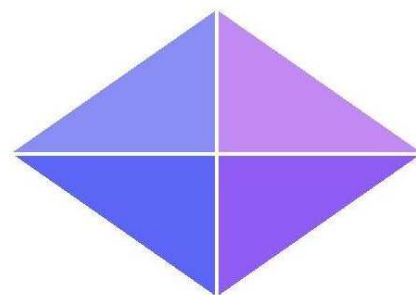


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A spectral sequence for
Iwasawa adjoints

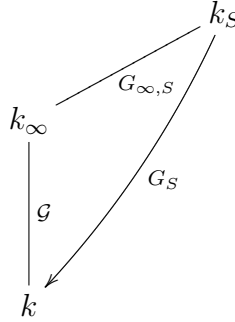
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A spectral sequence for Iwasawa adjoints

Uwe Jannsen, 1994 and July 29, 2003

Let k be a number field, fix a prime p , and let k_∞ be some Galois extension of k such that $\mathcal{G} = \text{Gal}(k_\infty/k)$ is a p -adic Lie-group (e.g., $\mathcal{G} \cong \mathbb{Z}_p^r$ for some $r \geq 1$). Let S be a finite set of primes containing all primes above p and ∞ , and all primes ramified in k_∞/k , and let k_S be the maximal S -ramified extension of k ; by assumption, $k_\infty \subseteq k_S$. Let $G_S = \text{Gal}(k_S/k)$ and $G_{\infty,S} = \text{Gal}(k_S/k_\infty)$.



Let A be a discrete G_S -module which is isomorphic to $(\mathbb{Q}_p/\mathbb{Z}_p)^r$ for some $r \geq 1$ as an abelian group (e.g., $A = \mathbb{Q}_p/\mathbb{Z}_p$ with trivial action, or $A = E[p^\infty]$, the group of p -power torsion points of an elliptic curve E/k with good reduction outside S). We are **not** assuming that $G_{\infty,S}$ acts trivially.

Let $\Lambda = \mathbb{Z}_p[[\mathcal{G}]]$ be the completed group ring. For a finitely generated Λ -module M we put

$$E^i(M) = \text{Ext}_\Lambda^i(M, \Lambda).$$

Hence $E^0(M) = \text{Hom}_\Lambda(M, \Lambda) =: M^+$ is just the Λ -dual of M . This has a natural structure of a Λ -module, by letting $\sigma \in \mathcal{G}$ act via

$$\sigma f(m) = \sigma f(\sigma^{-1}m)$$

for $f \in M^+$, $m \in M$. It is known that Λ is a noetherian ring (here we use that $\mathcal{G}$ is a p -adic Lie group), by results of Lazard [La]. Hence M^+ is a finitely generated Λ -module again (choose a projection $\Lambda^r \twoheadrightarrow M$; then we have an injection $M^+ \hookrightarrow (\Lambda^r)^+ = \Lambda^r$). By standard homological algebra, the $E^i(M)$ are finitely generated Λ -modules for all $i \geq 0$ which we call the (generalized) Iwasawa adjoints of M .

Examples

- a) If $\mathcal{G} = \mathbb{Z}_p$, then $\Lambda = \mathbb{Z}_p[[\mathcal{G}]] \cong \mathbb{Z}_p[[X]]$ is the classical Iwasawa algebra, and, for a Λ -torsion module M , $E^1(M)$ is isomorphic to the Iwasawa adjoint, which can be defined as

$$\text{ad}(M) = \varprojlim_n (M/\alpha_n M)^\vee$$

where $(\alpha_n)_{n \in \mathbb{N}}$ is a sequence of elements in Λ such that $\lim_{n \rightarrow \infty} \alpha_n = 0$ and (α_n) is prime to the support of M for every $n \geq 1$, and where

$$N^\vee = \text{Hom}(N, \mathbb{Q}_p/\mathbb{Z}_p)$$

is the Pontrjagin dual of a compact $\mathbb{Z}_p$ -module N . For any finitely generated Λ -module M , $E^1(M)$ is quasi-isomorphic to $\mathrm{Tor}_\Lambda(M)^\sim$, where $\mathrm{Tor}_\Lambda(M)$ is the Λ -torsion submodule of M , and $M^\sim$ is the "Iwasawa twist" of a Λ -module M : the action of $\gamma \in \mathcal{G}$ is changed to the action of γ^{-1} .

- b) If $\mathcal{G} = \mathbb{Z}_p^r$, $r \geq 1$, then the $E^i(M)$ are the standard groups considered in local duality. By duality for the ring $\mathbb{Z}_p[[\mathcal{G}]] = \mathbb{Z}_p[[x_1, \dots, x_r]]$, they can be computed in terms of local cohomology groups (with support) or by a suitable Koszul complex.

The topic of this note is the following observation.

Theorem 1 *There is a spectral sequence of finitely generated Λ -modules*

$$E_2^{p,q} = E^p(H^q(G_{\infty,S}, A)^\vee) \Rightarrow \varprojlim_{k',m} H^{p+q}(G_S(k'), A[p^m]) = \varprojlim_{k'} H^{p+q}(G_S(k'), T_p A).$$

Here the limit runs through the natural numbers m and the finite extensions k'/k contained in k_∞ , respectively, via the natural maps $H^n(G_S(k'), A[p^{m+1}]) \rightarrow H^n(G_S(k'), A[p^m])$ and the corestrictions. The rightmost group is the continuous cohomology of the Tate module $T_p A = \varprojlim_m A[p^m]$.

Before we give the proof of a slightly more general result (cf. Theorem 11 below), we discuss what this spectral sequence gives in more down-to-earth terms. First of all, we always have the 5-low-terms exact sequence

$$\begin{aligned} 0 &\rightarrow E^1(H^0(G_{\infty,S}, A)^\vee) \xrightarrow{\mathrm{inf}^1} \varprojlim_{k'} H^1(G_S(k'), T_p A) \\ &\rightarrow (H^1(G_{\infty,S}, A)^\vee)^+ \longrightarrow E^2(H^0(G_{\infty,S}, A)^\vee) \xrightarrow{\mathrm{inf}^2} \varprojlim_{k'} H^2(G_S(k'), T_p A). \end{aligned}$$

To say more, we consider some assumptions.

A.1 Assume that $p > 2$ or that k_∞ is totally imaginary. Then

$$H^r(G_{\infty,S}, A) = 0 = \varprojlim_{k'} H^r(G_S(k'), T_p A)$$

for all $r > 2$.

Corollary 2 *Assume that $H^2(G_{\infty,S}, A) = 0$ (This is the so-called "weak Leopoldt conjecture" for A . It is stated classically for $A = \mathbb{Q}_p/\mathbb{Z}_p$, and there are precise conjectures when this is expected for representations coming from algebraic geometry, cf. [Ja 2]).*

Then the cokernel of inf^2 is

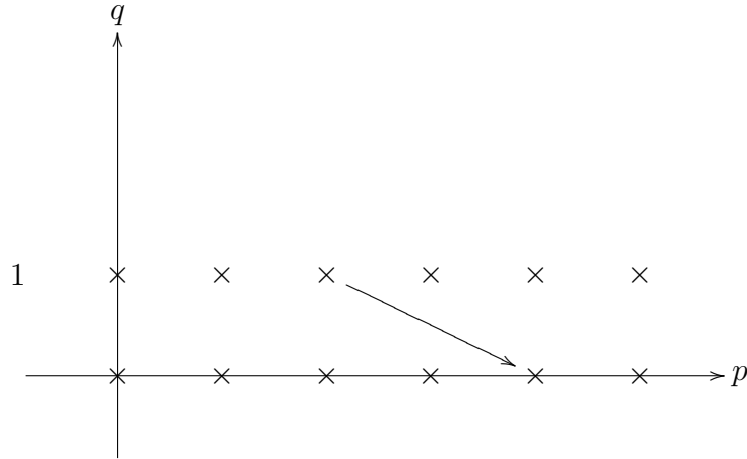
$$\ker(E^1(H^1(G_{\infty,S}, A)^\vee) \rightarrow E^3(H^0(G_{\infty,S}, A)^\vee)),$$

and there are isomorphisms

$$E^i(H^1(G_{\infty,S}, A)^\vee) \xrightarrow{\sim} E^{i+2}(H^0(G_{\infty,S}, A)^\vee)$$

for $i \geq 2$.

Proof This comes from A.1 and the following picture of the spectral sequence



Corollary 3 Assume that $H^0(G_{\infty, S}, A) = 0$. Then

(a)

$$\varprojlim_{k'} H^1(G_S(k'), T_p A) \xrightarrow{\sim} H^1(G_{\infty, S}, A)^{\vee+}$$

(b) There is an exact sequence

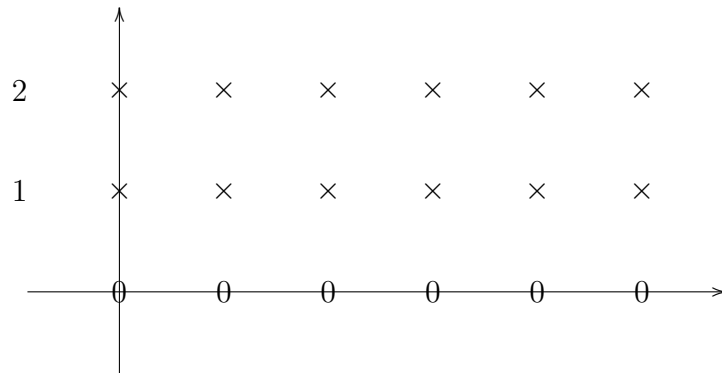
$$\begin{aligned} 0 \rightarrow E^1(H^1(G_{\infty, S}, A)^{\vee}) &\rightarrow \varprojlim_{k'} H^2(G_S(k'), T_p A) \\ &\rightarrow (H^2(G_{\infty, S}, A)^{\vee})^+ \rightarrow E^2(H^1(G_{\infty, S}, A)^{\vee}) \rightarrow 0 \end{aligned}$$

(c) There are isomorphisms

$$E^i(H^2(G_{\infty, S}, A)^{\vee}) \xrightarrow{\sim} E^{i+2}(H^1(G_{\infty, S}, A)^{\vee})$$

for $i \geq 1$.

Proof In this case, the spectral sequence looks like



Corollary 4 Assume that $\mathcal{G}$ is a p -adic Lie group of dimension 1 (equivalently: an open subgroup is $\cong \mathbb{Z}_p$). Then $E^i(-) = 0$ for $i \geq 3$. Let

$$B = \text{im} (\inf^2 : E^2(H^0(G_{\infty, S}, A)^\vee) \rightarrow \varprojlim_{k'} H^2(G_S(k'), T_p A))$$

Then B is finite, and there is an exact sequence

$$\begin{aligned} 0 \rightarrow E^1(H^1(G_{\infty, S}, A)^\vee) &\rightarrow \varprojlim_{k'} H^2(G_S(k'), T_p A)/B \rightarrow (H^2(G_{\infty, S}, A)^\vee)^+ \\ &\rightarrow E^2(H^1(G_{\infty, S}, A)^\vee) \rightarrow 0, \end{aligned}$$

and

$$E^1(H^2(G_{\infty, S}, A)^\vee) = 0 = E^2(H^2(G_{\infty, S}, A)^\vee),$$

i.e., $(H^2(G_{\infty, S}, A)^\vee)$ is a projective Λ -module.

Proof Quite generally, for a p -adic Lie group $\mathcal{G}$ of dimension n one has $\text{vcd}_p(\mathcal{G}) = n$ for the virtual cohomological p -dimension of $\mathcal{G}$, and hence $E^i(-) = 0$ for $i > n + 1$, cf. [Ja 3]. The finiteness of $E^2(M)$ (for a finitely generated Λ -module M) in our case is well-known, cf. [Ja 3]. The remaining claims follow from the following shape of the spectral sequence:

Lemma 5 Assume that $\mathcal{G}$ is a p -adic Lie group of dimension n (e.g., $\mathcal{G}$ contains an open subgroup $\cong \mathbb{Z}_p^n$). Then $E^i(H^0(G_{\infty, S}, A)^\vee) = 0$ for $i \neq n, n + 1$.

(a) If $H^0(G_{\infty, S}, A)$ is divisible (e.g., if $G_{\infty, S}$ acts trivially on A), then

$$E^i(H^0(G_{\infty, S}, A)^\vee) = \begin{cases} 0 & \text{for } i \neq n \\ \text{Hom}(D, H^0(G_{\infty, S}, A)), & \text{for } i = n, \end{cases}$$

where D is the dualising module for $\mathcal{G}$ ($D = \mathbb{Q}_p/\mathbb{Z}_p$ if $\mathcal{G} = \mathbb{Z}_p^n$).

(b) If $H^0(G_{\infty, S}, A)$ is finite, then

$$E^i(H^0(G_{\infty, S}, A)^\vee) = \begin{cases} 0 & \text{for } i \neq n + 1 \\ \text{Hom}(H^0(G_{\infty, S}, A), D)^\vee & \text{for } i = n + 1 \end{cases}$$

Proof This is well-known, cf. [Ja 3].

Corollary 6 *Let $\mathcal{G}$ be a p -adic Lie group of dimension 2 (e.g., $\mathcal{G}$ contains an open subgroup $\cong \mathbb{Z}_p^2$). If $G_{\infty, S}$ acts trivially on A , then there are exact sequences*

$$\begin{aligned} 0 \rightarrow \varprojlim_{k'} H^1(G_S(k'), T_p A) &\rightarrow (H^1(G_{\infty, S}, A)^\vee)^+ \\ &\rightarrow T_p A \xrightarrow{\inf^2} \varprojlim_{k'} H^2(G_S(k'), T_p A) \end{aligned}$$

and

$$\begin{aligned} 0 \rightarrow E^1(H^1(G_{\infty, S}, A)^\vee) &\rightarrow \varprojlim_{k'} H^2(G_S(k'), T_p A) / \text{im } \inf^2 \\ &\rightarrow (H^1(G_{\infty, S}, A)^\vee)^+ \rightarrow E^2(H^1(G_{\infty, S}, A)^\vee) \rightarrow 0, \end{aligned}$$

an isomorphism

$$E^1(H^2(G_{\infty, S}, A)^\vee) \xrightarrow{\sim} E^3(H^1(G_{\infty, S}, A)^\vee),$$

and one has

$$E^2(H^2(G_{\infty, S}, A)^\vee) = 0 = E^3(H^2(G_{\infty, S}, A)^\vee).$$

Proof The spectral sequence looks like

The diagram shows a spectral sequence grid. The vertical axis has labels 0, 1, 2. The horizontal axis has labels 0, 2, 3. Points marked with 'x' are located at coordinates (0,1), (0,2), (2,1), (2,2), (3,1), and (3,2). The origin is marked with a circle containing 0.

Corollary 7 *Let $\mathcal{G}$ be a p -adic Lie group of dimension 2 (So $E^i(-) = 0$ for $i \geq 4$). If $H^0(G_{\infty, S}, A)$ is finite, then*

$$\varprojlim_{k'} H^1(G_S(k'), T_p A) \cong (H^1(G_{\infty, S}, A)^\vee)^+.$$

If

$$d_2^{1,1} : E^1(H^1(G_{\infty, S}, A)^\vee) \rightarrow E^3(H^0(G_{\infty, S}, A)^\vee)$$

is the differential of the spectral sequence in the theorem, then one has an exact sequence

$$\begin{aligned} 0 \rightarrow \ker d_2^{1,1} &\rightarrow \varprojlim_{k'} H^2(G_S(k'), T_p A) \rightarrow \\ &\rightarrow \ker(d_2^{0,2} : (H^2(G_{\infty, S}, A)^\vee)^+ \rightarrow E^2(H^1(G_{\infty, S}, A)^\vee)) \rightarrow \text{coker } d_2^{1,1} \rightarrow 0, \end{aligned}$$

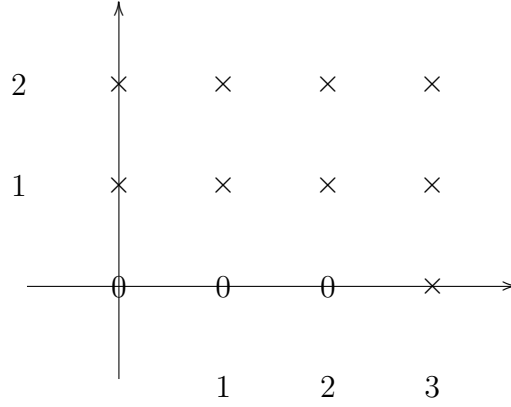
an isomorphism

$$E^1(H^2(G_{\infty,S}, A)^\vee) \xrightarrow{\sim} E^3(H^1(G_{\infty,S}, A)^\vee),$$

and the vanishing

$$E^2(H^2(G_{\infty,S}, A)^\vee) = 0 = E^3(H^2(G_{\infty,S}, A)^\vee).$$

Proof The spectral sequence looks like



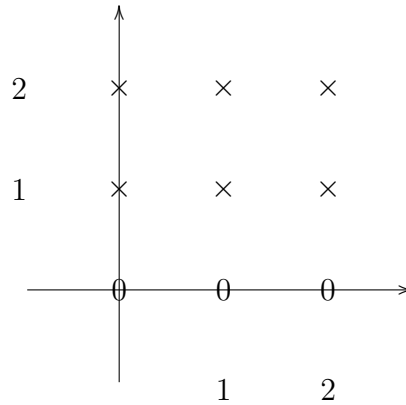
Remark In the situation of Corollary 5, one has an exact sequence up to *finite modules*:

$$\begin{aligned} 0 \rightarrow E^1(H^1(G_{\infty,S}, A)^\vee) &\rightarrow \varprojlim_{k'} H^2(G_S(k'), T_p A) \\ &\rightarrow (H^2(G_{\infty,S}, A)^\vee)^+ \rightarrow E^2(H^1(G_{\infty,S}, A)^\vee) \rightarrow 0. \end{aligned}$$

Corollary 8 Let $\mathcal{G}$ be a p -adic Lie group of dimension > 2 . Then

$$(H^1(G_{\infty,S}, A)^\vee)^+ \cong \varprojlim_{k'} H^1(G_S(k'), T_p A)$$

Proof The first three columns of the spectral sequence look like



We will now prove Theorem 1, by proving a somewhat more general result. For any profinite group G , let $\Lambda(G) = \mathbb{Z}_p[[G]]$ be the completed group ring over $\mathbb{Z}_p$, and let $M_G = M_{G,p}$ be the category of discrete (left) $\Lambda(G)$ -modules. These are the discrete G -modules A which are p -primary torsion abelian groups. For such a module A , its Pontrjagin dual $A^\vee = \text{Hom}(A, \mathbb{Q}_p/\mathbb{Z}_p)$ is a compact $\Lambda(G)$ -module. In fact, Pontrjagin duality gives an anti-equivalence between M_G and the category $C_G = C_{G,p}$ of compact (right) $\Lambda(G)$ -modules.

Let $M_G^{\mathbb{N}}$ be the category of inverse systems

$$(A_n) : \dots \rightarrow A_3 \rightarrow A_2 \rightarrow A_1$$

in M_G as in [Ja 1]. Denote by $H_{\text{cont}}^i(G, (A_n))$ the continuous cohomology of such a system and recall that one has an exact sequence for each i

$$0 \rightarrow \varprojlim_n H^{i-1}(G, A_n) \rightarrow H_{\text{cont}}^i(G, (A_n)) \rightarrow \varprojlim_n H^i(G, A_n) \rightarrow 0,$$

in which the $\lim^1$ -term vanishes if the groups $H^{i-1}(G, A_n)$ are finite for all n (cf. loc. cit.).

Definition 9 For a closed subgroup $H \leq G$ and a discrete G -module A in M_G define the relative cohomology $H^m(G, H; A)$ as the value at A of the m -th derived functor of the left exact functor (with Ab being the category of abelian groups)

$$\begin{aligned} H^0(G, H; -) : M_G &\rightarrow \text{Ab} \\ A &\mapsto \varprojlim_U H^0(U, A), \end{aligned}$$

where U runs through all open subgroups $U \subset G$ containing H , and the transition maps are the corestriction maps. For an inverse system (A_n) of modules in M_G define the continuous relative cohomology $H_{\text{cont}}^m(G, H; (A_n))$ as the value at (A_n) of the m -th right derivative of the functor

$$\begin{aligned} H_{\text{cont}}^0(G, H; -) : M_G^{\mathbb{N}} &\rightarrow \text{Ab} \\ (A_n) &\mapsto \varprojlim_n \varprojlim_U H^0(U, A_n), \end{aligned}$$

where the limit over U is as before, and the limit over n is induced by the transition maps $A_{n+1} \rightarrow A_n$.

Lemma 10 If G/H has a countable basis of neighbourhoods of identity, i.e., if there is a countable family U_ν of open subgroup, $H \leq U_\nu \leq G$, with $\bigcap_\nu U_\nu = H$, and if, in addition, $H^i(U, A_n)$ is finite for all these U and all n , then

$$H_{\text{cont}}^n(G, H; (A_n)) = \varprojlim_n \varprojlim_U H^n(U, A_n).$$

Proof In general, by deriving the inverse limit $\varprojlim_{n,U}$, one gets a Grothendieck spectral sequence

$$E_2^{p,q} = R^p \varprojlim_{n,U} H^q(U, A_n) \Rightarrow H_{\text{cont}}^{p+q}(G, H; (A_n)).$$

If the limit is over a countable family, then $R^p \varprojlim_{n,U} = 0$ for $p > 1$, and $R^1 \varprojlim_{n,U}$ has the usual description ([Ja 1]). If, in addition, all $H^q(U, A_n)$ are finite, then $R^1 \varprojlim_{n,U} H^q(U, A_n) = 0$, and we get the claimed isomorphisms.

Now we come to the spectral sequence in theorem 1. Any module A in M_G gives rise to two inverse systems, viz., the system $(A[p^n])$, where the transition maps $A[p^{n+1}] \rightarrow A[p^n]$ are induced by multiplication with p in A , and the system (A/p^n) , where the transition maps are induced by the identity of A . For reasons explained later, denote by $H_{\text{cont}}^m(G, H; RT_p A)$ the value at A of the m -th derived functor of the left exact functor

$$F : A \rightsquigarrow \varprojlim_n \varprojlim_U H^0(U, A[p^n])$$

where U runs through all open subgroups $U \subset G$ containing H , and the transition maps are the corestriction maps and those coming from $A[p^{n+1}] \rightarrow A[p^n]$, respectively. If H is a normal subgroup, then we may restrict to normal open subgroups $U \leq G$ containing H in the above inverse limit, and the limit is a (left) $\Lambda(G/H)$ -module in a natural way.

Theorem 11 *Let H be a closed subgroup of a profinite group G , and let A be a discrete $\Lambda(G)$ -module.*

(a) *There are short exact sequences*

$$0 \rightarrow H_{\text{cont}}^n(G, H; (A[p^n])) \rightarrow H_{\text{cont}}^n(G, H; RT_p A) \rightarrow H_{\text{cont}}^{n-1}(G, H; (A/p^n)) \rightarrow 0.$$

If H is a normal subgroup, then these are exact sequences of $\Lambda(G/H)$ -modules.

(b) *Let H' be a normal subgroup of G , with $H' \subset H$. There is a spectral sequence*

$$E_2^{p,q} = H_{\text{cont}}^p(G/H', H/H'; RT_p H^q(H', A)) \Rightarrow H_{\text{cont}}^{p+q}(G, H; RT_p A).$$

If H is a normal subgroup, too, this is a spectral sequence of $\Lambda(G/H)$ -modules.

(c) *If H is a normal subgroup of G , then for every discrete $\Lambda(G)$ -module A one has canonical isomorphisms of $\Lambda(G/H)$ -modules*

$$H^m(G, H; RT_p A) \cong \text{Ext}_{\Lambda(G)}^m(A^\vee, \Lambda(G/H))$$

for all $m \geq 0$, where $\Lambda(G/H)$ is regarded as a $\Lambda(G)$ -module via the ring homomorphism $\Lambda(G) \rightarrow \Lambda(G/H)$. More precisely, the δ -functor

$$M_G \rightarrow \text{Mod}_{\Lambda(G/H)} \quad , \quad A \rightsquigarrow (H^m(G, H; RT_p A) \mid m \geq 0)$$

is canonically isomorphic to the δ -functor

$$M_G \rightarrow \text{Mod}_{\Lambda(G/H)} \quad , \quad A \rightsquigarrow (\text{Ext}_{\Lambda(G)}^m(A^\vee, \Lambda(G/H)) \mid m \geq 0).$$

Here and in the following, the Ext-groups $\text{Ext}_{\Lambda(G)}(-, -)$ are taken in the category C_G of compact $\Lambda(G)$ -modules. We note that these Ext-groups are $\Lambda(G)$ -modules, but not necessarily compact.

(d) In particular, let H be a normal subgroup of G , and let $\mathcal{G} = G/H$. If A is a discrete $\Lambda(G)$ -module, then one has a spectral sequence of $\Lambda(\mathcal{G})$ -modules

$$E_2^{p,q} = \text{Ext}_{\Lambda(\mathcal{G})}^p(H^q(H, A)^\vee, \Lambda(\mathcal{G})) \Rightarrow H_{\text{cont}}^{p+q}(G, H; R\underline{T}_p A) = \text{Ext}_{\Lambda(\mathcal{G})}^{p+q}(A^\vee, \Lambda(\mathcal{G})).$$

Before we give the proof of Theorem 11, we note that it implies Theorem 1. In fact, we apply Theorem 11 to $G = G_S$ and $H = G_{\infty, S}$. If A is a G_S -module of cofinite type as in Theorem 1, then $A/p^n = 0$ and $A[p^n]$ is finite, for all n . Moreover, $H^i(U, B)$ is known to be finite for all open subgroups $U \leq G_S$ and all finite U -modules B . By (a) and Lemma 10 we deduce

$$H_{\text{cont}}^m(G_S, G_{\infty, S}; R\underline{T}_p A) = \varprojlim_{n, U} H^m(U, A[p^n]) = \varprojlim_{n, k'} H^m(G_S(k'), A[p^n]),$$

where k' runs through all finite subextensions of k_∞/k . Moreover, one has canonical isomorphisms

$$\varprojlim_n H^m(U, A[p^n]) \cong H^m(U, T_p A)$$

where the latter group is continuous cochain group cohomology, cf. [Ja 1]. By applying Theorem 11 (d) we thus get the desired spectral sequence. Finally, $H^m(H, A)^\vee$ is a finitely generated $\Lambda(\mathcal{G})$ -module for all $m \geq 0$, so that the initial terms of the spectral sequence are finitely generated $\Lambda(\mathcal{G})$ -modules as well, and so are the limit terms. In fact, let N be the kernel of the homomorphism $G_S \rightarrow \text{Aut}(A)$ given by the action of G_S on A , and let $H' = H \cap N$. Then G/H' is a p -adic analytic Lie group, since G/H and G/N are. It is well-known that $H^m(H', \mathbb{Q}_p/\mathbb{Z}_p)$ is a cofinitely generated discrete $\Lambda(G/H')$ -module for all $m \geq 0$; hence the same is true for $H^m(H', A) \cong H^m(H', \mathbb{Q}_p/\mathbb{Z}_p) \otimes T_p A$. The claim then follows from the Hochschild-Serre spectral sequence $H^p(H/H', H^q(H', A)) \Rightarrow H^{p+q}(H, A)$.

Proof of Theorem 11 (a): We can write F as the composition of the two left exact functors

$$\begin{aligned} \underline{T}_p : M_G &\rightarrow M_G^\mathbb{N} \\ A &\mapsto (A[p^n]) \end{aligned}$$

and

$$\begin{aligned} H_{\text{cont}}^0(G, H; -) : M_G^\mathbb{N} &\rightarrow \text{Ab} \\ (A_n) &\rightsquigarrow \varprojlim_n \varprojlim_U H^0(U, A_n), \end{aligned}$$

where U runs through all open (normal) subgroups of G containing H . Because $\underline{T}_p$ maps injectives to $H_{\text{cont}}^0(G, H; -)$ -acyclics we get a spectral sequence

$$E_2^{p,q} = H_{\text{cont}}^p(G, H; R^q \underline{T}_p A) \Rightarrow H_{\text{cont}}^{p+q}(G, H; R\underline{T}_p A).$$

From the snake lemma one immediately gets

$$R^q \underline{T}_p A = \begin{cases} (A/p^m A) & q = 1 \\ 0 & q > 1 \end{cases}$$

and hence short exact sequences

$$0 \rightarrow H_{\text{cont}}^n(G, H; \underline{T}_p A) \rightarrow H_{\text{cont}}^n(G, H; R\underline{T}_p A) \rightarrow H_{\text{cont}}^{n-1}(G, H; R^1 \underline{T}_p A) \rightarrow 0.$$

This shows (a) and also explains the notation for $R^n F$. In fact, $H_{\text{cont}}^n(G, H; RT_p A)$ is the hypercohomology with respect to $H_{\text{cont}}^0(G, H; -)$ of a complex $RT_p A$ in $M_G^{\mathbb{N}}$ computing the $R^i T_p A$.

(b): If H is a normal subgroup, we can regard the functor F as a functor from M_G to the category $\text{Mod}_{\Lambda(G/H)}$ of $\Lambda(G/H)$ -modules. On the other hand, we can also write F as the composition of the left exact functors

$$H^0(H, -) : M_G \rightarrow M_{G/H} \quad , \quad A \rightsquigarrow A^H$$

and

$$\tilde{F} : M_{G/H} \rightarrow \text{Mod}_{\Lambda(G/H)} \quad , \quad B \rightsquigarrow \varprojlim_n \varprojlim_{U/H} H^0(U/H, B[p^n]) = H^0(G/H, \{1\}; RT_p B).$$

(Note that U/H runs through all open (normal) subgroups of G/H .) This immediately gives the spectral sequence in (b).

(c): We claim that the functor F is isomorphic to the functor

$$\begin{aligned} M_G &\rightarrow \text{Mod}_{\Lambda(G/H)} \\ B &\rightsquigarrow \text{Hom}_{\Lambda(G)}(B^\vee, \Lambda(G/H)). \end{aligned}$$

In fact, writing $\text{Hom}_{\Lambda(G)}(-, -)$ for the homomorphism groups of compact $\Lambda(G)$ -modules, we have (cf. [Ja 3] p. 179)

$$\begin{aligned} \text{Hom}_{\Lambda(G)}(B^\vee, \Lambda(G/H)) &= \varprojlim_U \text{Hom}_{\Lambda(G)}(B^\vee, \mathbb{Z}_p[G/U]) \\ &= \varprojlim_n \varprojlim_U \text{Hom}_{\text{cont}}(H^0(U, B)^\vee, \mathbb{Z}/p^n \mathbb{Z}) \\ &= \varprojlim_n \varprojlim_U \text{Hom}_{\text{cont}}(H^0(U, B[p^n])^\vee, \mathbb{Z}/p^n \mathbb{Z}) \\ &= \varprojlim_n \varprojlim_U H^0(U, B[p^n]), \end{aligned}$$

where U runs through all open subgroups of G containing H , and hence

$$\text{Hom}_{\Lambda(G)}(B^\vee, \Lambda(G/H)) = H^0(G, H; RT_p B).$$

Since taking Pontrjagin duals is an exact functor $M_G \rightarrow C_G$ taking injectives to projectives, the derived functors of the functor $B \rightsquigarrow \text{Hom}_{\Lambda(G)}(B^\vee, \Lambda(G/H))$ are the functors $B \rightsquigarrow \text{Ext}_{\Lambda(G)}^i(B^\vee, \Lambda(G/H))$, and we get (c). Finally, by applying (b) for $H' = H$ and (c) for $H = \{1\}$ we get (d).

Let us note that the proof of theorem 11 gives the following $\mathbb{Z}/p^n$ -analogue (by 'omitting the inverse limits over n '). For a profinite group G let $\Lambda_n(G) = \Lambda(G)/p^n = \mathbb{Z}/p^n[[G]]$ be the completed group ring over $\mathbb{Z}/p^n$.

Theorem 12 *Let H and H' be normal subgroups of a profinite group G , with $H' \subset H$, and let A be a discrete $\Lambda_n(G)$ -module.*

(a) *There is a spectral sequence of $\Lambda_n(G/H)$ -modules*

$$E_2^{p,q} = H^p(G/H', H/H'; H^q(H', A)) \Rightarrow H^{p+q}(G, H; A).$$

(b) On the category of discrete $\Lambda_n(G)$ -modules the δ -functor $A \rightsquigarrow (H^m(G, H; A) \mid m \geq 0)$ with values in the category of $\Lambda_n(G/H)$ -modules is canonically isomorphic to the δ -functor $A \rightsquigarrow (Ext_{\Lambda_n(G)}^m(A^\vee, \Lambda_n(G/H)) \mid m \geq 0)$, where the Ext-groups are taken in the category of compact $\Lambda_n(G)$ -modules.

(c) In particular, if $\mathcal{G} = G/H$, and A is a discrete $\Lambda_n(G)$ -module, then one has a spectral sequence of $\Lambda_n(\mathcal{G})$ -modules

$$E_2^{p,q} = Ext_{\Lambda_n(\mathcal{G})}^p(H^q(H, A)^\vee, \Lambda_n(\mathcal{G})) \Rightarrow H^{p+q}(G, H; A) = Ext_{\Lambda_n(G)}^{p+q}(A^\vee, \Lambda(\mathcal{G})).$$

Corollary 13 With the notations as for Theorem 1, let A be a finite $\Lambda_n(G_S)$ -module, and $\Lambda_n = \Lambda(\mathcal{G})$. Then there is a spectral sequence of finitely generated Λ_n -modules

$$E_2^{p,q} = Ext_{\Lambda_n}^p(H^q(G_{\infty,S}, A)^\vee, \Lambda_n) \Rightarrow \varprojlim_{k'} H^{p+q}(G_S(k'), A) = Ext_{\Lambda_n(G_S)}^{p+q}(A^\vee, \Lambda_n),$$

where k' runs through the finite subextensions k'/k of k_∞/k .

On the other hand, Theorem 1 also has the following counterpart for finite modules.

Theorem 14 With notations as for Theorem 1, let A be a finite p -primary G_S -module, of exponent p^n . Then there is a spectral sequence

$$E_2^{p,q} = Ext_{\Lambda}^p(H^q(G_{\infty,S}, A)^\vee, \Lambda) \Rightarrow \varprojlim_{k'} H^{p+q-1}(G_S(k'), A) = Ext_{\Lambda_n(G_S)}^{p+q-1}(A^\vee, \Lambda_n),$$

where, in the inverse limit, k' runs through the finite extension k' of k inside k_∞ and the transition maps are the corestrictions.

Proof As in the proof of Theorem 1, Theorem 11 (d) applies to $G = G_S$ and $H = G_{\infty,S}$. But now the inverse system $(A[p^n])$ is Mittag-Leffler-zero in the sense of [Ja 1]: if the exponent of A is p^d , then the transition maps $A[p^{n+d}] \rightarrow A[p^n]$ are zero. This implies that $H_{cont}^m(G_S, (A[p^n])) = 0$ for all $m \geq 0$, cf. [Ja 1]. On the other hand it is clear that the system (A/p^n) is essentially constant ($A/p^n = A$ for $n \geq d$). From Theorem 11 (a) and Lemma 10 we immediately get

$$H_{cont}^m(G_S, G_{\infty,S}; RT_p A) \cong H_{cont}^{m-1}(G_S, G_{\infty,S}; (A/p^n)) \cong \varprojlim_{k'} H^{p+q-1}(G_S(k'), A),$$

and hence the claim, by applying Theorem 12 (b) in addition.

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