \dots, s\} = \emptyset} \mathcal{A}_{\tilde{X}, \emptyset}^{?D} dx_J .$$

The assertion then follows immediately by induction.

By assumption

$$(4.5) \quad 0 = \nabla\omega = \sum_{\#K=p+2} \left(\sum_{k \in K} \text{sgn}_K(k) (Q_k w_{K \setminus \{k\}}) \right) dx_K .$$

Taking $k < s$ and $s \in J$ and examining the summand of (4.5) corresponding to a set of the form $K := \{q\} \cup J$ we see that

$$(4.6) \quad Q_q w_J = 0 \quad \text{for all such } q = 1, \dots, s-1 .$$

Now, consider the system (Σ_J) of partial differential equations for the unknown function u_J , where J is a fixed subset $J \subset \{s, \dots, d\}$ of cardinality $p+1$ with $s \in J$:

$$(\Sigma_J) : \begin{cases} Q_q u_J = 0 & \text{for all } q = 1, \dots, s-1 \\ Q_s u_J = w_J , \end{cases}$$

together with the integrability assumption (4.6). Systems of this type had been studied by Majima (cf. [11]) before. In the case of $\mathcal{A}_{\tilde{X}, \emptyset}^{<D}$, the result follows from [19], Appendix A, in the moderate case we refer to [9], Theorem A.1 (which is formulated in dimension two only but generalizes without difficulties to the case of arbitrary dimension).

In each case, we can always find a solution $u_J \in \mathcal{A}_{\tilde{X}, \emptyset}^{?D}$ for any such $J \subset \{s, \dots, d\}$ of cardinality $p+1$ and $s \in J$ and if we let

$$\eta := \sum_{J \text{ as above}} u_J dx_J ,$$

we easily see that (4.4) is satisfied. $\square$

In particular, the proposition applies to the trivial connection $(\mathcal{O}_X, d)$ for any (X, D) such that D has normal crossings. The only interesting cohomology sheaf is

$$\mathcal{H}^0 \text{DR}_{\tilde{X}}^{<D}(\mathcal{O}_X, d) = \tilde{j}_! \mathbb{C}_{U^{\text{an}}} ,$$

where $\tilde{j}: U^{\text{an}} \hookrightarrow \tilde{X}$ again denotes the inclusion. Therefore, the local duality pairing can be written as

$$(4.7) \quad \text{DR}_{\tilde{X}}^{\text{mod} D}(\nabla^\vee) \otimes_{\mathbb{C}} \text{DR}_{\tilde{X}}^{<D}(\nabla) \rightarrow \tilde{j}_! \mathbb{C}_{U^{\text{an}}} .$$

We are now ready to state the following local duality theorem which lies in the heart of this work:

Theorem 4.4. — *The local duality pairing (4.1) is perfect in the sense that the induced morphisms*

$$\mathrm{DR}_{\tilde{X}}^{\mathrm{mod} D}(\nabla^\vee) \rightarrow \mathbf{R}\mathrm{Hom}_{\tilde{X}}(\mathrm{DR}_{\tilde{X}}^{< D}(\nabla), \tilde{j}_! \mathbb{C})$$

and

$$\mathrm{DR}_{\tilde{X}}^{< D}(\nabla) \rightarrow \mathbf{R}\mathrm{Hom}_{\tilde{X}}(\mathrm{DR}_{\tilde{X}}^{\mathrm{mod} D}(\nabla^\vee), \tilde{j}_! \mathbb{C})$$

are isomorphisms in the derived category.

We give the proof of this theorem in the next section.

4.1. Proof of Theorem 4.4

According to Mochizuki's results, (E, ∇) admits a good lattice after some regular birational morphism $b : Y \rightarrow X$. Let $Z := b^{-1}(D)$ be the resulting divisor in Y and let $\pi_X : \tilde{X} \rightarrow X^{\mathrm{an}}$ and $\pi_Y : \tilde{Y} \rightarrow Y^{\mathrm{an}}$ denote the oriented real blow-ups. Let $\tilde{b} : \tilde{Y} \rightarrow \tilde{X}$ be a lift of b . We will also use the following notation for the embeddings $\tilde{j} : X^{\mathrm{an}} \setminus D \hookrightarrow \tilde{X}$ and $\tilde{i} : Y^{\mathrm{an}} \setminus Z \hookrightarrow \tilde{Y}$. The de Rham complexes on the real oriented blow-ups behave functorial with respect to this situation:

Lemma 4.5. — *We have natural isomorphisms*

$$\mathbf{R}\tilde{b}_*(\mathrm{DR}_{\tilde{Y}}^{\mathrm{mod} Z}(b^* \nabla)) \cong \mathrm{DR}_{\tilde{X}}^{\mathrm{mod} D}(\nabla)$$

and

$$\mathbf{R}\tilde{b}_*(\mathrm{DR}_{\tilde{Y}}^{< Z}(b^* \nabla)) \cong \mathrm{DR}_{\tilde{X}}^{< D}(\nabla) .$$

Proof. — Via the projection formula – note that E is locally free – we obtain

$$\begin{aligned} \mathbf{R}\tilde{b}_*(\mathrm{DR}_{\tilde{Y}}^{?Z}(b^* \nabla)) &= \mathbf{R}\tilde{b}_*(\mathcal{A}_{\tilde{Y}}^{?Z} \otimes_{\pi_Y^{-1}(\mathcal{O}_{Y^{\mathrm{an}}})}^{\mathbb{L}} \pi_Y^{-1} \mathrm{DR}_{Y^{\mathrm{an}}}(b^* \nabla)) \cong \\ &\cong \mathbf{R}\tilde{b}_*(\mathcal{A}_{\tilde{Y}}^{?Z} \otimes_{\pi_X^{-1}(\mathcal{O}_{X^{\mathrm{an}}})}^{\mathbb{L}} \pi_X^{-1} \mathrm{DR}_{X^{\mathrm{an}}}(\nabla)) , \end{aligned}$$

where $?$ stands for $<$ or mod respectively. Hence, it suffices to prove that $\mathbf{R}\tilde{b}_*(\mathcal{A}_{\tilde{Y}}^{?Z}) = \mathcal{A}_{\tilde{X}}^{?D}$.

Since the restriction of $b : Y^{\mathrm{an}} \rightarrow X^{\mathrm{an}}$ to $Y^{\mathrm{an}} \setminus Z$ is an isomorphism $Y^{\mathrm{an}} \setminus Z \rightarrow X^{\mathrm{an}} \setminus D = U^{\mathrm{an}}$, we see that the sheaves of flat C^∞ -functions on each space, i.e. all of whose derivations vanish on the boundary, are compatible with $\tilde{b}$, i.e.

$$\tilde{b}_* \mathcal{P}_{\tilde{Y}}^{< Z} = \mathcal{P}_{\tilde{X}}^{< D} .$$

Similarly, we have $\tilde{b}_* \mathcal{P}_{\tilde{Y}}^{\mathrm{mod} Z} = \mathcal{P}_{\tilde{X}}^{\mathrm{mod} D}$ for the C^∞ -functions with moderate growth for arguments in U^{an} approaching the boundary. Computing $\mathbf{R}\tilde{b}_*(\mathcal{A}_{\tilde{Y}}^{?Z})$ via the flat resolutions (2.2) and (2.3) respectively, the assertions of the lemma follow immediately. $\square$ $\square$

Now $\tilde{y}_! \mathbb{C}_{U^{\text{an}}}[2d]$ is the dualizing sheaf on the manifold $\tilde{Y}$ with boundary $\tilde{Y} \setminus U^{\text{an}}$ and similarly $\tilde{y}_! \mathbb{C}_{U^{\text{an}}}[2d]$ for $\tilde{X}$. By Proposition 4.3 for the trivial connection $(\mathcal{O}_Y, d)$, we deduce from (2.2) the fine resolution

$$\tilde{y}_! \mathbb{C}_{U^{\text{an}}} \simeq (\mathcal{P}_{\tilde{Y}}^{<Z} \otimes_{\pi_Y^{-1}(C_{Y^{\text{an}}}^{\infty})} \pi_Y^{-1} \Omega_{Y^{\text{an}}}^{\infty, (\cdot, \cdot)}, \partial, \bar{\partial}) ,$$

where the right hand is to be understood as the simple complex associated to the indicated double complex of Dolbeault-type. Applying $\tilde{b}_*$ thus yields an isomorphism

$$\begin{aligned} \alpha : \tilde{b}_* \tilde{y}_! \mathbb{C}_{U^{\text{an}}} &= \tilde{b}_* (\mathcal{P}_{\tilde{Y}}^{<Z} \otimes_{\pi_Y^{-1}(C_{Y^{\text{an}}}^{\infty})} \pi_Y^{-1} \Omega_{Y^{\text{an}}}^{\infty, (\cdot, \cdot)}) = \\ &= \tilde{b}_* \mathcal{P}_{\tilde{Y}}^{<Z} \otimes_{\pi_X^{-1}(C_{X^{\text{an}}}^{\infty})} \pi_X^{-1} \Omega_{X^{\text{an}}}^{\infty, (\cdot, \cdot)} \cong \tilde{y}_! \mathbb{C}_{U^{\text{an}}} . \end{aligned}$$

Since the isomorphisms of Lemma 4.5 had been constructed in the same way, the diagram

$$(4.8) \quad \begin{array}{ccc} \tilde{b}_* \text{DR}_{\tilde{Y}}^{\text{mod}Z}(b^* \nabla) \otimes_{\mathbb{C}} \tilde{b}_* \text{DR}_{\tilde{Y}}^{<Z}(b^* \nabla^{\vee}) & \longrightarrow & \tilde{b}_* \tilde{y}_! \mathbb{C}_{U^{\text{an}}} \\ \text{Lemma 4.5} \downarrow \cong & & \cong \downarrow \alpha \\ \text{DR}_{\tilde{X}}^{\text{mod}D}(\nabla) \otimes_{\mathbb{C}} \text{DR}_{\tilde{X}}^{<D}(\nabla^{\vee}) & \longrightarrow & \tilde{y}_! \mathbb{C}_{U^{\text{an}}} \end{array}$$

commutes. Hence, the morphism

$$\text{DR}_{\tilde{X}}^{\text{mod}D}(\nabla) \rightarrow \mathbf{R}\mathcal{H}om_{\tilde{X}}(\text{DR}_{\tilde{X}}^{<D}(\nabla^{\vee}), \tilde{y}_! \mathbb{C}_{U^{\text{an}}})$$

induced by the lower row of (4.8) factors as

$$(4.9) \quad \begin{array}{ccc} \tilde{b}_* \text{DR}_{\tilde{Y}}^{\text{mod}Z}(b^* \nabla) & \xrightarrow{\beta} & \tilde{b}_* \mathbf{R}\mathcal{H}om_{\tilde{Y}}(\text{DR}_{\tilde{Y}}^{<Z}(b^* \nabla^{\vee}), \tilde{y}_! \mathbb{C}_{U^{\text{an}}}) \\ \cong \downarrow & & \downarrow \gamma \\ \text{DR}_{\tilde{X}}^{\text{mod}D}(\nabla) & \longrightarrow & \mathbf{R}\mathcal{H}om_{\tilde{X}}(\text{DR}_{\tilde{X}}^{<D}(\nabla^{\vee}), \tilde{y}_! \mathbb{C}_{U^{\text{an}}}) , \end{array}$$

where γ is given by the composition of the natural morphism

$$\tilde{b}_* \mathbf{R}\mathcal{H}om_{\tilde{Y}}(\text{DR}_{\tilde{Y}}^{<Z}, \tilde{y}_! \mathbb{C}_{U^{\text{an}}}) \rightarrow \mathbf{R}\mathcal{H}om_{\tilde{X}}(\tilde{b}_* \text{DR}_{\tilde{Y}}^{<Z}, \tilde{b}_* \tilde{y}_! \mathbb{C}_{U^{\text{an}}})$$

with the morphism α from above. By Poincaré-Verdier duality (Proposition 3.1.10 in [10]), γ is an isomorphism. The same arguments apply in the case of the moderate de Rham complex. Consequently, in order to prove Theorem 4.4 namely the fact that the bottom row is an isomorphism, it suffices to do so for β .

In other words, we may a priori assume that (E, ∇) has a good lattice on (X, D) . Then we know by Proposition 4.3 that both complexes in the local duality pairing have cohomology in degree 0 only, i.e. we are reduced to prove that the pairing of sheaves

$$\mathcal{S}^{\text{mod}D} \otimes {}^{\vee} \mathcal{S}^{<D} \rightarrow \tilde{y}_! \mathbb{C}_{U^{\text{an}}}$$

is perfect, where we define $\mathcal{S}^{\text{mod}D} := \mathcal{H}^0(\text{DR}_{\tilde{X}}^{\text{mod}D}(\nabla^{\vee}))$ and ${}^{\vee} \mathcal{S}^{<D} := \mathcal{H}^0(\text{DR}_{\tilde{X}}^{<D}(\nabla))$.

Exactly as in the proof of Proposition 4.3, due to the existence of a good lattice on (X, D) and the asymptotic decomposition given by Theorem 3.2, we can reduce to the case where ∇ is a rank one connection of the elementary form $\nabla = e^{\alpha} \otimes x^{\lambda}$ for some $\alpha \in \mathcal{O}_X(*D)$ and $\lambda \in \mathbb{C}^d$.

We restrict to a small enough open polysector $V \subset \tilde{X} \xrightarrow{\pi} X$ where locally $D = \{x_1 \cdots x_k = 0\}$. We will write $\alpha(x) = x_1^{-m_1} \cdot x_k^{-m_k} \cdot u(x)$ with non-vanishing $u(x)$. Let us define the Stokes multi-directions of α along D inside V to be

$$(4.10) \quad \Sigma_\alpha := St^{-1}\left(\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)\right),$$

with $St : \tilde{D} \cap V \rightarrow \mathbb{R}/2\pi\mathbb{Z}$, $St(r_i, \vartheta_i) := -\sum_{i=1}^k m_i \vartheta_i + \arg(u \circ \pi(r_i, \vartheta_i))$. Let $V_\alpha := (V \setminus \tilde{D}) \cup \Sigma_\alpha$, i.e. $V_\alpha \cap \tilde{D}$ are the directions in which $e^{\alpha(x)}$ has rapid decay for x radially approaching D . If $\tilde{j}_\alpha : V_\alpha \hookrightarrow V$ denotes the inclusion, one obviously has (possibly after shrinking V):

$$(4.11) \quad \mathcal{S}^{\text{mod}D}|_V = \tilde{j}_{-\alpha}(e^{-\alpha(x)} \cdot \mathbb{C}_{U^{\text{an}}})|_V \quad \text{and} \quad {}^V\mathcal{S}^{<D}|_V = \tilde{j}_\alpha(e^{\alpha(x)} \cdot \mathbb{C}_{U^{\text{an}}})|_V.$$

Consequently we have the following commutative diagram:

$$(4.12) \quad \begin{array}{ccc} \mathcal{R}\mathcal{H}om({}^V\mathcal{S}^{<D}, \tilde{j}_! \mathbb{C}_{U^{\text{an}}})|_V & \longrightarrow & \mathcal{S}^{\text{mod}D}|_V \\ \cong \downarrow & & \downarrow \cong \\ \mathcal{R}\mathcal{H}om((\tilde{j}_{-\alpha})! \mathbb{C}_{V \setminus D}, \tilde{j}_! \mathbb{C}_{V \cap U^{\text{an}}})|_V & \longrightarrow & (\tilde{j}_\alpha)! \mathbb{C}_{V_\alpha} \end{array}$$

By the factorization $\tilde{j} = \tilde{j}_{-\alpha} \circ \tilde{l}_{-\alpha}$ with $\tilde{l}_{-\alpha} : V \cap U^{\text{an}} \hookrightarrow V_{-\alpha}$, we see that

$$\begin{aligned} \mathcal{R}\mathcal{H}om((\tilde{j}_{-\alpha})! \mathbb{C}_{V_{-\alpha}}, \tilde{j}_! \mathbb{C}_{U^{\text{an}}}) &\cong \\ &\cong (\tilde{j}_{-\alpha})_* \mathcal{R}\mathcal{H}om(\mathbb{C}_{V_{-\alpha}}, (\tilde{l}_{-\alpha})! \mathbb{C}_{V \cap U^{\text{an}}}) \cong \\ &\cong (\tilde{j}_{-\alpha})_* \mathcal{H}om(\mathbb{C}_{V_{-\alpha}}, (\tilde{l}_{-\alpha})! \mathbb{C}_{V \cap U^{\text{an}}}) = (\tilde{j}_{-\alpha})! \mathbb{C}_{V_\alpha}, \end{aligned}$$

since $(V \setminus V_\alpha) \cap \tilde{D}$ coincides with the closure of $V_{-\alpha} \cap \tilde{D}$ inside $\tilde{D}$. Hence, the bottom line of (4.12) is an isomorphism and thus

$$\mathcal{S}^{\text{mod}D} \cong \mathcal{R}\mathcal{H}om_{\tilde{X}}({}^V\mathcal{S}^{<D}, \tilde{j}_! \mathbb{C}_{U^{\text{an}}})$$

locally on $\tilde{X}$ over an arbitrary point of D . Interchanging ${}^V\mathcal{S}^{<D}$ and $\mathcal{S}^{\text{mod}D}$ gives the analogous isomorphism.

This completes the proof of the local duality result, Theorem 4.4. $\square$

5. Period integrals

5.1. Definition of rapid decay homology

We now want to interpret the local duality pairing as a pairing via period integrals. Let us consider the following geometric set-up. We start with the given flat algebraic connection (E, ∇) on the smooth quasi-projective variety U . Due to Hironaka's resolution of singularities, we can compactify U by some smooth projective X with a normal crossing divisor $D := X \setminus U$ as complement. Due to Mochizuki's Theorem, quoted as Theorem 3.1 above, we can further pull-back the connection with respect to some regular birational map $b : Y \rightarrow X$ in order to obtain a good lattice for the connection. In other words, by replacing X with Y as a different choice of compactification, we may assume that the given connection admits a good lattice on the chosen compactification (X, D) itself.

Let $\pi : \tilde{X} \rightarrow X^{\text{an}}$ denote the real oriented blow-up of the normal crossing divisor $D \subset X^{\text{an}}$. We will write $\tilde{D} := \pi^{-1}(D)$ and denote by $j : U^{\text{an}} \rightarrow X$ and $\tilde{j} : U^{\text{an}} \rightarrow \tilde{X}$ the inclusions. For any $p \in \mathbb{N}_0$, we will write $S_p(\tilde{X})$ for the free $\mathbb{Q}$ -vector space over all piecewise smooth maps $c : \Delta^p \rightarrow \tilde{X}$ and we will consider singular homology with $\mathbb{Q}$ -coefficients and piecewise smooth chains in the following.

Let $\mathcal{C}_{\tilde{X}, \tilde{D}}^{-p}$ denote the sheaf of relative p -chains of the pair $\tilde{D} \subset \tilde{X}$, i.e. the sheaf associated to the presheaf

$$V \mapsto S_p(\tilde{X}, (\tilde{X} \setminus V) \cup \tilde{D}) .$$

Let $\mathcal{E} := \ker(\nabla|_U)$ be the local system on U of flat sections of ∇ . In the same way as in [9], Definition 2.3, a section $c \otimes \varepsilon \in \Gamma(V, \mathcal{C}_{\tilde{X}, \tilde{D}}^{-\bullet} \otimes_{\mathbb{Q}} \tilde{j}_* \mathcal{E})$ is called a **rapid decay chain**, if for any point $y \in c(\Delta^p) \cap \tilde{D} \cap V$ the following holds: Let $x_1, \dots, x_d$ be local coordinates of X around $y = 0$ such that $D = \{x_1 \cdots x_k = 0\}$ for some $1 \leq k \leq d$. We chose a meromorphic basis $e : E \cong (\mathcal{O}_{X^{\text{an}}}(*D))^r$ of E at y and require that if we develop $\varepsilon = \sum_{i=1}^d f_i \cdot e_i$ in this basis with analytic functions $f_i \in j_* \mathcal{O}_{U^{\text{an}}}$, these coefficients $f_i(x)$ decrease faster than any monomial for x approaching D , i.e. that for all $N \in \mathbb{N}^k$ there is a $C_N > 0$ such that for all small enough x we have

$$|f_i(x)| \leq C_N \cdot |x_1|^{N_1} \cdots |x_k|^{N_k} .$$

For chains $c \otimes \varepsilon$ inside U^{an} , we do not impose any condition. The sheaves of all these rapid decay p -chains will be denoted by $\mathcal{C}_{\tilde{X}}^{\text{rd}, -p}(\nabla)$. Together with the usual boundary operator ∂ of singular chains they define the **complex of rapid decay chains** $\mathcal{C}_{\tilde{X}}^{\text{rd}}(\nabla)$. The **rapid decay homology** of $(E(*D), \nabla)$ is the hypercohomology

$$H_k^{\text{rd}}(U^{\text{an}}, E, \nabla) := \mathbb{H}^{-k}(\tilde{X}, \mathcal{C}_{\tilde{X}}^{\text{rd}}(\nabla)) ,$$

which can be computed as the cohomology of the global sections, since the usual barycentric subdivision operator can be defined on the rapid decay chains and thus one deduces that the complex of rapid decay chains, similar to the sheaf complex of singular chains (cp. [23], p. 87) is homotopically fine.

5.2. The local duality pairing and periods

In order to obtain the desired interpretation of the local duality pairing (4.1) in terms of period integrals, we will examine the relation between the asymptotically flat de Rham complex of the dual connection $(E^\vee, \nabla^\vee)$ and its rapid decay complex. Let $\mathcal{E}^\vee$ denote the local system of the dual connection.

Recall that we chose (X, D) in a way such that (E, ∇) admits a good lattice on (X, D) . According to Proposition 4.3, the natural inclusion

$$(5.1) \quad \text{DR}_{\tilde{X}}^{\leq D}(\nabla^\vee) \xleftarrow{\sim} \mathcal{H}^0(\text{DR}_{\tilde{X}}^{\leq D}(\nabla^\vee)) =: {}^\vee \mathcal{S}^{< D}$$

is a quasi-isomorphism in this situation. Note that by definition and the properties of $\mathcal{A}_{\tilde{X}}^{\leq D}$, the sheaf ${}^\vee \mathcal{S}^{< D}$ consists of local solutions of $\nabla^\vee$ which have rapid decay along D . More precisely, given an open subset $V \subset \tilde{X}$ and a section $\sigma \in \Gamma(V, {}^\vee \mathcal{S}^{< D})$, the

section $\sigma \in {}^v\mathcal{S}^{<D}(V) \subset \tilde{j}_*(\mathcal{E}^\vee)$ is rapidly decaying along any chain in $c \in \mathcal{C}_{\tilde{X}, \tilde{D}}^{-\bullet}(V)$, i.e.

$$c \otimes \sigma \in \Gamma(V, \mathcal{C}_{\tilde{X}}^{\text{rd}}(\nabla^\vee)) .$$

Thus, we have a natural morphism

$$(5.2) \quad \mathcal{C}_{\tilde{X}, \tilde{D}}^{-\bullet} \otimes_{\mathbb{Q}} {}^v\mathcal{S}^{<D} \longrightarrow \mathcal{C}_{\tilde{X}}^{\text{rd}}(\nabla^\vee) .$$

Now, by standard arguments (e.g. [24]), the complex $\mathcal{C}_{\tilde{X}, \tilde{D}}^{-\bullet}$ is a homotopically fine resolution of the constant sheaf $\mathbb{C}_{\tilde{X}}[2d]$: to see this, note first that $U^{\text{an}} = \tilde{X} \setminus \tilde{D}$ is a real manifold of dimension $2d$ and since we take homology relative to the boundary $\tilde{D}$ we deduce that for small enough $V \subset \tilde{X}$ such that $V \subset \Omega$ for some contractible $\Omega \subset \tilde{X}$ we have by excision:

$$\Gamma(V, \mathcal{H}^{-p}(\mathcal{C}_{\tilde{X}, \tilde{D}}^{-\bullet})) = H_p(\Omega, (\Omega \setminus V) \cup \tilde{D}) \cong \begin{cases} 0 & \text{for } p \neq 2d \\ \mathbb{C} & \text{for } p = 2d . \end{cases}$$

Consequently, we end up with a natural morphism

$${}^v\mathcal{S}^{<D}[2d] \xrightarrow{\sim} \mathbb{C}_{\tilde{X}}[2d] \otimes {}^v\mathcal{S}^{<D} \xrightarrow{\sim} \mathcal{C}_{\tilde{X}, \tilde{D}}^{-\bullet} \otimes {}^v\mathcal{S}^{<D} \xrightarrow{(5.2)} \mathcal{C}_{\tilde{X}}^{\text{rd}}(\nabla^\vee) .$$

Proposition 5.1. — *If (E, ∇) admits a good lattice on (X, D) , there is a natural isomorphism*

$$\mathcal{C}^{\text{rd}}(\nabla^\vee) \cong \text{DR}_{\tilde{X}}^{<D}(\nabla^\vee)[2d]$$

in the derived category $D^b(\mathbb{C}_{\tilde{X}})$.

Proof. — Due to (5.1), it remains to prove that (5.2) is a quasi-isomorphism. Restricted to U^{an} , this is clear since there is no rapid decay condition involved and the rapid decay homology is nothing but the usual singular homology with values in the local system $\mathcal{E}^\vee$ which coincides with the restriction ${}^v\mathcal{S}^{<D}|_U$ on U^{an} .

As for the situation locally around $\tilde{D}$: By assumption on the existence of a good lattice, there is a cyclic ramification map $\rho : Y \rightarrow X$ such that $\rho^*\nabla^\vee$ admits an unramified good lattice. Since (5.2) obviously is compatible with direct image by ρ , it is enough to prove the assertion on Y , i.e. we can assume without loss of generality that $\nabla^\vee$ itself has an unramified good lattice. Exactly as in the proof of Proposition 4.3, we can apply Mochizuki's lifting theorem, Theorem 3.2 above, and reduce the statement to the case $\nabla^\vee = e^{\mathfrak{a}} \otimes \mathcal{R}$ for some regular singular connection $\mathcal{R}$. Now, since the solutions of a regular singular connection are moderate (see [7]), we have

$$\mathcal{S}^{<D}(e^{\mathfrak{a}} \otimes \mathcal{R}) = \mathcal{S}^{<D}(e^{\mathfrak{a}}) \otimes \tilde{j}_*\mathcal{R}$$

with the obvious notation $\mathcal{S}^{<D}(\nabla) := \mathcal{H}^0(\text{DR}_{\tilde{X}}^{<D}(\nabla))$ for a given connection ∇ .

Now, consider a small open polysector V around some direction $\vartheta \in \pi^{-1}(y)$ with $y \in D$. Let $\Sigma_{\mathfrak{a}} \subset \pi^{-1}(y)$ denote the set of Stokes-directions of $\mathfrak{a}$ as in (4.10).

We distinguish the following cases with respect to the direction ϑ : If $\vartheta \in \Sigma_{\mathfrak{a}}$ then for V being a small enough polysector, we have $\mathcal{S}^{<D}(e^{\mathfrak{a}})|_V = \tilde{j}_!(e^{\mathfrak{a}}\mathbb{C}_{U^{\text{an}}})|_V$. For a

smooth topological chain c in $\tilde{X}$, the local section e^a will not have rapid decay along c in V as required by the definition unless the chain does not meet $\tilde{D} \cap V$. Hence

$$\mathcal{C}_{\tilde{X}}^{\text{rd}}(e^a)|_V = \mathcal{C}_{\tilde{X}, \tilde{D}}^{-\bullet} \otimes \mathcal{S}^{<D}(e^a)|_V .$$

If $\vartheta \in \Sigma_a$, we can assume that V is an open polysector such that all the arguments of points in V are contained in Σ_a . Then $\mathcal{S}^{<D}(e^a)|_V \cong \tilde{\mathcal{J}}_*(e^a \mathbb{C}_V)$. Similarly, all twisted chains $c \otimes e^a$ will have rapid decay inside V and again both complexes considered are equal to $\mathcal{C}_{\tilde{X}, \tilde{D}}^{-\bullet} \otimes \tilde{\mathcal{J}}_*(e^a \mathbb{C}_{U^{\text{an}}})$.

Finally, if ϑ separates the Stokes regions of e^a and e^{-a} , we have with the notation of (4.11):

$$\mathcal{S}^{<D}(e^a)|_V \cong (\tilde{\mathcal{J}}_a)!(e^a \mathbb{C}_{U^{\text{an}}})|_V .$$

The subspace V_a is characterized by the property that $V_a \cap \tilde{D}$ consists of those directions along which $e^{a(x)}$ has rapid decay for x approaching $\tilde{D}$. In particular, $c \otimes e^a$ is a rapid decay chain on V if and only if the topological chain c in $\tilde{X}$ approaches $\tilde{D} \cap V$ in V_a at most. Hence

$$\mathcal{C}_{\tilde{X}}^{\text{rd}}(e^{-a})|_V = \mathcal{C}_{\tilde{X}, \tilde{D}}^{-\bullet} \otimes (\tilde{\mathcal{J}}_a)!(e^a \mathbb{C}_{U^{\text{an}}}) = \mathcal{C}_{\tilde{X}, \tilde{D}}^{-\bullet} \otimes {}^v\mathcal{S}^{<D}|_V .$$

Thus, we have seen that (5.2) is a quasi-isomorphism in all three cases. $\square \quad \square$

As a consequence, we also deduce

Corollary 5.2. — *The rapid decay homology groups $H^{rd}(U^{\text{an}}, E, \nabla)$ do not depend on the choice of the compactification X such that $D := X \setminus U$ is a normal crossing divisor and (E, ∇) admits a good lattice on X .*

Proof. — If X' is another such compactification with $D' := X' \setminus U$, we can resolve the singularities and apply Mochizuki's Theorem 3.1 to the closure of U inside $X \times X'$ and obtain a common compactification Y such that (E, ∇) admits a good lattice on Y . Lemma 4.5 and Proposition 5.1 prove that the rapid decay complexes on $\tilde{X}$ and $\tilde{X}'$ are isomorphic in the derived category. $\square \quad \square$

We remain in the situation given above, namely (E, ∇) admitting a good lattice on (X, D) . Due to Proposition 4.3, the local duality pairing reduces to the pairing

$$(5.3) \quad {}^v\mathcal{S}^{<D} \otimes \mathcal{S}^{\text{mod}D} \longrightarrow \tilde{\mathcal{J}}! \mathbb{C}_{U^{\text{an}}}$$

given by multiplication.

In order to be able to apply Proposition 5.1, consider the resolution

$$\mathcal{A}_{\tilde{X}}^{<D} \otimes_{\pi^{-1}\mathcal{O}_{X^{\text{an}}}} \pi^{-1}\Omega_{X^{\text{an}}}^r \hookrightarrow (\mathcal{P}_{\tilde{X}}^{<D} \otimes_{\pi^{-1}C_{X^{\text{an}}}^{\infty}} \pi^{-1}\Omega_{X^{\text{an}}}^{\infty, (r, \bullet)}, \bar{\partial}) ,$$

induced by the resolution (2.2), where $\Omega_{X^{\text{an}}}^{\infty, (r, s)}$ denotes the sheaf of C^∞ forms on X of degree (r, s) . Let $\mathcal{R}d^\bullet$ denote the total complex associated to the bicomplex

$$\mathcal{R}d^{\bullet, \bullet} := (\mathcal{P}_{\tilde{X}}^{<D} \otimes_{\pi^{-1}C_{X^{\text{an}}}^{\infty}} \pi^{-1}\Omega_{X^{\text{an}}}^{\infty, (\bullet, \bullet)}, \partial, \bar{\partial}) .$$

Then, $\mathcal{R}d^\bullet$ is a fine resolution of $\tilde{\mathcal{J}}! \mathbb{C}_{U^{\text{an}}}$.

Next, consider the canonical quasi-isomorphism $\beta : \mathcal{C}_{\tilde{X}, \tilde{D}}^{-\bullet} \otimes \mathcal{R}d^\bullet \xrightarrow{\simeq} \mathfrak{Db}_{\tilde{X}}^{\text{rd}, -\bullet}$ which maps an element $c \otimes \rho \in \mathcal{C}_{\tilde{X}, \tilde{D}}^{-r} \otimes \mathcal{R}d^s(V)$ of the left hand side over some open $V \subset \tilde{X}$ to the distribution given by $\eta \mapsto \int_c \eta \wedge \rho$ for a test form η with compact support in V . Because of the rapid decay property of ρ , this distribution has the property expressed in (2.4), hence is indeed a local section of $\mathfrak{Db}_{\tilde{X}}^{\text{rd}, s-r}$.

Now, the main step towards the interpretation as period integrals is to define the following pairing:

$$(5.4) \quad \begin{array}{ccc} \gamma : \mathcal{C}_{\tilde{X}}^{\text{rd}, -r}(\nabla^\vee) \otimes \text{DR}_{\tilde{X}}^{\text{mod}D, s}(\nabla) & \rightarrow & \mathfrak{Db}_{\tilde{X}}^{\text{rd}, s-r} \\ (c \otimes \varepsilon, \omega) & \mapsto & (\eta \mapsto \int_c \eta \wedge \langle \varepsilon, \omega \rangle) \end{array},$$

where we observe that since ε is rapidly decaying along c and ω is of moderate growth at most, the integral first of all converges and it satisfies the estimate (2.4) for all test forms η with compact support on $\tilde{X}$. The resulting pairing (5.4) is called the **period pairing** of the connection ∇ .

Now, the inclusion $\mathcal{S}^{\text{mod}D} \rightarrow \text{DR}_{\tilde{X}}^{\text{mod}D}(\nabla)$ is a quasi-isomorphism and the period pairing and the local duality pairing (in the form of (5.3) and with the appropriate shift) fit into the commutative diagram

$$(5.5) \quad \begin{array}{ccc} {}^\vee \mathcal{S}^{<D}[2d] \otimes \mathcal{S}^{\text{mod}D} & \longrightarrow & \tilde{\mathcal{H}}! \mathbb{C}_{U^{\text{an}}}[2d] \\ \simeq \downarrow & & \downarrow \simeq \\ \mathcal{C}_{\tilde{X}, \tilde{D}}^{-\bullet} \otimes {}^\vee \mathcal{S}^{<D} \otimes \mathcal{S}^{\text{mod}D} & \longrightarrow & \mathcal{C}_{\tilde{X}, \tilde{D}}^{-\bullet} \otimes \mathcal{R}d^\bullet \\ \simeq \downarrow (5.2) & & \simeq \downarrow \beta \\ \mathcal{C}_{\tilde{X}}^{\text{rd}}(\nabla^\vee) \otimes \text{DR}_{\tilde{X}}^{\text{mod}D}(\nabla) & \xrightarrow{\gamma} & \mathfrak{Db}_{\tilde{X}}^{\text{rd}, -\bullet} \end{array}.$$

Applying the local duality result, Theorem 4.4, together with Poincaré-Verdier duality gives an isomorphism

$$(5.6) \quad \begin{aligned} R\Gamma(\text{DR}_{\tilde{X}}^{\leq D}(\nabla^\vee)[2d]) &\cong R\Gamma R\mathcal{H}om_{\mathbb{C}_{\tilde{X}}}(\text{DR}_{\tilde{X}}^{\text{mod}D}(\nabla), \tilde{\mathcal{H}}! \mathbb{C}_{U^{\text{an}}}[2d]) \cong \\ &\cong \text{Hom}_{\mathbb{C}}^\bullet(R\Gamma(\text{DR}_{\tilde{X}}^{\text{mod}D}(\nabla)), \mathbb{C}) \end{aligned}.$$

In summary, we have obtained a description of the perfect duality

$$\mathbb{H}^{2d-p}(\tilde{X}, \text{DR}_{\tilde{X}}^{\leq D}(\nabla^\vee)) \otimes \mathbb{H}^p(\tilde{X}, \text{DR}_{\tilde{X}}^{\text{mod}D}(\nabla)) \longrightarrow \mathbb{C}$$

induced by the local duality pairing, in terms of the period pairing (5.4). We summarize this result:

Theorem 5.3. — *The period pairing (5.4) induces a perfect pairing*

$$H_p^{\text{rd}}(U^{\text{an}}, E^\vee, \nabla^\vee) \otimes_{\mathbb{C}} H_{\text{dR}}^p(U; E, \nabla) \longrightarrow \mathbb{C}$$

between the algebraic de Rham cohomology of (E, ∇) and the rapid decay homology of the dual connection.

5.3. The period determinant

Let $k \subsetneq \mathbb{C}$ be a subfield of the field of complex numbers and assume that the given geometric data, the variety U as well as the vector bundle and the connection are defined over k already. Then the de Rham cohomology inherits this k -structure, i.e. $H_{dR}(U, E, \nabla)$ is naturally a k -vector space.

Now, let $F \subsetneq \mathbb{C}$ denote another subfield of $\mathbb{C}$ and let us assume that the local system $\mathcal{E}$ on U^{an} comes equipped with a given F -structure. In analogy to [22] (for the regular singular case), we consider the category $W_{k,F}(U)$ – which we already defined in [9], 2.5 – of triples $\mathcal{M} = ((E, \nabla), \mathcal{E}_F, \rho)$ with:

- i) a vector bundle E on U with rank r together with a flat connection $\nabla : E \rightarrow E \otimes_{\mathcal{O}_U} \Omega_U^1$,
- ii) a local system $\mathcal{E}_F$ of F -vector spaces on the analytic manifold U^{an} ,
- iii) a morphism $\rho : \mathcal{E}_F \rightarrow E^{an}$ of sheaves on U^{an} inducing an isomorphism $\mathcal{E}_F \otimes_F \mathbb{C} \xrightarrow{\sim} \ker(\nabla^{an})$ of local systems of $\mathbb{C}$ -vector spaces on U^{an} .

A morphism between $((E, \nabla), \mathcal{E}_F, \rho)$ and $((E', \nabla'), \mathcal{E}_F', \rho')$ is given by a morphism $E \rightarrow E'$ respecting the connections together with a morphism $\mathcal{E}_F \rightarrow \mathcal{E}_F'$ of F -local systems with the natural compatibility condition with respect to ρ and ρ' .

Let $((E, \nabla), \mathcal{E}_F, \rho) \in W_{k,F}(U)$ be an object in this category for given subfields $k, F \subsetneq \mathbb{C}$. Then the local system $\mathcal{E}^\vee$ inherits an F -structure from the given F -structure on $\mathcal{E}$. Consequently, we can consider the F -lattice

$$\mathcal{C}_{\tilde{X}}^{rd, -p}(\nabla^\vee)_F \subset \mathcal{C}_{\tilde{X}, \tilde{D}}^{-p} \otimes_{\mathbb{Q}} \tilde{J}_* \mathcal{E}_F^\vee$$

of all rapidly decaying chains in $\mathcal{C}_{\tilde{X}, \tilde{D}}^{-p} \otimes_{\mathbb{Q}} \tilde{J}_* \mathcal{E}_F^\vee$ and end up with a natural F -lattice $H_p^{rd}(U^{an}, E, \nabla)_F$ inside the rapid decay homology:

$$H_p^{rd}(U^{an}, E, \nabla)_F \otimes_F \mathbb{C} \xrightarrow{\cong} H_p^{rd}(U^{an}, E, \nabla),$$

the isomorphism induced by ρ . The duality between the algebraic de Rham cohomology and the rapid decay homology via the period pairing enables us to compare these lattices and to generalize [9], Definition 2.7, unconditionally to the case of arbitrary dimension of U :

Definition 5.4. — *For $((E, \nabla), \mathcal{E}_F, \rho) \in W_{k,F}(U)$, we define its **period determinant** to be the element*

$$\det((E, \nabla), \mathcal{E}_F, \rho) := \prod_{p \geq 0} \det(\langle \gamma_j^{(p)}, \omega_i^{(p)} \rangle)_{i,j}^{(-1)^p} \in \mathbb{C}^\times / k^\times F^\times,$$

where $\omega_i^{(p)}$ denotes a basis of $H_{dR}^p(U, E, \nabla)$ over k and $\gamma_j^{(p)}$ a basis of the F -vector space $H_k^{rd}(U^{an}, E^\vee, \nabla^\vee)_F$.

We hope that this will be the starting point of further examinations in the spirit of T. Saito and T. Terasoma's work [22] in the case of regular singular connections including the irregular singular case.

Appendix A

Asymptotic lift of formal lattices

In this appendix, we want to include a short overview of the proof of T. Mochizuki's result (Theorem 3.2 above) on the asymptotic lift of the good formal properties. We emphasize that the necessary steps are completely contained in T. Mochizuki's paper [17], however – since Mochizuki's main objective lies in different applications as the one we give here – they are scattered over several paragraphs in loc. cit. and are intertwined with further results and ideas which aim at different applications. Therefore the proof of Theorem 3.2, though being completely contained in [17], is not easy to be traced back. This led to our decision to reproduce a straight path to the proof in this appendix to the reader's convenience. We will, however, give a roadmap only and for the individual proofs of the various steps we will refer to the original source [17].

A.1. Irregular decompositions and the minimal order

We assume that (E, ∇) admits an unramified good lattice $\mathbb{E}$ on (X, D) . We are in the local situation and write $D = \bigcup_{i=1}^{\ell} D_i$ for the decomposition in irreducible components. Then one has the formal decomposition (3.1) for the completion along all the intersections D_I for some $I \subset \{1, \dots, \ell\}$ (in the notation given for (3.1)).

In this situation, Mochizuki lifts the formal decompositions coming from the good lattice successively with respect to the increasing orders. Proceeding in this way makes it possible to define and examine the *Stokes filtration* in a very precise way (see [17], Chapt. 7). Since we do not need the full information on the latter, we will apply Mochizuki's arguments to give a proof by induction which allows us to be more economic (e.g. with respect to notation).

Let $\mathbf{m}(0) := \min\{\text{ord}(\mathbf{a}) \mid \mathbf{a} \in \text{Irr}(\nabla)\}$. Since the set $\{\text{ord}(\mathbf{a} - \mathbf{b}) \mid \mathbf{a}, \mathbf{b} \in \text{Irr}(\nabla)\}$ is totally ordered by definition, there is an $i \in \{1, \dots, \ell\}$ such that $\mathbf{m}(1) := \mathbf{m}(0) + (0, \dots, 1, \dots, 0)$, with the 1 on i -th place, is smaller than or equal to the all orders $\text{ord}(\mathbf{a} - \mathbf{b}) > \mathbf{m}(0)$. We may assume that $i = 1$. Let us define the following truncation of Laurent series (where $z^{\mathbf{n}} := z_1^{n_1} \cdots z_{\ell}^{n_{\ell}}$)

$$\bar{\eta} : \mathcal{O}_{X^{\text{an}}}(*D)/\mathcal{O}_{X^{\text{an}}} \rightarrow \mathcal{O}_{X^{\text{an}}}(*D)/\mathcal{O}_{X^{\text{an}}}, \mathbf{c} = \sum_n \mathbf{c}_n z^{\mathbf{n}} \mapsto \sum_{\mathbf{n} \geq \mathbf{m}(1)} \mathbf{c}_n z^{\mathbf{n}}$$

(denoted by $\bar{\eta}_{\mathbf{m}(0)}$ in Mochizuki's [17]) and the **set of irregular values in the minimal level** to be

$$\overline{\text{Irr}}(\nabla, \mathbf{m}(0)) := \bar{\eta}(\text{Irr}(\nabla)) = \{\bar{\eta}(\mathbf{c}) \mid \mathbf{c} \in \text{Irr}(\nabla)\}.$$

Let k be given by $\mathbf{m}(0) \in \mathbb{Z}_{\leq 0}^k \times \{0\}^{n-k}$ and let $D(\leq k) := \bigcup_{\nu \leq k} D_{\nu}$. Due to the definition of an unramified good lattice, we have for each $I \subset \{1, \dots, \ell\}$ the decomposition (3.1)

$$(A.1) \quad (\mathbb{E}, \nabla)|_{\widehat{D}_I} = \bigoplus_{\mathbf{c} \in S(I)} ({}^I \widehat{\mathbb{E}}_{\mathbf{c}}, {}^I \widehat{\nabla}_{\mathbf{c}})$$

over $\widehat{D}_I$ with the appropriate property (3.2), in a compatible way for $J \subset I$, i.e. ${}^J\widehat{\mathbb{E}}_{\mathfrak{b}}|_{\widehat{D}_I} = \bigoplus_{\mathfrak{c}} {}^I\widehat{\mathbb{E}}_{\mathfrak{c}}$ where the sum runs over all $\mathfrak{c} \in S(J)$ mapping to $\mathfrak{b}$ in the natural projection $S(J) \rightarrow S(I)$. In particular, for $I = \underline{\ell} := \{1, \dots, \ell\}$:

$$(A.2) \quad (\mathbb{E}, \nabla)|_{\widehat{D}_{\underline{\ell}}} = \bigoplus_{\mathfrak{c} \in \text{Irr}(\nabla)} ({}^{\underline{\ell}}\widehat{\mathbb{E}}_{\mathfrak{c}}, {}^{\underline{\ell}}\widehat{\nabla}_{\mathfrak{c}}) .$$

It is straightforward to see that this family of decompositions induces a decomposition

$$(A.3) \quad (\mathbb{E}, \nabla)|_{\widehat{D}(\leq k)} = \bigoplus_{\mathfrak{a} \in \overline{\text{Irr}}(\nabla, \mathbf{m}(0))} (\widehat{\mathbb{E}}_{\mathfrak{a}}^{\mathbf{m}(0)}, \widehat{\nabla}_{\mathfrak{a}}^{\mathbf{m}(0)})$$

(see Lemma 5.7 in [17]), the **irregular decomposition in the minimal level $\mathbf{m}(0)$** .

Further restricting to the intersection $D_{\underline{\ell}} := \bigcap_{i=1}^{\ell} D_i$ gives

$$(A.4) \quad \widehat{\mathbb{E}}_{\mathfrak{a}}^{\mathbf{m}(0)}|_{\widehat{D}_{\underline{\ell}}} = \bigoplus_{\mathfrak{c} \in \overline{\eta}^{-1}(\mathfrak{a})} ({}^{\underline{\ell}}\widehat{\mathbb{E}}_{\mathfrak{c}}, {}^{\underline{\ell}}\widehat{\nabla}_{\mathfrak{c}})$$

for $\mathfrak{a} \in \overline{\text{Irr}}(\nabla, \mathbf{m}(0))$.

A.2. Mochizuki's Lifting Theorem

Mochizuki's result (Theorem 3.2 above) now asserts that for each small enough poly-sector $S \subset X \setminus D(\leq k)$ there is a decomposition

$$\mathbb{E}|_{\overline{S}} = \bigoplus_{\mathfrak{c} \in \text{Irr}(\nabla)} (\mathcal{G}_{\mathfrak{c}, S}, \nabla_{\mathfrak{c}})$$

such that $\nabla_{\mathfrak{c}} - d\mathfrak{c}$ is logarithmic.

The claim will be proved by (decreasing) induction on the minimal order $\mathbf{m}(0)$. For $\mathbf{m}(0) = 0$, there is nothing to prove, since then $\text{Irr}(\nabla)$ is trivial and ∇ itself is logarithmic. The induction step will be carried out in the following section by lifting the irregular decomposition in the minimal level $\mathbf{m}(0)$ for $\mathbf{m}(0) < 0$ to obtain a good lattice with minimal order $\mathbf{m}(1) > \mathbf{m}(0)$. As a preparation, note that the decomposition (A.1) for $I = \{i\}$ makes it possible to define the **residue** of ∇ along D_i (with respect to the lattice $\mathbb{E}$) as

$$\text{Res}_i(\nabla) := \sum_{\mathfrak{c} \in S(i)} \text{Res}_{D_i}({}^i\widehat{\nabla}_{\mathfrak{c}} - d\mathfrak{c}) \in \bigoplus_{\mathfrak{c} \in S(i)} \text{End}({}^i\widehat{\mathbb{E}}_{\mathfrak{c}}|_{D_i}) \subset \text{End}(\mathbb{E}|_{D_i}) .$$

Now, for fixed i , a standard argument shows that one can find a meromorphic transformation such that the difference $\lambda - \mu$ of two distinct eigenvalues of $\text{Res}_i(\nabla)$ is not an integer (see [12], p. 39, or [21], Proposition 2.12). Transforming $\mathbb{E}$ accordingly, we can a priori assume that for all $i = 1, \dots, \ell$:

$$(A.5) \quad \lambda - \mu \notin (\mathbb{Z} \setminus \{0\}) \quad \text{for all eigenvalues of } \text{Res}_i(\nabla) .$$

A.3. Lift at the minimal order

Due to the definition of $\bar{\eta}$ and $\mathbf{m}(1)$, we see that, given an element $\mathbf{a} = \bar{\eta}(\mathbf{c}) \in \overline{\text{Irr}}(\nabla, \mathbf{m}(0))$, one has $\mathbf{c} - \mathbf{a} = z^{\mathbf{m}(1)} \mathbf{g}$ where $\mathbf{g}$ is holomorphic. Then $d\mathbf{c} = d\mathbf{a} + d(z^{\mathbf{m}(1)} \mathbf{g})$. Now, by definition of a good lattice, if $\mathbf{u}$ is a basis of $\mathbb{E}|_{\widehat{D}_\ell}$ respecting the decomposition (A.2) and if $\mathbf{u}_\alpha$ denotes the part of the basis corresponding to $\widehat{\mathbb{E}}_\alpha^{\mathbf{m}(0)}|_{\widehat{D}_\ell}$ as in (A.4), then

$$\widehat{\nabla}_\alpha \mathbf{u}_\alpha = \mathbf{u}_\alpha \cdot \left(\bigoplus_{\mathbf{c} \in \bar{\eta}^{-1}(\mathbf{a})} (d\mathbf{c} + T_\mathbf{c}) \right)$$

where $T_\mathbf{c}$ is a matrix of log-1-forms. The relation between $\mathbf{a}$ and $\mathbf{c}$ gives

$$\widehat{\nabla}_\alpha \mathbf{u}_\alpha = \mathbf{u}_\alpha (d\mathbf{a} + T'_\alpha)$$

where now $z^{-\mathbf{m}(1)} T'_\alpha$ consists of log-forms. In other words, the decomposition (A.3) fulfills

$$(\widehat{\nabla}_\alpha^{\mathbf{m}(0)} - d\mathbf{a}) \widehat{\mathbb{E}}_\alpha^{\mathbf{m}(0)} \subset z^{\mathbf{m}(1)} \cdot \widehat{\mathbb{E}}_\alpha^{\mathbf{m}(0)} \otimes \Omega_{X^{\text{an}}}^1(\log D),$$

i.e. $\text{ord}(\widehat{\nabla}_\alpha^{\mathbf{m}(0)} - d\mathbf{a}) \geq \mathbf{m}(1)$.

Additionally, all the elements $\mathbf{a} \in \overline{\text{Irr}}(\nabla, \mathbf{m}(0))$ over which the sum in (A.3) is taken are of the form $z^{\mathbf{m}(0)} \cdot \tilde{\mathbf{a}}$ with $\tilde{\mathbf{a}}$ being holomorphic and independent of the variable z_1 . Now, consider a given polysector (which may be shrunk in the following) and let $\bar{S}$ be the interior of the closure of S in $\tilde{X}$ and $Z := \bar{S} \cap \pi^{-1}(D(\leq k))$. The irregular decomposition (A.3) induces the decomposition

$$(A.6) \quad \mathbb{E}|_{\bar{Z}} := \mathbb{E} \otimes_{\pi^{-1}\mathcal{O}_{X^{\text{an}}}} \mathcal{A}_{\bar{S}|Z} = \bigoplus_{\mathbf{a} \in \overline{\text{Irr}}(\nabla, \mathbf{m}(0))} (\widehat{\mathbb{E}}_\alpha^{\mathbf{m}(0)}, \widehat{\nabla}_\alpha^{\mathbf{m}(0)})|_{\bar{Z}}.$$

Now, there is a partial order $\leq_S$ on $\overline{\text{Irr}}(\nabla, \mathbf{m}(0))$, where for $\mathbf{a} \neq \mathbf{b}$, the relation $\mathbf{a} \leq_S \mathbf{b}$ holds, if $\text{Re}(\mathbf{a} - \mathbf{b}) > 0$ on S . It induces the filtration $\mathcal{F}^Z$ indexed by $(\overline{\text{Irr}}(\nabla, \mathbf{m}(0)), \leq_S)$ given by $\mathcal{F}_\alpha^Z := \bigoplus_{\mathbf{b} \leq_S \alpha} \widehat{\mathbb{E}}_\mathbf{b}^{\mathbf{m}(0)}$.

Lemma A.1. — *Up to shrinking S , there is a decomposition*

$$\mathbb{E}|_{\bar{S}} = \bigoplus_{\mathbf{a} \in \overline{\text{Irr}}(\nabla, \mathbf{m}(0))} {}^1\mathbb{E}_{\mathbf{a}, S}$$

on $\bar{S} \subset \tilde{X}$, which is flat with respect to $\nabla_{z_1} := \nabla(\partial_{z_1})$ and which induces (A.6).

Proof. — This is Lemma 7.6 in [17]. □ □

Consequently, one defines the lift of the filtration $\mathcal{F}^Z$ to $\bar{S}$ by $\mathcal{F}_\alpha^S := \bigoplus_{\mathbf{b} \leq_S \alpha} {}^1\mathbb{E}_{\mathbf{b}, S}$. This filtration is indeed ∇ -flat, see [17], Lemma 7.2. Moreover, one has

Proposition A.2. — *There exists a ∇ -flat splitting of the filtration*

$$\mathcal{F}_\alpha^S = \bigoplus_{\mathbf{b} \leq_S \alpha} \mathbb{E}_{\mathbf{b}, S} \text{ on } \bar{S}.$$

Proof. — Let $\nabla_{\leq k}$ denote the connection with respect to the first k variables $z_1, \dots, z_k$ (i.e. we consider the local decomposition $X = \Delta^k \times Y$, Δ^k a polydisc in $\mathbb{C}^k$ and $\nabla_{\leq k}$ denotes the relative connection with respect to the projection $X \rightarrow Y$). Recall that $\mathbf{m}(0) \in \mathbb{Z}_{<0}^k \times 0$. Then Proposition 7.5 in [17] gives a $\nabla_{\leq k}$ -flat splitting which can be modified to give a ∇ -flat splitting by the assumption on the eigenvalues of the residue as in [17], 7.13. $\square$

By Proposition A.2, we have a decomposition

$$\mathbb{E}|_{\overline{S}} = \bigoplus_{\mathbf{a} \in \overline{\text{Irr}}(\nabla, \mathbf{m}(0))} (\mathbb{E}_{\mathbf{a}, S}, \nabla_{\mathbf{a}, S})$$

such that $\mathbb{E}_{\mathbf{a}, S}|_{\widehat{Z}} = \mathbb{E}_{\mathbf{a}}^{\mathbf{m}(0)}|_{\widehat{Z}}$, the right hand side as in (A.3). Now, varying the polysector S , the resulting $\mathbb{E}_{\mathbf{a}, S}$ can be glued to obtain an $\mathcal{O}_{X^{\text{an}}}$ -module $\mathbb{G}_{\mathbf{a}}$ (allowing to shrink X^{an} – we are interested in a local situation anyway) equipped with a flat meromorphic connection $\nabla_{\mathbf{a}}$, such that

$$\mathbb{G}_{\mathbf{a}}|_{\widehat{D}(\leq k)} = \widehat{\mathbb{E}}_{\mathbf{a}}^{\mathbf{m}(0)} \quad \text{and} \quad \text{ord}(\nabla_{\mathbf{a}} - d\mathbf{a}) \geq \mathbf{m}(1) .$$

For the construction of the glueing, we refer to [17], 7.4.

Now, consider the $\mathcal{O}_{X^{\text{an}}}$ -module $\mathbb{G}_{\mathbf{a}}$ together with the flat meromorphic connection $\nabla_{\mathbf{a}} - d\mathbf{a}$. It is again a good lattice with irregular values

$$(A.7) \quad \text{Irr}(\nabla_{\mathbf{a}} - d\mathbf{a}) = \{\mathbf{b} - \mathbf{a} \mid \mathbf{b} \in \text{Irr}(\nabla) \text{ und } \bar{\eta}(\mathbf{b}) = \mathbf{a}\} .$$

Due to the properties of the set $\text{Irr}(\nabla)$, one sees that $\mathbf{m}(1)$ is the minimal order appearing in $\text{Irr}(\nabla_{\mathbf{a}} - d\mathbf{a})$. Since $\mathbf{m}(1) > \mathbf{m}(0)$, we can apply the induction hypotheses to $\mathbb{G}_{\mathbf{a}}$ and obtain a decomposition (for a sufficiently small polysector S):

$$(\mathbb{G}_{\mathbf{a}}, \nabla_{\mathbf{a}} - d\mathbf{a})|_{\overline{S}} = \bigoplus_{\mathbf{b} \in \text{Irr}(\nabla_{\mathbf{a}} - d\mathbf{a})} (\mathbb{H}_{\mathbf{b}}^{\mathbf{a}}, \widetilde{\nabla}_{\mathbf{b}}^{\mathbf{a}})$$

such that $\widetilde{\nabla}_{\mathbf{b}}^{\mathbf{a}} - d\mathbf{b}$ is logarithmic. Summing over all $\mathbf{a} \in \overline{\text{Irr}}(\nabla, \mathbf{m}(0))$, we finally obtain

$$(A.8) \quad \mathbb{E}|_{\overline{S}} = \bigoplus_{\mathbf{a} \in \overline{\text{Irr}}(\nabla, \mathbf{m}(0))} \bigoplus_{\mathbf{b} \in \text{Irr}(\nabla_{\mathbf{a}} - d\mathbf{a})} (\mathbb{H}_{\mathbf{b}}^{\mathbf{a}}, \widetilde{\nabla}_{\mathbf{b}}^{\mathbf{a}} + d\mathbf{a}) .$$

Now, we define $\nabla_{\mathbf{b}}^{\mathbf{a}} := \widetilde{\nabla}_{\mathbf{b}}^{\mathbf{a}} + d\mathbf{a}$. Then $\nabla_{\mathbf{b}}^{\mathbf{a}} - d(\mathbf{a} + \mathbf{b}) = \widetilde{\nabla}_{\mathbf{b}}^{\mathbf{a}} - d\mathbf{b}$ is logarithmic. Furthermore, the set

$$\{\mathbf{c} = \mathbf{a} + \mathbf{b} \mid \mathbf{a} \in \overline{\text{Irr}}(\nabla, \mathbf{m}(0)) \text{ und } \mathbf{b} \in \text{Irr}(\nabla_{\mathbf{a}} - d\mathbf{a})\} = \text{Irr}(\nabla)$$

by (A.7) and for given $\mathbf{c}$ the summands $\mathbf{a}, \mathbf{b}$ are uniquely defined, since then $\mathbf{a} = \bar{\eta}(\mathbf{c})$. Therefore, the decomposition (A.8) has the desired properties and hence completes the proof of Mochizuki's Lifting Theorem.

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M.HIEN, Marco Hien, NWF I – Mathematik, Universität Regensburg, 93040 Regensburg, Germany
E-mail : `marco.hien@mathematik.uni-regensburg.de`