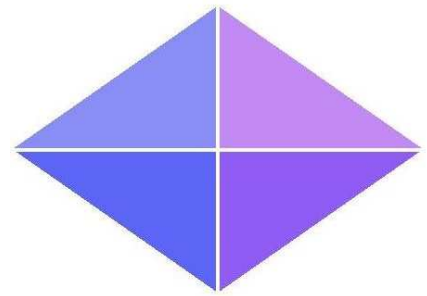


**DFG-Forschergruppe
Regensburg/Leipzig**
Algebraische Zyklen und
L-Funktionen



**On fibre sequences in motivic
homotopy theory**

Matthias Wendt

Preprint Nr. 10/2007

ON FIBRE SEQUENCES IN MOTIVIC HOMOTOPY THEORY

DER FAKULTÄT FÜR MATHEMATIK UND INFORMATIK
DER UNIVERSITÄT LEIPZIG
EINGEREICHTE

DISSERTATION

ZUR ERLANGUNG DES AKADEMISCHEN GRADES

DOCTOR RERUM NATURALIUM
(DR. RER. NAT.)

IM FACHGEBIET

MATHEMATIK

VORGELEGT VON

DIPLOM-INFORMATIKER MATTHIAS WENDT

GEBOREN AM 22. FEBRUAR 1980
IN LUTHERSTADT WITTENBERG

LEIPZIG, DEN 1. JUNI 2007

Bibliographische Daten

Wendt, Matthias

On Fibre Sequences in Motivic Homotopy Theory

Dissertation, Universität Leipzig 2007

x+165 Seiten, 122 Literaturangaben

Mathematics Subject Classification (2000): 14F35, (18G55, 55U35)

Keywords: simplicial presheaves, Bousfield-Kan localization, $\mathbb{A}^1$ -homotopy theory

Wissenschaftlicher Werdegang

- 1998 Abitur am Johann-Wolfgang-von-Goethe-Gymnasium in Chemnitz
- 1999 – 2002 Studium an der TU Dresden im Studiengang Diplom Informatik
- 2002 – 2004 Studium an der TU Dresden im Internationalen Studiengang Computational Logic
- 2004 Diplom Informatik, Note 1.1 mit Auszeichnung
- seit Juli 2004 Promotionsstudium an der Universität Leipzig, gefördert durch ein DFG-Stipendium im Graduiertenkolleg “Analysis, Geometrie und ihre Verbindung zu den Naturwissenschaften”

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Leipzig, den 1. Juni 2007

(Matthias Wendt)

Danksagung

Mein Dank gilt zuallererst meiner Betreuerin Annette Huber-Klawitter. Für ihre Geduld, fortwährende Ermutigung, und vor allem dafür, daß sie mir die Möglichkeit gegeben hat, mich überhaupt mit der Homotopietheorie von Schemata zu beschäftigen. Ohne sie wäre die vorliegende Arbeit nicht entstanden.

Vielen Dank auch an die Menschen, die mir bei verschiedenen Gelegenheiten mit wertvollen mathematischen Einsichten weitergeholfen haben: An Fabien Morel für seine Erläuterungen zu $\mathbb{A}^1$ -Homotopietheorie und zu Homotopietypen von algebraischen Gruppen, an Jens Hornbostel für Gespräche zur Lokalisierungstheorie, sowie an Marc Levine und Oliver Röndigs für ihre Erläuterungen zur Serre-Spektralsequenz für die motivische Kohomologie. Die mit Denis-Charles Cisinski geführte Diskussion zu Fragen der Eigentlichkeit der lokalen Modellstruktur haben meine Darstellung dieser Problematik stark geprägt.

Meinem Wissen auf dem Gebiet der algebraischen Geometrie auf die Sprünge geholfen haben in Leipzig vor allem Shahram Biglari, Bernd Herzog, Sascha Orlik und Malte Witte.

Bjørn Mäurer, Hilke Reiter, Christoph Sachse, Cornelia Schneider und Malte Witte haben verschiedene Teile der Dissertation korrekturgelesen und zur Beseitigung vieler Fehler sowie zur Verbesserung der Darstellung beigetragen.

Den größten Teil der Zeit, die ich mit der Arbeit an dieser Dissertation verbracht habe, war ich Stipendiat im DFG-Graduiertenkolleg “Analysis, Geometrie und ihre Verbindung zu den Naturwissenschaften”, wodurch mir verschiedene Konferenzbesuche ermöglicht wurden. In diesem Zusammenhang ebenso vielen Dank an Prof. Rademacher und Frau Leißner. Zusätzliche Finanzhilfen habe ich außerdem vom Clay Mathematics Institute für den Besuch der Sommerschule “Arithmetic Geometry” im Jahr 2006, sowie von der DFG-Forschergruppe “Algebraische Zykel und L -Funktionen” für den Besuch des Workshops “Algebraic Cycles, Motives and $\mathbb{A}^1$ -Homotopy Theory over General Bases” im Februar 2007 erhalten, auf dem ich auch einen Teil meiner Ergebnisse vorstellen durfte.

“Viel schon ist getan,
mehr noch bleibt zu tun,”
sprach der Wasserhahn
zu dem Wasserhuhn.

Robert Gernhardt

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Chapter 1

Introduction

In this thesis, we will examine the motivic or $\mathbb{A}^1$ -homotopy theory of schemes, a very recent field of algebraic geometry which applies methods and constructions of algebraic topology and the theory of model categories to schemes. Since the early days of Grothendieck-style algebraic geometry, intuition and methods from topology have always played an important role in the investigation of algebraic varieties and their structure. To give just one example, the study of algebraic K-theory and vector bundles on schemes, which has led to deep insights into the nature of linear algebra over general rings, would never have taken the shape it nowadays has without an existing theory of vector bundles on manifolds, and the use of elaborate methods from algebraic topology.

The motivation for developing a homotopy theory of schemes is the analogy between smooth schemes and manifolds, and homotopy theoretic arguments are ubiquitous in the theory of manifolds: In order to decide if there exist vector bundles with certain properties on a manifold, one translates this geometric problem into a homotopy theory problem, which then in turn is solved by computing some homotopy groups. It is reasonable to hope that a similar procedure can be used in algebraic geometry: Take a geometric problem, translate it into a question about $\mathbb{A}^1$ -homotopy theory, and then solve it by computing some homotopy groups. However, all of the steps seem to be more difficult than the corresponding ones in differential topology, since the main difference between schemes and manifolds is, that the arithmetic of the ring over which the scheme is defined intervenes in basically every question concerning the structure of schemes.

Both the above abstract motivation and a couple of already existing computations of $\mathbb{A}^1$ -homotopy groups make it seem worthwhile to study motivic homotopy theory. This is where we enter the scope of the thesis: Already in classical algebraic topology, there are only very few spaces whose homotopy groups can be computed directly; for all further computations, one has to use fibre sequences, which are sequences of spaces $F \rightarrow E \rightarrow B$ which induce long exact sequences for the corresponding homotopy groups. The main geometric examples of such fibre sequences are fibre bundles like $SO(n)$ -bundles or projective space bundles.

Hence, the goal of this thesis is an investigation of the theory of fibre sequences in $\mathbb{A}^1$ -homotopy theory. The definition of $\mathbb{A}^1$ -homotopy theory uses Quillen's homotopical algebra and model categories, which is an abstract framework to do algebraic topology and homotopy theory. This framework provides a notion of

fibre sequences, and it also follows from the model category machinery that every morphism of schemes $p : E \rightarrow B$ sits in a fibre sequence $F \rightarrow E \rightarrow B$ for some F . However, the homotopy fibre F of p need not be the same as the point-set fibre $p^{-1}(b)$ for a base point $b \in B$. Moreover, the description of the space F is a very abstract one, and what happens in the course of its construction is hard to keep track of. In the thesis, we will prove general criteria to decide when the homotopy fibre really can be identified with the point-set fibre. These general criteria allow to construct interesting fibre sequences which can be used to compute some $\mathbb{A}^1$ -homotopy groups. As an example, the following is a fibre sequence in $\mathbb{A}^1$ -homotopy theory:

$$\mathbb{P}^1 \rightarrow \mathbb{P}^\infty \vee \mathbb{P}^\infty \rightarrow \mathbb{P}^\infty.$$

The space $\mathbb{P}^1$ is the projective line, say over a field k , and $\mathbb{P}^\infty$ is the infinite projective space. The space $\mathbb{P}^\infty$ is the classifying space for line bundles, and therefore, its homotopy groups of $\mathbb{P}^\infty$ are known. The fibre sequence can then be used to give a presentation of the motivic fundamental group $\pi_1^{\mathbb{A}^1}(\mathbb{P}^1)$. Both the methods of computation and the presentation obtained are different from the methods that Morel has used to describe $\pi_1^{\mathbb{A}^1}(\mathbb{P}^1)$.

Other examples of fibre sequences that we can construct using our general criteria are provided by fibre bundles whose structure groups are the general linear groups GL_n or the symplectic groups Sp_{2n} . In particular, any projective space bundle $E \rightarrow B$ induces a fibre sequence

$$\mathrm{Sing}_\bullet^{\mathbb{A}^1}(\mathbb{P}^n) \rightarrow \mathrm{Sing}_\bullet^{\mathbb{A}^1}(E) \rightarrow \mathrm{Sing}_\bullet^{\mathbb{A}^1}(B),$$

where $\mathrm{Sing}_\bullet^{\mathbb{A}^1}(X)$ is an analogue of the singular complex of a topological space, using the affine spaces $\mathbb{A}^n$ as standard simplices.

In contrast to topology, it is not clear that any fibre bundle also yields a fibre sequence. To obtain the above fibre sequence for a projective space bundle, one needs to know that all projective space bundles on $U \times \mathbb{A}^n$ already come from bundles on U for any affine scheme U . This is a consequence of the Serre problem, whose solution by Quillen and Suslin is one of the deep facts in commutative algebra. In particular, the above reasoning can not be extended to, say, orthogonal bundles because for these, the Serre problem is known to fail.

From the known solutions of (variations of) the Serre problem, we can also derive descriptions of the $\mathbb{A}^1$ -homotopy groups of Chevalley groups like the classical groups over Dedekind rings. In particular, we can identify the $\mathbb{A}^1$ -homotopy groups of Chevalley groups with unstable K-groups, which are known to encode complicated information about the behaviour of matrices over general rings.

We are now ready to go a little more into detail. Before describing our main results, we explain what $\mathbb{A}^1$ -homotopy theory actually is.

$\mathbb{A}^1$ -Homotopy Theory: In its present form, $\mathbb{A}^1$ -homotopy theory was started off by Morel and Voevodsky. There are preliminary accounts of the theory in [Voe98, Mor99] leading to the final definitions (and proofs) in [MV99].

Precursors of this theory, which in particular emphasize the importance of the affine spaces $\mathbb{A}^n$, can be found in the definition of K-theory after Karoubi and Villamayor [KV69], Bloch's higher Chow groups or Suslin homology. In all of these constructions, the affine spaces $\mathbb{A}^n$ are used as a standard cosimplicial

object, similar to the standard simplices in classical algebraic topology. In general, $\mathbb{A}^1$ -invariance plays a strong role in K-theory and intersection theory.

The problem in setting up a homotopy theory of schemes similar to the homotopy theory of topological spaces is, that regular morphisms of schemes are much more rigid than continuous maps of topological spaces. One expects the proper analogue of the unit interval to be the affine line $\mathbb{A}^1$, but the naive definition of homotopies as morphisms $X \times \mathbb{A}^1 \rightarrow Y$ does not yield a well-behaved theory. Therefore, one has to use a rather abstract approach, constructing a model category which has enough good properties to be called a homotopy theory of schemes.

There are several steps in the construction of $\mathbb{A}^1$ -homotopy: We start with a scheme S , and consider the category Sm_S of smooth schemes X over S . The topological analogue of this would be morphisms of manifolds $X \rightarrow S$, whose fibres are again manifolds. To do the constructions of algebraic topology, one has to be able to form arbitrary colimits, which is not possible for schemes. Therefore, we pass to the category of sheaves on Sm_S . In this category, we can consider simplicial objects, and then formally do homotopy theory of simplicial presheaves on a Grothendieck site. The right way to extend the homotopy theory of simplicial sets to simplicial sheaves and presheaves has been known for some time, due to the work of Jardine [Jar87]. Since there are several Grothendieck topologies on the category of smooth schemes, there are still some degrees of freedom. It seems like the so-called Nisnevich topology is the right Grothendieck topology for the homotopy theory of schemes.

The resulting homotopy theory has the problem that the affine line $\mathbb{A}^1$ over the base S is not contractible. Therefore, one has to apply yet another abstract process, the so-called Bousfield localization, to force the affine line to be contractible. This formal construction is one of the reasons why it is rather difficult to compute homotopy groups of varieties in the $\mathbb{A}^1$ -homotopy theory: Both the fibrant replacement in model categories of presheaves as well as Bousfield localization are hard to keep track of. The goal of the thesis is to investigate how one can deal with these problems.

Since the invention of $\mathbb{A}^1$ -homotopy in the abovementioned paper [MV99], several papers have been written on the homotopy theory of simplicial sheaves in general and on $\mathbb{A}^1$ -homotopy in particular. Most important for this thesis is the work of Goerss and Jardine [GJ98], which constructs localization functors for general model categories of simplicial sheaves. Most of the recent work on $\mathbb{A}^1$ -homotopy deals with the stable $\mathbb{A}^1$ -homotopy theory, which is an analogue of the category of spectra. Using suitable models of this stable $\mathbb{A}^1$ -homotopy, one can investigate the various cohomology theories for schemes, such as motivic cohomology and higher Chow groups, algebraic K-theory or algebraic cobordism. In this setting, however, our results are not really interesting, since the Bousfield localization for spectra is much more well-behaved than for simplicial sheaves.

The only work on unstable $\mathbb{A}^1$ -homotopy theory is done in a series of preprints by Fabien Morel [Mor07, Mor06b], in which some interesting applications of unstable $\mathbb{A}^1$ -homotopy theory, like a theory of the Euler class for vector bundles over smooth affine schemes, are developed. The results in these papers are, however, restricted to smooth schemes over fields. In our work, we try to study homotopy theory of schemes over a general base scheme S , only restricting to Dedekind rings or fields when it comes to concrete computations.

Main Results: We now give an exposition of the basic theorems of the thesis. As mentioned earlier, the goal of the thesis is a discussion of what we think are the most important aspects of the theory of fibrations in model categories of simplicial presheaves. We will describe how to patch fibre sequences together, how fibre sequences can be classified and how they behave under localization functors, which gives us some insight into fibre sequences in the $\mathbb{A}^1$ -local model category. We also examine some aspects of the cohomological theory of fibre sequences, in particular the Serre spectral sequence, and finally investigate how the geometry of schemes determines if a scheme is fibrant resp. $\mathbb{A}^1$ -local.

One of the key properties of simplicial presheaves is that we have homotopy distributivity, a homotopy generalization of the point-set distributivity known to hold for topoi. This homotopy distributivity as well as some of its important consequences will be discussed in Chapter 3: In classical algebraic topology, the work of Dold and Thom [DT58] allowed to glue fibrations over a CW-complex to *quasi-fibrations*, which still give rise to fibre sequences. Something similar works for general simplicial presheaves. We show that it is possible to glue fibre sequences of simplicial presheaves, which allows to construct many interesting fibre sequences, among them torsors under linear algebraic groups. The possibility to glue fibre sequences is a consequence of homotopy distributivity, which allows to commute homotopy colimits with finite homotopy limits. Combining this with the Brown representability theorem, one can then also construct classifying spaces for fibre sequences of simplicial presheaves, i.e. it is possible to construct spaces which completely describe the behaviour of fibre sequences $F \rightarrow E \rightarrow B$ with a given fibre F .

THEOREM 1 (CLASSIFYING SPACES) *For a fixed space F , there exists a classifying space $B^f F$ for fibre sequences, i.e. for any other space B , the fibre sequences $F \rightarrow E \rightarrow B$ over B are in bijection with homotopy classes of maps $B \rightarrow B^f F$. Moreover, there is a universal fibre sequence of simplicial presheaves with fibre F :*

$$F \rightarrow E^f F \rightarrow B^f F.$$

Any other fibre sequence is a pull-back of the universal one.

Having the above theorem, one of the favourite constructions from geometric topology can now be carried over to an algebraic geometric setting: The space classifying spherical fibrations $S^{p,q} \rightarrow E \rightarrow B$, where the fibre we use is one of the many motivic spheres $S^{p,q} = (S_s^1)^{\wedge(p-q)} \wedge (\mathbb{G}_m)^{\wedge q}$. The first factor in this is one of the usual spheres, a smash power of the standard simplicial circle S^1 , and the latter is a smash power of the multiplicative group. That the latter should be considered as a sphere is clear from complex realization, where $\mathbb{G}_m(\mathbb{C}) = \mathbb{C}^*$ also has the homotopy type of the circle. As in classical topology, one can associate via a Thom-type construction to each vector bundle over a scheme $E \rightarrow X$ a new fibre sequence $\tilde{E} \rightarrow X$ whose fibre is a sphere $S^{2n,n}$. If the motivic situation is only remotely similar to the differential topology of smooth manifolds, the study of the classifying space of spherical fibrations and its relation with algebraic K-theory could yield interesting insights into the structure of schemes.

Summing up, the theory of fibre sequences of simplicial presheaves is rather similar to the situation for simplicial sets, and this is the main result of Chapter 3.

However, motivic homotopy theory is not just homotopy theory of simplicial presheaves, there is also the Bousfield localization at the affine line $\mathbb{A}^1$. Therefore, Chapter 4 is concerned with the behaviour of fibre sequences under the Bousfield localization. In general, a fibre sequence is not preserved by a Bousfield localization, so our aim is to obtain a sharp criterion when this is the case. Such questions have been investigated for simplicial sets already, and we show in this chapter, that a great deal of the localization theory for simplicial sets can be carried over to the setting of simplicial sheaves. As one example, we can show that the various constructions of fibrewise localizations for simplicial sets can be defined for simplicial presheaves. Fibrewise localization produces for a given fibre sequence $F \rightarrow E \rightarrow B$ and a Bousfield localization L a new fibre sequence $LF \rightarrow \overline{E} \rightarrow B$, which is basically obtained by fibrewise replacing F by its localization LF . This allows to keep track of the behaviour of the fibre sequence under special localizations. The main result of Chapter 4 is the following sharp criterion for locality of fibre sequences.

THEOREM 2 (LOCALIZATION FOR FIBRE SEQUENCES) *Let T be a site, and let $f : X \rightarrow Y$ be a null-homotopic morphism of simplicial presheaves. Furthermore, let $F \rightarrow E \xrightarrow{p} B$ be a fibre sequence. Let $L_f F \rightarrow E' \rightarrow A_{L_f} B$ be the pullback of the fibrewise localization $L_f F \rightarrow \overline{E} \rightarrow B$ of p to the fibre $A_{L_f} B$ of the localization $B \rightarrow L_f B$. Then $E' \simeq L_f F \times A_{L_f} B$ iff the fibre sequence p is preserved by L_f .*

This means that there is some global object controlling if a given fibre sequence descends to the Bousfield localization: The homotopy fibre $A_{L_f} B$ above is the part of the base space B which is not local. If the fibre sequence over this non-local space $A_{L_f} B$ is trivial, then the fibre localizes. Otherwise, there is nontrivial information living over the non-local part which would necessarily be killed by localization – therefore localization does not preserve such a fibre sequence. This yields a sharp criterion for fibre sequences of simplicial presheaves to descend to a localized model structure, and in particular allows to construct interesting fibre sequences in $\mathbb{A}^1$ -homotopy theory over general bases.

Note that the condition that $f : X \rightarrow Y$ is null-homotopic is essential. This criterion does not work in full generality, it uses some assertions that only hold for nullification functors, i.e. localization with respect to null-homotopic morphisms. Our investigations in Chapter 4 also yield a new proof of the relation between nullification functors and properness of the local model structure: Localization with respect to a morphism of connected simplicial presheaves $f : X \rightarrow Y$ is a nullification if and only if the local model structure is proper.

The main application of the locality criterion above is however the description of interesting fibre sequences in motivic homotopy theory. One particularly beautiful such fibre sequence is the one we mentioned earlier:

$$\mathbb{P}^1 \rightarrow \mathbb{P}^\infty \vee \mathbb{P}^\infty \rightarrow \mathbb{P}^\infty.$$

It describes the homotopy type of the projective line $\mathbb{P}^1$ as the homotopy fibre of the fold map on the infinite projective space. This allows to reduce the computation of the fundamental group sheaf $\pi_1^{\mathbb{A}^1}(\mathbb{P}^1)$ to an application of the van-Kampen theorem to the wedge sum $\mathbb{P}^\infty \vee \mathbb{P}^\infty$. The above fibre sequence therefore implies that the fundamental group of the projective line $\mathbb{P}^1$ is a rather complicated and non-abelian object.

Another application of our general results is the computation of homotopy groups of a split algebraic group: It turns out that such groups can have a quite complicated fundamental groupoid related to the connected components or the solvable radical, but the higher homotopy groups only depend on the root system of the group. These higher homotopy groups can then be shown to be (related to) unstable K-theory groups, in the definition of Karoubi and Villamayor.

For concrete computations, the behaviour of the spaces involved with respect to polynomial rings is of fundamental importance: For a projective space bundle $\mathbb{P}^n \rightarrow E \rightarrow B$ to induce a fibre sequence in $\mathbb{A}^1$ -homotopy theory, it is necessary that all projective space bundles on $X \times \mathbb{A}^1$ are extended from bundles on X . Such questions are not at all easy to answer, as the vast literature on the Serre problem shows. Most of Chapter 5 deals with such questions. Our main interest is in the homotopy groups of algebraic groups. To describe these homotopy groups, one of the crucial notions that has to be investigated is the affine Brown-Gersten property. The main result of Chapter 5 is the following

THEOREM 3 *Let Φ be a root system of rank $\text{rk } \Phi > 1$, and let R be an excellent Dedekind ring. Then the simplicial presheaf $\text{Sing}_{\bullet}^{\mathbb{A}^1}(G(\Phi))$ has the affine Brown-Gersten property for the Nisnevich topology, where $G(\Phi)$ denotes the universal Chevalley group scheme of type Φ .*

This generalizes results of Morel, which deal with the case of GL_n over a field. We find that we can identify the $\mathbb{A}^1$ -homotopy groups of a Chevalley group as the corresponding unstable Karoubi-Villamayor K-groups. Similar results could also be obtained for varieties related to GL_n , such as the Grassmann and Stiefel varieties. The $\mathbb{A}^1$ -homotopy groups of these varieties encode a great deal of information on the behaviour of vector bundles and their generators. This is the main difference to stable $\mathbb{A}^1$ -homotopy, and motivation to further study unstable phenomena as well: In stable $\mathbb{A}^1$ -homotopy, we have algebraic K-theory, which encodes information about vector bundles. On the other hand, the unstable K-theories describe vector bundles with a given number of explicit generating sections. The difference between stable and unstable K-groups then describes the behaviour of generating sections.

We hope that the results in this thesis will enable other interesting computations of $\mathbb{A}^1$ -homotopy groups, and an understanding of the relation between the geometry of schemes and their $\mathbb{A}^1$ -homotopy types.

Structure of the Thesis: The thesis is structured as follows:

We start out in Chapter 2 with a short overview of the general notions of homotopical algebra, giving all the necessary definitions of model categories and their properties which will be needed throughout the thesis.

In Chapter 3, we will start with a very general class of model categories, namely model categories of simplicial presheaves on some Grothendieck site, and investigate what can be said about the theory of fibre sequences in such a model category. The two most important properties of model categories of simplicial presheaves turn out to be the existence of canonical homotopy colimit decompositions and homotopy distributivity. Then we show how to construct classifying spaces for fibre sequences, and give some examples of interesting fibre sequences of simplicial presheaves. At the end of the chapter, we also discuss some prospects for further work: Without going into the messy technical details, we outline the construction of a Serre-type spectral sequences for model categories of simplicial sheaves.

The last two chapters then deal with localized model categories, in particular with the $\mathbb{A}^1$ -local model categories. We discuss fibrewise localization and its consequences in Chapter 4. From the existence of fibrewise localization it is possible to derive a sharp criterion when a fibre sequence in a simplicial model category is preserved by a nullification functor. We give some applications, i.e. some local fibre sequences that allow a couple of computations in motivic homotopy theory.

Finally, Chapter 5 describes what sort of geometric structures are needed to determine if $\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(X)$ is almost fibrant. This allows to show that certain fibre bundles induce fibre sequences, and to determine the homotopy groups of Chevalley groups over affine schemes over a Dedekind ring.

Chapter 2

Preliminaries and Notation

Throughout the thesis, we will freely use the definitions and results of category and topos theory. The general references used will be Mac Lane's book [Mac98] for category theory and [MM92] for topos theory.

The category of sets is denoted by $\mathcal{Set}$, the category of simplicial objects in a category $\mathcal{C}$ by $\Delta^{\text{op}}\mathcal{C}$. A general Grothendieck site will usually be denoted by T , and the category of sheaves on T by $\mathcal{Shv}(T)$. The sheafification functor will usually be denoted by L^2 .

Recall that a *point of a topos* is a pair of adjoint functors

$$p_* : \mathcal{Set} \rightleftarrows \mathcal{Shv}(T) : p^*,$$

such that p^* preserves finite limits. We will occasionally confuse the point p with its inverse image part p^* . Recall that a topos $\mathcal{Shv}(T)$ *has enough points* if points separate distinct morphisms, i.e. for any two distinct morphisms $f, g : X \rightarrow Y$ in $\mathcal{Shv}(T)$ there exists a point p such that $p^*(f) \neq p^*(g)$. As is always the case, the definitions and results we use are rather easy to write down in case the topos of sheaves on T has enough points [MM92, p. 523]. The same results still hold but are rather tedious to work out if there are not enough points, in which case one has to use a Boolean localization argument [MM92, Corollary IX.11.2]. Since all the sites we are dealing with in $\mathbb{A}^1$ -homotopy do have enough points, the standard way of dealing with this will be to give the precise definitions and proofs of the results under the assumption that we have enough points, and only give short explanations concerning the general case. If a topos has enough points, then there is already a set of points which is conservative, cf. [Tho85, Construction 1.31] and [AGV73, Exposé IV, Proposition 6.5.b].

The main reference for algebraic geometry will be Hartshorne [Har77]. Whenever we consider a category of schemes over a base scheme S , this base scheme will usually be separated and of finite type. The category of smooth, separated, finite type schemes over a base S is denoted by Sm_S . The category of all separated, finite type schemes over a base S is denoted by Sch_S .

We denote by $\Delta^\bullet$ the standard cosimplicial object, with $\Delta_S^n = S \times_{\mathbb{Z}} \Delta_{\mathbb{Z}}^n$ and

$$\Delta_{\mathbb{Z}}^n = \text{Spec } \mathbb{Z}[t_0, \dots, t_n] / \sum_i t_i = 1.$$

For a presheaf of sets F on $\mathbf{Sm}_S$, we denote by $\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(F)$ the $\mathbb{A}^1$ -simplicial resolution of F , which is the simplicial presheaf of sets on $\mathbf{Sm}_S$ given by

$$U \mapsto F(\Delta^{\bullet} \times U).$$

2.1 Homotopical Algebra

We first recall the basic notions of homotopical algebra. The conception of homotopical algebra is entirely due to Quillen [Qui67]. For an idea of what homotopical algebra has become nowadays, as well as general reference for the definitions presented, we refer to [DS95, GJ99, Hir03, Hov98]. The central objects of interest in homotopical algebra are model categories, which are categories with enough structure to do all the constructions known from algebraic topology:

DEFINITION 2.1.1 (MODEL CATEGORY) *A (closed) model category $\mathcal{C}$ is a category equipped with three classes of morphisms, called cofibrations, fibrations and weak equivalences, which satisfy the following conditions:*

- (MC1) $\mathcal{C}$ has all small limits and colimits.
- (MC2) If f and g are composable morphisms and two of f , g and $g \circ f$ are weak equivalences, then so is the third.
- (MC3) If the morphism f is a retract of g and g is a weak equivalence, cofibration or fibration, then so is f . Recall that f is a retract of g if the following diagram is commutative:

$$\begin{array}{ccccc}
 & & Y & & \\
 & \nearrow & \downarrow & \searrow & \\
 X & \xrightarrow{\mathrm{id}} & X & & \\
 \downarrow f & & \downarrow g & & \downarrow f \\
 & \nearrow & Y' & \searrow & \\
 X' & \xrightarrow{\mathrm{id}} & X' & &
 \end{array}$$

- (MC4) Any fibration has the right lifting property w.r.t. trivial cofibrations, i.e. for any commutative solid square

$$\begin{array}{ccc}
 A & \longrightarrow & X \\
 c \downarrow & \nearrow g & \downarrow f \\
 B & \longrightarrow & Y,
 \end{array}$$

with f a fibration and c a trivial cofibration and weak equivalence, the dotted lift g exists, and makes the diagram commute.

Similarly, trivial fibrations have the right lifting property w.r.t. cofibrations.

(MC5) Any morphism f has functorial factorizations $f = p' \circ c$ and $p \circ c'$ with p resp. p' a fibration resp. trivial fibration, and c resp. c' a cofibration resp. trivial cofibration.

As most of the model categories we consider have monomorphisms as cofibrations, we will find it convenient to denote cofibrations by $i : X \hookrightarrow Y$. A weak equivalence need not be a homotopy equivalence, therefore usually has a direction, and we therefore denote weak equivalences by $X \xrightarrow{\simeq} Y$. If there is a zig-zag of weak equivalences between two objects X and Y , we write $X \simeq Y$.

REMARK 2.1.2 (PHILOSOPHY) The structure of a model category can be divided into a left and a right part, making all definitions appear in two dual forms. The left part is the cofibration part, which (as we will see later on) interacts nicely with left adjoint functors, in particular with products. The right part of the model structure is the one governed by the fibrations and interacts nicely with right adjoint functors such as hom-complexes.

The main examples of model categories are simplicial sets, topological spaces, and categories of chain complexes of modules. We will discuss in more detail the model category of simplicial sets, and its sheafification, the model categories of simplicial sheaves on a Grothendieck site.

EXAMPLE 2.1.3 The category of simplicial sets has a model category structure. A comprehensive overview of the subtleties of simplicial topology is given in [GJ99]. Recall that the simplicial category Δ is the category having as objects finite ordinals $\mathbf{n}$ and morphisms between them order-preserving maps. There are distinguished such maps, the *coface maps* $d^i : \mathbf{n} - 1 \rightarrow \mathbf{n}$ and the *codegeneracy maps* $s_j : \mathbf{n} + 1 \rightarrow \mathbf{n}$, satisfying the *cosimplicial identities*. A *simplicial set* is then a functor $X : \Delta^{\text{op}} \rightarrow \text{Set}$. The set $X(\mathbf{n})$ is called the *set of n -simplices*, $X(s^j)$ are the *degeneracy maps*, and $X(d^i)$ the *face maps*. A *morphism of simplicial sets* is a natural transformation between such functors.

Now for the model structure part: The *cofibrations* of simplicial are simply the monomorphisms, and the *fibrations* are morphisms which have the right lifting property w.r.t. all morphisms $\Lambda_k^n \hookrightarrow \Delta^n$, with Δ^n being the *standard n -simplex*, and Λ_k^n being the subcomplex of Δ^n generated by all faces except the k -th face.

A *simplicial homotopy* of simplicial maps $f, g : K \rightarrow X$, denoted by $h : f \sim g$, is a map $h : K \times \Delta^1 \rightarrow X$ sitting in the following commutative diagram:

$$\begin{array}{ccc}
 K \times \Delta^0 = K & & \\
 \text{id} \times d^1 \downarrow & \searrow f & \\
 K \times \Delta^1 & \xrightarrow{h} & X \\
 \text{id} \times d^0 \uparrow & \nearrow g & \\
 K \times \Delta^0 = K & &
 \end{array}$$

For a fibrant simplicial set (or Kan set) X , i.e. one for which the structure morphism $X \rightarrow \text{pt}$ is a fibration, homotopy is an equivalence relation and one defines the *simplicial homotopy groups* $\pi_n(X, v)$ with a chosen base point $v \in X_0$ to be the sets of homotopy classes of maps $\alpha : \Delta^n \rightarrow X$ sitting in the following commutative diagram:

$$\begin{array}{ccc}
\Delta^n & \xrightarrow{\alpha} & X \\
\uparrow & & \uparrow v \\
\partial\Delta^n & \longrightarrow & \Delta^0
\end{array}$$

This is well defined, and gives group structures for $n \geq 1$ which are abelian in case $n \geq 2$. A *weak equivalence* of simplicial sets is then a map $f : X \rightarrow Y$ which induces an isomorphism on simplicial homotopy groups and an isomorphism on sets of connected components.

The last important structure of the model category of simplicial sets we want to mention is the cartesian closed structure. This in particular allows to consider categories enriched in simplicial sets. The cartesian product of simplicial sets is of course the pointwise cartesian product of set-valued functors. The right adjoint is an internal Hom-functor, and for simplicial sets is defined by

$$\mathbf{Hom}(X, Y)_n = \text{hom}_{\Delta^{\text{op}}\text{Set}}(X \times \Delta^n, Y).$$

These function complexes interact well with the model structure on the category of simplicial sets, i.e. for any cofibration of simplicial sets $i : K \hookrightarrow L$ and any fibration $p : X \rightarrow Y$, the morphism

$$\mathbf{Hom}(L, X) \xrightarrow{(i^*, p_*)} \mathbf{Hom}(K, X) \times_{\mathbf{Hom}(K, Y)} \mathbf{Hom}(L, Y)$$

is a fibration. This morphism is induced by the diagram

$$\begin{array}{ccc}
\mathbf{Hom}(L, X) & \xrightarrow{p_*} & \mathbf{Hom}(L, Y) \\
i^* \downarrow & & \downarrow i^* \\
\mathbf{Hom}(K, X) & \xrightarrow{p_*} & \mathbf{Hom}(K, Y).
\end{array}$$

This well-behavedness of the internal hom-functor w.r.t. the model structure is essential for the definition of simplicial model categories which will be given later.

The internal hom-complexes can be used to give another definition of homotopy groups which generalizes nicely to categories of sheaves. This definition can be found in [Jar96]. Consider the following pullback diagram, assuming that X is fibrant:

$$\begin{array}{ccc}
F_n X & \longrightarrow & \mathbf{Hom}(\Delta^n, X) \\
\downarrow & & \downarrow i^* \\
X_0 & \longrightarrow & \mathbf{Hom}(\partial\Delta^n, X).
\end{array}$$

The simplicial set X_0 is the set of 0-simplices of X , and the connected components of $F_n X$ over $x \in X_0$ are in bijection with the n -th homotopy group of X with base point x . $\square$

EXAMPLE 2.1.4 We now describe the *injective model structure* on the category of simplicial (pre-)sheaves on an arbitrary Grothendieck site, a construction due to Jardine [Jar87]. For this reason, we will also refer to this model structure as *Jardine model structure*. Let T be a Grothendieck site, let $\text{Shv}(T)$ be the corresponding

category of sheaves on T , and let $\Delta^{\text{op}}\mathcal{Shv}(T)$ denote the category of simplicial sheaves of sets.

A *cofibration* of simplicial sheaves is a monomorphism in the category of simplicial sheaves, which basically means that it is a pointwise monomorphism.

The weak equivalences are defined using the Grothendieck topology on the site T . In the general case, one has to use a Boolean localization to define local weak equivalences and to prove that they provide a model structure [Jar96]. We will restrict ourselves to the case of a topos with enough points.

A morphism $f : X \rightarrow Y$ is a *weak equivalence* of simplicial sheaves if for any point $p^* : \mathcal{Shv}(T) \rightarrow \mathcal{Set}$ of the topos, the induced morphism of simplicial sets $p^*(f) : p^*(X) \rightarrow p^*(Y)$ is a weak equivalence of simplicial sets.

Finally, a *fibration* of simplicial sheaves is a morphism which has the right lifting property w.r.t. all trivial cofibrations. It is proved in [Jar87] that the above definitions provide $\Delta^{\text{op}}\mathcal{Shv}(T)$ with the structure of a model category. The same is true for the category of $\Delta^{\text{op}}P\mathcal{Shv}(T)$ of simplicial presheaves on the site. Note that weak equivalences still take into account the topology of the site, since they are defined via the set of points. This implies that $\Delta^{\text{op}}\mathcal{Shv}(T)$ and $\Delta^{\text{op}}P\mathcal{Shv}(T)$ are Quillen-equivalent model categories, and the sheafification morphism is a weak equivalence in $\Delta^{\text{op}}P\mathcal{Shv}(T)$.

We want to make a remark on homotopy groups of simplicial sheaves: Let X be a fibrant simplicial sheaf, and let $F_n X$ be the simplicial sheaf defined by the following pullback:

$$\begin{array}{ccc} F_n X & \longrightarrow & \mathbf{Hom}(\Delta^n, X) \\ \downarrow & & \downarrow i^* \\ X_0 & \longrightarrow & \mathbf{Hom}(\partial\Delta^n, X). \end{array}$$

The n -th homotopy group sheaf is then defined to be the sheaf of connected components $\pi_0 F_n X$.

The following definition of weak equivalence is equivalent to the one given above [Jar96, Lemma 26]: A morphism $f : X \rightarrow Y$ is a weak equivalence iff it induces an isomorphism on connected components $f_* : \pi_0 L^2 \text{Ex}^\infty X \rightarrow \pi_0 L^2 \text{Ex}^\infty Y$ and the squares

$$\begin{array}{ccc} \pi_n L^2 \text{Ex}^\infty X & \xrightarrow{f_*} & \pi_n L^2 \text{Ex}^\infty Y \\ \downarrow & & \downarrow \\ (L^2 \text{Ex}^\infty X)_0 & \xrightarrow{f_*} & (L^2 \text{Ex}^\infty Y)_0 \end{array}$$

are pullback squares. In the above L^2 is the sheafification functor and Ex^∞ is a sheafification of the Ex^∞ -construction for simplicial sets. $\square$

There are other model structures on categories of simplicial sheaves, like the projective [Bla01, Dug01b], the flasque model structure [Isa05] and many other intermediate ones [Jar06b]. They all share the same class of weak equivalences, but differ in the choice of fibrations resp. cofibrations. Each do have their respective advantages: While the projective model structure has quite easy characterizations

of fibrant objects, the injective model structure has the property that all objects are cofibrant.

In any model category, one can define cylinder and path objects, which are the analogues of the topological $X \times [0, 1]$ resp. $\text{map}([0, 1], X)$. They can be used to define left and right homotopies and to give a direct construction of the homotopy category.

DEFINITION 2.1.5 *Let $\mathcal{C}$ be a model category, and A an object of $\mathcal{C}$. A cylinder object for A is a commutative triangle*

$$\begin{array}{ccc} A \sqcup A & & \\ \downarrow i & \searrow \nabla & \\ \tilde{A} & \xrightarrow{\sigma} & A. \end{array}$$

In the above, $\nabla : A \sqcup A \rightarrow A$ is the fold map which is the identity on each summand, i is a cofibration, and σ is a weak equivalence. A cylinder object is usually denoted by $A \times I$, and there are maps $i_0, i_1 : A \rightarrow A \times I$.

Dually, a path object is a commutative triangle

$$\begin{array}{ccc} & \tilde{A} & \\ s \nearrow & \downarrow p & \\ A & \xrightarrow{\Delta} & A \times A. \end{array}$$

In the above, $\Delta : A \rightarrow A \times A$ is the diagonal, s is a weak equivalence and p is a fibration. The path object is usually denoted by A^I .

Note that the factorization axiom CM5 produces cylinder resp. path objects in any model category. The above cylinder resp. path objects allow to define left resp. right homotopies. Being left resp. right homotopic are equivalence relations for cofibrant and fibrant objects, which moreover are the same. The *homotopy category* $\text{Ho}\mathcal{C}$ of the model category $\mathcal{C}$ is defined to be the category whose objects are the cofibrant and fibrant objects of $\mathcal{C}$ and whose morphisms are homotopy classes of maps. For more details on the construction of the homotopy category and the proofs of the above statements, see e.g. [GJ99, Section II.1].

We continue with two important properties of model categories, which will be satisfied in particular by model categories of simplicial sheaves.

DEFINITION 2.1.6 (PROPERNESS) *A model category $\mathcal{C}$ is called left resp. right proper if every pushout of a weak equivalence along a cofibration is a weak equivalence resp. if every pullback of a weak equivalence along a fibration is a weak equivalence. The model category $\mathcal{C}$ is called proper if it is both left and right proper.*

DEFINITION 2.1.7 *A category $\mathcal{C}$ is a simplicial category if it is enriched in the category of simplicial sets, i.e. there is a mapping space functor*

$$\mathbf{Hom}_{\mathcal{C}}(-, -) : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \Delta^{\text{op}}\mathcal{S}et$$

such that the following conditions are satisfied:

$$(i) \mathbf{Hom}_{\mathcal{C}}(A, B)_0 = \mathrm{hom}_{\mathcal{C}}(A, B),$$

(ii) the functor $\mathbf{Hom}_{\mathcal{C}}(A, -)$ has a left adjoint $A \otimes - : \Delta^{\mathrm{op}}\mathrm{Set} \rightarrow \mathcal{C}$, which satisfies associativity

$$A \otimes (K \times L) \cong (A \otimes K) \otimes L$$

for $A \in \mathcal{C}$ and $K, L \in \Delta^{\mathrm{op}}\mathrm{Set}$,

(iii) the functor $\mathbf{Hom}_{\mathcal{C}}(-, B) : \mathcal{C}^{\mathrm{op}} \rightarrow \Delta^{\mathrm{op}}\mathrm{Set}$ has a left adjoint $\mathbf{hom}_{\mathcal{C}} : \mathcal{C}^{\mathrm{op}} \rightarrow \Delta^{\mathrm{op}}\mathrm{Set}$.

Note that this makes sense because the category of simplicial sets is a monoidal closed category as shown in Example 2.1.3

A model category $\mathcal{C}$ is called *simplicial* if it is a simplicial category and additionally Quillen's axiom SM7 is satisfied: For any cofibration $j : A \rightarrow B \in \mathcal{C}$ and fibration $q : X \rightarrow Y \in \mathcal{C}$ the morphism

$$\mathbf{Hom}_{\mathcal{C}}(B, X) \xrightarrow{(j^*, q_*)} \mathbf{Hom}_{\mathcal{C}}(A, X) \times_{\mathbf{Hom}_{\mathcal{C}}(A, Y)} \mathbf{Hom}_{\mathcal{C}}(B, Y)$$

is a fibration which is trivial if j or q is trivial.

Pointed Model Categories and Fibre Sequences: Pointed model categories are those model categories which have an object, denoted by $*$ or pt , which is both terminal and initial.

There are some additional structures one can define in a pointed model category: The disjoint sum of objects X and Y becomes the *wedge sum* $X \vee Y$, which is the pushout of $X \leftarrow * \rightarrow Y$. There also exists another product for pointed spaces X and Y , the *smash product* $X \wedge Y$, which is defined as the cokernel of $(\mathrm{id} \times *) \vee (* \times \mathrm{id}) : X \vee Y \rightarrow X \times Y$.

It is also possible to define suspension and loop space functors, which is why fibre sequences are treated in the setting of pointed model categories. Let X be an object in a pointed model category $\mathcal{C}$, denote by QX resp. RX a cofibrant resp. fibrant replacement. The *suspension* ΣX of X is defined to be the cokernel of $QX \vee QX \rightarrow QX \times I$. Dually, the *loop space* ΩX of X is defined to be the kernel of $(RX)^I \rightarrow RX \times RX$.

We now repeat the standard definition of fibre sequences in model categories, taken from [Hov98]. For details of the proof see [Hov98, Theorem 6.2.1].

DEFINITION 2.1.8 *Given a fibration $p : E \rightarrow B$ of fibrant objects with fibre $i : F \rightarrow E$. There is an action of ΩB on F , given as follows. Let $h : A \times I \rightarrow B$ represent $[h] \in [A, \Omega B]$ and let $u : A \rightarrow F$ represent $[u] \in [A, F]$. We define $\alpha : A \times I \rightarrow E$ as the lift in the following diagram:*

$$\begin{array}{ccc} A & \xrightarrow{iou} & E \\ i_0 \downarrow & & \downarrow p \\ A \times I & \xrightarrow{h} & B \end{array}$$

Then define $[u].[h] = [w]$ with $w : A \rightarrow F$ to be the unique map satisfying $i \circ w = \alpha \circ i_1$.

This defines a natural right action of $[A, \Omega B]$ on $[A, F]$ for any A , which suffices to provide an action of ΩB on F .

This motivates the definition of fibre sequences [Hov98, Definition 6.2.6], given as follows:

DEFINITION 2.1.9 *Let $\mathcal{C}$ be a pointed model category. A fibre sequence is a diagram $X \rightarrow Y \rightarrow Z$ together with a right action of ΩZ on X that is isomorphic in $\text{Ho } \mathcal{C}$ to a diagram $F \xrightarrow{i} E \xrightarrow{p} B$ where p is a fibration of fibrant objects with fibre i and F has the right ΩB -action of Definition 2.1.8.*

The importance of fibre sequences lies in the fact that they give rise to long exact homotopy sequences. We actually give a construction of the long exact sequence of homotopy group sheaves for a fibre sequence of simplicial sheaves. Assuming the topos has enough points, all we need to do is construct the boundary map.

DEFINITION 2.1.10 *Let $p : E \rightarrow B$ be a fibration of fibrant simplicial sheaves, let $x \in B$ be a base point, and let $F = p^{-1}(x)$ with chosen base point $y \in F$. Then there is a boundary map*

$$\partial : \pi_n(B, x) \rightarrow \pi_{n-1}(F, y),$$

given as $d_0\theta \in \pi_{n-1}(F, y)$ where θ is the lift in the following diagram:

$$\begin{array}{ccc} \Lambda_0^n & \xrightarrow{(\cdot, y, \dots, y)} & E \\ \downarrow & \nearrow \theta & \downarrow p \\ \Delta^n & \xrightarrow{\alpha} & B. \end{array}$$

The morphism $\alpha : \Delta^n \rightarrow B$ represents a homotopy class in $\pi_n(B, x)$.

Once we have the existence of the boundary map, we can check exactness of the sequence by just checking exactness at the points, where it is just the usual long exact homotopy sequence for a fibration of simplicial sets [GJ99, Lemma 7.3].

PROPOSITION 2.1.11 *Let $p : E \rightarrow B$ be a fibre sequence of simplicial presheaves. Then the following sequence is an exact sequence of sheaves:*

$$\begin{aligned} \cdots \rightarrow \pi_n(F, y) &\xrightarrow{i_*} \pi_n(E, y) \xrightarrow{p_*} \pi_n(B, x) \xrightarrow{\partial} \pi_{n-1}(F, y) \rightarrow \cdots \\ &\cdots \rightarrow \pi_0(F) \xrightarrow{i_*} \pi_0(E) \xrightarrow{p_*} \pi_0(B) \end{aligned}$$

For $n \geq 1$ it is an exact sequence of sheaves of groups, and for the π_0 -part it is an exact sequence of sheaves of sets, in the sense that at the points the image of a map equals the kernel of the following map.

Although the above formulation is for sheaves of homotopy groups, a similar version can be formulated for sheaves of homotopy groups fibred over the sheaf of 0-simplices.

Functors between Model Categories: Functors between model categories which are compatible with the model structures are called Quillen functors. Note that these functors only preserve one part of the model structure, which is why we distinguish between left resp. right Quillen functors.

DEFINITION 2.1.12 *Let $\mathcal{C}$ and $\mathcal{D}$ be two model categories. A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is called left Quillen functor if F is a left adjoint functor, and preserves cofibrations and trivial cofibrations. Dually, a functor $G : \mathcal{D} \rightarrow \mathcal{C}$ is called a right Quillen functor if G is a right adjoint functor, and preserves fibrations and trivial fibrations. A Quillen adjunction between $\mathcal{C}$ and $\mathcal{D}$ is an adjoint pair of functors*

$$F : \mathcal{C} \rightleftarrows \mathcal{D} : G,$$

where F is a left Quillen functor and G is a right Quillen functor.

A Quillen adjunction

$$F : \mathcal{C} \rightleftarrows \mathcal{D} : G$$

is called a Quillen equivalence if for all cofibrant X in $\mathcal{C}$ and all fibrant Y in $\mathcal{D}$ a morphism $f : FX \rightarrow Y$ is a weak equivalence in $\mathcal{D}$ iff its adjoint $g : X \rightarrow GY$ is a weak equivalence in $\mathcal{C}$.

Quillen functors are the homotopical analogues of left resp. right exact functors: It is possible to define derived functors on the level of the homotopy categories. Quillen equivalences are pairs of functors that induce equivalences of the homotopy categories. Note however that the existence of the functors on the level of the model categories is much stronger than just a simple categorical equivalence of homotopy categories, all the higher homotopy structure present in the model category is preserved by a Quillen equivalence.

Let $\mathcal{C}$ be a simplicial model category, let $\mathcal{A}$ be any category and let $F : \mathcal{C} \rightarrow \mathcal{A}$ be a functor which takes weak equivalences between cofibrant objects to isomorphisms. Then we can define a *total left derived functor* $\mathbf{L}F : \mathrm{Ho} \mathcal{C} \rightarrow \mathcal{A}$ by setting $\mathbf{L}F(X) = F(QX)$ for QX a cofibrant replacement of X . Dually we can define total right derived functors $\mathbf{R}F : \mathrm{Ho} \mathcal{C} \rightarrow \mathcal{A}$ for any functor F which takes weak equivalences between fibrant objects to isomorphisms by setting $\mathbf{R}F(X) = F(RX)$ for a fibrant replacement RX of X .

THEOREM 2.1.13 (TOTAL DERIVED FUNCTOR THEOREM) *Let*

$$F : \mathcal{C} \rightleftarrows \mathcal{D} : G$$

be a Quillen adjunction. Then there is an adjunction between the total derived functors on the level of the homotopy categories:

$$\mathbf{L}F : \mathrm{Ho} \mathcal{C} \rightleftarrows \mathrm{Ho} \mathcal{D} : \mathbf{R}G.$$

If the above Quillen adjunction (F, G) is a Quillen equivalence, the total derived functors form an adjoint equivalence as well.

One natural adjunction to apply this theorem to is the hom-tensor adjunction, which yields adjunctions between the derived product and the derived hom-spaces. Another application of the total derived functor theorem is the abstract definition of homotopy limits and colimits, which will be explained in Section 3.1.

EXAMPLE 2.1.14 A natural example of a Quillen equivalence is the equivalence between the model category $\Delta^{\text{op}}\mathcal{S}hv(T)$ of simplicial sheaves on a site T and the model category $\Delta^{\text{op}}\mathcal{S}hv(T)_*$ of *pointed* simplicial sheaves on T . The corresponding adjoint functors are $U : \Delta^{\text{op}}\mathcal{S}hv(T)_* \rightarrow \Delta^{\text{op}}\mathcal{S}hv(T)$, which forgets the base point, and $F : \Delta^{\text{op}}\mathcal{S}hv(T) \rightarrow \Delta^{\text{op}}\mathcal{S}hv(T)_* : X \mapsto X_+$, which adds a disjoint base point. $\square$

Size Matters: In this paragraph we recall some of the smallness conditions one can impose on model categories. These are homotopical generalizations of categorical properties like local presentability and describe how good homotopy types can be approximated by “small” homotopy types. Such smallness properties will be particularly important for canonical decompositions of objects and the Brown representability theorem which we will use.

A cardinal λ is called *regular* if for any A with $|A| < \lambda$ and any A -indexed collection $(B_a)_{a \in A}$ with $|B_a| < \lambda$ for all $a \in A$, we have

$$\left| \bigcup_{a \in A} B_a \right| < \lambda.$$

We consider a cocomplete category $\mathcal{C}$. For an ordinal λ , a λ -sequence in $\mathcal{C}$ is a colimit-preserving diagram $X : \lambda \rightarrow \mathcal{C}$, using the usual interpretation of ordered sets as categories. Colimit-preserving means that for any limit ordinal $\gamma < \lambda$ the induced morphism

$$\text{colim}_{\beta < \gamma} X(\beta) \rightarrow X(\gamma)$$

is an isomorphism. The morphism $X(0) \rightarrow \text{colim}_{\beta < \lambda} X(\beta)$ is called the *transfinite composition* of the λ -sequence X .

For a cardinal γ , an ordinal α is called γ -*filtered* if it is a limit ordinal and for any $A \subseteq \alpha$ with $|A| \leq \gamma$ we have $\sup A < \alpha$.

DEFINITION 2.1.15 (SMALL OBJECTS) *Let $\mathcal{C}$ be a cocomplete category and let $\mathcal{D}$ be a subcategory of $\mathcal{C}$. If κ is a cardinal, an object $W \in \mathcal{C}$ is κ -small relative to $\mathcal{D}$ if for every regular cardinal $\lambda \geq \kappa$ and every λ -sequence $X : \lambda \rightarrow \mathcal{D}$ the canonical morphism is an isomorphism:*

$$\text{colim}_{\beta < \lambda} \text{Hom}_{\mathcal{C}}(W, X(\beta)) \xrightarrow{\cong} \text{Hom}_{\mathcal{C}}(W, \text{colim}_{\beta < \lambda} X(\beta)).$$

The property of being cofibrantly generated for a model category states that all the trivial fibrations and cofibrations are already determined by a sub-set of cofibrations, and a similar statement holds for the fibrations and trivial cofibrations as well. We will introduce the necessary terminology in the next definition. For more information on the material see [Hir03, Section 10.5].

DEFINITION 2.1.16 *Let I be a set of maps of a category $\mathcal{C}$.*

The subcategory of I -injectives is the subcategory of morphisms having the right lifting properties w.r.t. every element of I .

The subcategory of I -cofibrations is the subcategory of maps that have the left lifting property w.r.t. every I -injective.

The subcategory of relative I -cell complex is the subcategory of those morphisms which can be constructed as transfinite compositions of pushouts of elements of I .

For a category $\mathcal{C}$, a set I of maps of $\mathcal{C}$ permits the small object argument if the domains of the elements of I are small relative to I .

The small object argument as described e.g. in [Hir03, Proposition 10.5.16] states that any map can be factored into a relative I -cell complex and an I -injective map.

DEFINITION 2.1.17 *A cofibrantly generated model category is a model category $\mathcal{C}$ such that*

- (i) *there exists a set I of generating cofibrations that permits the small object argument and such that a map is a trivial fibration iff it has the right lifting property w.r.t. every element of I , and*
- (ii) *there exists a set J of generating trivial cofibrations that permits the small object argument and such that a map is a fibration iff it has the right lifting property w.r.t. every element of J .*

Combinatoriality of a model category is a slightly stronger property than being cofibrantly generated, requiring also that the underlying category be locally presentable. For more informations on such smallness properties of categories, see [Bor94, Chapter 5].

DEFINITION 2.1.18 *A category $\mathcal{C}$ is called locally presentable if it has all small colimits and there exists a regular cardinal λ and a set of objects $\mathcal{A} \subseteq \text{Ob}(\mathcal{C})$ such that*

- (i) *every object in $\mathcal{A}$ is small w.r.t. λ -filtered colimits, and*
- (ii) *every object of $\mathcal{C}$ can be expressed as a λ -filtered colimit of elements of $\mathcal{A}$.*

A model category is called combinatorial if it is cofibrantly generated and the underlying category is locally presentable.

In other words, a locally presentable category $\mathcal{C}$ has all small colimits and a dense subcategory $\mathcal{D} \hookrightarrow \mathcal{C}$ which is small. This means that any object of $\mathcal{C}$ is a small colimit of objects from $\mathcal{D}$. Therefore to understand any object in the category $\mathcal{C}$, one only needs to look at objects that are small compared to some large enough regular cardinal λ . The corresponding model category notion of combinatoriality is the homotopy analogue: To understand any homotopy type in the model category $\mathcal{C}$, one only needs homotopy types that are small w.r.t. the regular cardinal λ , and there is only a set of those. This is a statement that holds for all model categories of simplicial sheaves on a Grothendieck site, and it will imply the existence of rather canonical homotopy colimit decompositions in those. This will be discussed in Section 3.1.

Obviously we only introduced all those model category properties because we are going to use them and because they do hold for model categories of simplicial sheaves. Here's the well-known result:

PROPOSITION 2.1.19 *Let T be any Grothendieck site. Then the injective model structure of Jardine on the category of presheaves of simplicial sets on T is a proper simplicial and cofibrantly generated model structure.*

PROOF: The favourite references for the proofs are [Jar87, Theorem 2.3, Corollary 2.7] and [Jar96, Theorem 18]. Properness and simpliciality are proven in [Jar96, Theorem 24]. The fact that the model categories are cofibrantly generated is implicit in Jardine's proofs, though not explicitly stated. The combinatoriality follows since categories of sheaves on a Grothendieck site are locally presentable [Bor94]. ■

REMARK 2.1.20 *We give some more details on cofibrant generation in model categories of simplicial presheaves. The key step is [Jar87, Lemma 2.4].*

Let T be a Grothendieck site, and assume for simplicity it is small, not just essentially small. Then there exists a regular cardinal α greater than the set of all subsets of the set of morphisms of T . A simplicial presheaf is called α -bounded if the cardinality of $X_n(U)$ is smaller than α for all $U \in T$ and $n \geq 0$. Lemma 2.4 in [Jar87] then states that to check if $p : E \rightarrow B$ is a global fibration it suffices to check if it is I -injective with I the set of cofibrations $i : A \rightarrow B$ with α -bounded codomain B . A modification of the proof of Lemma 2.4 shows the same for cofibrations and trivial global fibrations. This implies that the corresponding model category is cofibrantly generated.

Note also that the above is stronger: There is not just a bound on the size of the domain of the generating (trivial) cofibrations, but also on the codomains. The Jardine model structure does not seem to be cellular, however.

REMARK 2.1.21 *Note that such smallness conditions depend strongly on the choice of the class of cofibrations. This implies that properties like being cofibrantly generated are not necessarily preserved under Quillen equivalences of model categories.*

In particular, choosing a smaller class of cofibrations like in the projective model structures of [Bla01, Dug01b], it is possible to obtain even stronger smallness properties like cellularity.

Cellularity and its consequences are explained in [Hir03, Chapter 12], and the proof that the projective model structure on a category of simplicial sheaves is proper, simplicial and cellular is given in [Bla01, Theorem 1.5, Theorem 2.1].

This implies that a result slightly stronger than Proposition 2.1.19 can be obtained for the projective model structures on simplicial presheaves.

Compactness in Model Categories: There is yet another size condition for model categories which we need for Jardine's version of Brown representability [Jar06c] to work.

DEFINITION 2.1.22 *A model category $\mathcal{C}$ is called compactly generated, if there is a set of compact cofibrant objects $\{K_i\}$ such that a map $f : X \rightarrow Y$ is a weak equivalence iff it induces a bijection*

$$[K_i, X] \xrightarrow{\cong} [K_i, Y]$$

for all objects K_i in the generating set.

This is a size condition that does not hold for all model categories of simplicial sheaves. In the next section we will discuss the Grothendieck sites we will study in this thesis, and prove that the model categories of sheaves on these sites are compactly generated in the above sense.

2.2 Sites for Motivic Homotopy Theory

In this section, we will describe the Grothendieck topologies that are most relevant for this thesis, namely the Zariski, Nisnevich and cdh-topologies on the categories of smooth schemes over a base scheme S . They are all given by cd-structures in the sense of Voevodsky, and they allow to prove some strong descent and compactness theorems. The compactness statements are needed to apply Jardine's Brown representability result, and the descent results can be used to check if a simplicial sheaf is fibrant.

Completely Decomposed Topologies: We first recall the definition of a cd-structure due to Voevodsky [Voe00a].

DEFINITION 2.2.1 *Let $\mathcal{C}$ be a small category. A cd-structure on $\mathcal{C}$ is a class P of commutative squares in $\mathcal{C}$, which is closed under isomorphisms. The elements of P are called distinguished squares.*

For such a cd-structure, the associated Grothendieck topology t_P on $\mathcal{C}$ is defined to be the smallest Grothendieck topology for which the distinguished squares are coverings.

DEFINITION 2.2.2 *Let P be a cd-structure on a category $\mathcal{C}$.*

The cd-structure P is called complete if any covering sieve of an object $\emptyset \neq X \in \mathcal{C}$ contains a sieve generated by a simple covering. The class of simple coverings is the smallest class of covering families $U_i \rightarrow U$ containing isomorphisms, such that for any distinguished square

$$\begin{array}{ccc} B & \xrightarrow{f} & Y \\ \downarrow & & \downarrow p \\ A & \xrightarrow{e} & X \end{array}$$

and simple coverings $q_j : A_j \rightarrow A$ and $p_i : Y_i \rightarrow Y$ the covering family $\{e \circ q_j : A_j \rightarrow A \rightarrow X, p \circ p_i : Y_i \rightarrow Y \rightarrow X\}$ is a simple covering of X .

The cd-structure P is called regular if any distinguished square as above is a pullback square with $e : A \rightarrow X$ a monomorphism such that the following morphism of the represented sheaves is surjective:

$$\Delta \coprod (f \times f) : Y \coprod (B \times_A B) \rightarrow Y \times_X Y.$$

REMARK 2.2.3 *There is also the notion of a bounded cd-structure: For any cd-structure one can define the dimension of objects in a category with cd-structure, and boundedness describes cd-structures such that the objects are locally finite-dimensional. The only facts we will use about these properties is however that the*

standard cd-structures discussed below are bounded, complete and regular. We will therefore not discuss the precise definition of bounded cd-structure which is a bit lengthy. For details see [Voe00a, Section 2].

The following examples describe the cd-structures and Grothendieck topologies that are most interesting for $\mathbb{A}^1$ -homotopy.

EXAMPLE 2.2.4 (ZARISKI TOPOLOGY) The distinguished squares for the Zariski topology are of the form

$$\begin{array}{ccc} U \times_X V & \longrightarrow & V \\ \downarrow & & \downarrow i \\ U & \xrightarrow{j} & X, \end{array}$$

where i and j are open immersions. The corresponding cd-structure is called the *plain upper cd-structure*. $\square$

EXAMPLE 2.2.5 (NISNEVICH TOPOLOGY) Recall from [MV99, Definition 3.1.3], that an *elementary distinguished square* of the *upper cd-structure* in the category of smooth schemes over S is a cartesian square

$$\begin{array}{ccc} U \times_X V & \longrightarrow & V \\ \downarrow & & \downarrow p \\ U & \xrightarrow{j} & X, \end{array}$$

such that p is étale, j is an open immersion and $p^{-1}(X \setminus U) \rightarrow X \setminus U$ is an isomorphism for the reduced induced structures. The Nisnevich topology is the Grothendieck topology generated by the upper cd-structure. $\square$

EXAMPLE 2.2.6 (CDH- RESP. SCDH-TOPOLOGY) The cdh-topology is the smallest Grothendieck topology on Sch_S generated by the *combined cd-structure*, whose distinguished squares are the distinguished squares for the Nisnevich topology and squares of the form

$$\begin{array}{ccc} Y \times_X X' & \longrightarrow & X' \\ \downarrow & & \downarrow \pi \\ Y & \xrightarrow{i} & X, \end{array}$$

where i is a closed immersion and π is a proper morphisms such that $\pi^{-1}(X \setminus Y) \rightarrow X \setminus Y$ is an isomorphism. Such squares are called *abstract blow-up squares*.

The scdh-topology is the smallest Grothendieck topology on Sm_S generated by the distinguished squares for the Nisnevich topology and abstract blow-up squares isomorphic to blow-ups of smooth schemes along smooth centres. The scdh-topology is actually the restriction of the cdh-topology to the category Sm_S of smooth schemes. $\square$

PROPOSITION 2.2.7 *The above cd-structures on Sch_S are complete, bounded and regular. On Sm_S , the upper and plain upper cd-structure are bounded, complete and regular for any base. The combined cd-structure is bounded, complete and regular if S is a field admitting resolution of singularities.*

PROOF: The statement is proven in various Lemmas in [Voe00b]. Lemma 2.2 shows that the standard cd-structures are complete on Sch_S for any base, and that the upper and plain upper cd-structures are complete on Sm_S . Lemma 4.3 shows that the combined cd-structure is complete for Sm_S if S is a field admitting resolution of singularities.

Lemma 2.13 and Lemma 4.5 show that the standard cd-structures on Sch_S and Sm_S are regular, and Proposition 2.12 and Lemma 4.4 show boundedness of the standard cd-structures on Sch_S and Sm_S . $\blacksquare$

Descent and Compactness: In general it is rather complicated to check if a given simplicial presheaf X is fibrant. Even checking if X is quasi-fibrant, i.e. if the homotopy groups of the sections $\pi_n X(U)$ for $U \in T$ are really the homotopy groups of a fibrant replacement of X is difficult. For the Grothendieck topologies induced by cd-structures on schemes, there is however an easier way to do this. The Brown-Gersten property is a good criterion to check if simplicial sheaves are quasi-fibrant. It was first used in the work of Brown and Gersten [BG73] to show that the K-theory presheaf is quasi-fibrant for the Zariski-topology.

DEFINITION 2.2.8 (BROWN-GERSTEN PROPERTY) *Let $\mathcal{C}$ be a category with a cd-structure P .*

A simplicial presheaf $\mathcal{F}$ has the Brown-Gersten property for P if each distinguished square gives rise to a homotopy cartesian square of simplicial sets:

$$\begin{array}{ccc} \mathcal{F}(X) & \longrightarrow & \mathcal{F}(V) \\ \downarrow & & \downarrow \\ \mathcal{F}(U) & \longrightarrow & \mathcal{F}(W), \end{array}$$

The importance of the Brown-Gersten property lies in the fact that for a fibrant replacement $\tilde{\mathcal{F}}$ of $\mathcal{F}$, the induced morphisms $\mathcal{F}(U) \rightarrow \tilde{\mathcal{F}}(U)$ are weak equivalences. This allows direct computation of some homotopy groups without the ado of fibrant replacement. The following important descent theorem from [CHSW06, Theorem 3.4] makes precise the above remarks.

THEOREM 2.2.9 *Let $\mathcal{C}$ be a category with a bounded, complete and regular cd-structure P . Then a presheaf of simplicial sets is quasi-fibrant iff it satisfies the Brown-Gersten property for P .*

Recall that a simplicial presheaf $\mathcal{F}$ on T is called *quasi-fibrant* if the fibrant replacement $\tilde{\mathcal{F}}$ in the injective model structure induces weak equivalences on all spaces of sections, i.e. $\mathcal{F}(U) \rightarrow \tilde{\mathcal{F}}(U)$ is a weak equivalence for any $U \in T$.

This generalizes earlier descent-type results for the Zariski topology [BG73, Theorem 4] resp. the Nisnevich topology [MV99, Proposition 3.1.16].

Using the above descent result, it is possible to prove that the injective model structures on a category of simplicial sheaves for a site with a cd-structure is compactly generated:

PROPOSITION 2.2.10 *Let T be a site with a topology coming from a complete, bounded and regular cd-structure. Then the model category $\Delta^{\text{op}}\mathcal{Shv}(T)$ with the injective model structure is compactly generated in the sense of Definition 2.1.22.*

In particular, this holds for the site $T = \text{Sm}_S$ of all smooth schemes over a finite type scheme S with either the Zariski or Nisnevich. Similarly, it holds for the cdh-topology on the category Sch_S .

PROOF: It is clear that for any inductive system

$$Y_0 \rightarrow Y_1 \rightarrow Y_2 \rightarrow \cdots$$

of globally fibrant simplicial presheaves on a cd-topology, the colimit $\text{colim}_i Y_i$ has the Brown-Gersten property, and therefore $\text{colim}_i Y_i$ is quasi-fibrant. Using the same argument as given in [Jar06c, Section 3] for the examples (1) and (2) given there, it follows that the collection of $K \otimes U_+$ with K a finite simplicial set and $U \in T$ contains a set of compact generators of $\Delta^{\text{op}}\mathcal{Shv}(T)$. Recall that $\otimes$ denoted one of the adjoint functors of the simplicial structure on $\Delta^{\text{op}}\mathcal{Shv}(T)$. ■

REMARK 2.2.11 *We note, that from Voevodsky's work on completely decomposed topologies it follows that his Brown-Gersten-type model structure as well as the local projective model structure of Blander also are compactly generated. See [Voe00a, Hov01b].*

2.3 Remarks

This section contains some paragraphs dealing with simple technicalities that are used all over the place. The first paragraph discusses why a sheaf of constant simplicial sets is fibrant, the second discusses consequences of properness, and finally we discuss different notions of connectedness of simplicial presheaves.

Fibrancy of Constant Simplicial Sheaves: We first discuss the homotopy types of representable objects. These are rather simple:

LEMMA 2.3.1 *For any Grothendieck site T , a presheaf of constant simplicial sets is fibrant iff it is a sheaf.*

We give several ways of seeing this: Let X be a presheaf of constant simplicial sets.

The first way is to look at the stalks of a presheaf of constant simplicial sets. For any point p of $\mathcal{Shv}(T)$, $p^*(X)$ is a directed colimit of constant simplicial sets, therefore is a constant simplicial set. The sheaves of homotopy groups therefore have trivial stalks, so they have to be trivial. There is no higher homotopy and the sheaf of connected components is the sheafification of X .

If the topology on T comes from a cd-structure, one can also use the characterization of quasi-fibrant sets via the Brown-Gersten type properties. A morphism of

constant simplicial sets is a fibration, since the lifting properties begin in degree 2 with $\Lambda_i^2 \hookrightarrow \Delta^2$, and all such simplices are degenerate. Thus a presheaf of constant simplicial sets satisfies the Brown-Gersten property iff it is a sheaf, because all the morphisms are fibrations, so the distinguished square is a homotopy pullback iff it is a pullback of sets.

Finally, one can use the characterization of fibrancy using hypercover descent from [DHI04]. Any simplicial presheaf is objectwise fibrant because a constant simplicial set is fibrant. Moreover, for simplicial dimension zero, hypercover descent is the same as Čech descent [DHI04, Corollary A.9], and the latter reduces to the sheaf axiom for simplicial dimension zero.

This implies that a sheaf of constant simplicial sets is $\mathbb{A}^1$ -local as soon as it is $\mathbb{A}^1$ -invariant. The $\mathbb{A}^1$ -invariant schemes are therefore cofibrant and fibrant in the $\mathbb{A}^1$ -local model structure, so morphisms between them in the homotopy category are just regular morphisms.

The proper topological analogue of this is the rigidity of hyperbolic manifolds: One way to define hyperbolic manifolds is via non-existence of holomorphic maps $\mathbb{C} \rightarrow X$. This is basically $\mathbb{A}^1$ -rigidity. The famous rigidity theorems of Mostow and others show that any two hyperbolic manifolds which are homotopy equivalent are already homeomorphic. Therefore the functor from the category of hyperbolic manifolds to the homotopy category of topological spaces is a full embedding. However, although there is a formal analogy between these two settings, the statement in the $\mathbb{A}^1$ -world is rather easy to obtain, since the $\mathbb{A}^1$ -local homotopy types are basically simplicial sheaves of simplicial dimension zero, the sections over a smooth scheme U being exactly the morphism $U \rightarrow X$. Therefore we see that $\mathbb{A}^1$ -homotopy is not very interesting for hyperbolic or anabelian varieties.

Although the simplicial homotopy of representable objects is rather simple, there are some situations where the simplicial homotopy is already interesting *before* localizing $\mathbb{A}^1$.

As a simple example, we can consider pointed homotopy colimits of the $\mathbb{A}^1$ -rigid variety $\mathbb{G}_m$, like the suspension $\Sigma\mathbb{G}_m$. This is not obviously $\mathbb{A}^1$ -local and it is also no longer of simplicial dimension zero. The value at a henselian local ring $\mathcal{O}_{X,x}^h$ is given by $\Sigma(\mathcal{O}_{X,x}^h)^*$, with $G = (\mathcal{O}_{X,x}^h)^*$ viewed as a group. This has lots of interesting higher homotopy, coming from the wedge $BG \vee BG$. It is easy to imagine that one can build arbitrarily complicated simplicial sheaves by just taking such pointed homotopy colimits. We will see that some of these constructions can be identified with concrete smooth schemes in the $\mathbb{A}^1$ -local model structure. As an example $\Sigma\mathbb{G}_m \simeq \mathbb{P}^1$.

Another interesting situation is the case of simplicial schemes, as long as they are not hypercoverings of representable things: By the hypercover descent characterization of model structures on simplicial presheaves in [DHI04], a hypercover $V_\bullet \rightarrow U$ of a representable U is a weak equivalence. One example is to take the Nisnevich topology and consider the simplicial scheme coming from de Jong's theorem applied to any singular variety. It is not clear if this is homotopy equivalent to the corresponding singular variety, but still it seems an interesting object of study.

On the Non-fibrant Base Problem: We found it necessary to include the following statement on fibre sequences in proper model categories, as it is not clear whether any fibration in an arbitrary model category gives rise to a fibre sequence in the sense of Definition 2.1.9.

We first repeat one important consequence of right properness, namely the cogluing lemma [GJ99, Corollary II.8.13]:

LEMMA 2.3.2 (COGLUING LEMMA) *Suppose that $\mathcal{C}$ is a proper model category. Consider a diagram*

$$\begin{array}{ccccc} X_1 & \longrightarrow & Y_1 & \xleftarrow{p_1} & Z_1 \\ \simeq \downarrow & & \simeq \downarrow & & \simeq \downarrow \\ X_2 & \longrightarrow & Y_2 & \xleftarrow{p_2} & Z_2, \end{array}$$

where the maps p_1 and p_2 are fibrations and the three vertical maps are weak equivalences. Then the induced map $X_1 \times_{Y_1} Z_1 \rightarrow X_2 \times_{Y_2} Z_2$ is a weak equivalence.

Note that this in particular implies that products of weak equivalences of fibrant objects are again weak equivalences in any proper model category.

PROPOSITION 2.3.3 *Let $\mathcal{C}$ be a proper pointed model category in which all objects are cofibrant. Denoting $F = p^{-1}(*)$, the sequence $F \rightarrow E \xrightarrow{p} B$ is a fibre sequence for any fibration $p : E \rightarrow B$, in particular for B not fibrant.*

PROOF: We plan to apply the cogluing lemma to the following diagram:

$$\begin{array}{ccccc} * & \longrightarrow & B & \xleftarrow{p} & E \\ \downarrow & & \simeq \downarrow g & & \simeq \downarrow h \\ * & \longrightarrow & \tilde{B} & \xleftarrow{\tilde{p}} & \tilde{E} \end{array}$$

In this diagram, g is a fibrant replacement of B . Then we obtain the right-hand square by factoring $g \circ p$ as trivial cofibration h followed by a fibration $\tilde{p}$.

By the cogluing lemma, we obtain that the canonical map $k : F \rightarrow \tilde{F}$ between the pullbacks of the two relevant diagrams above is a weak equivalence. Moreover, since p and $\tilde{p}$ are fibrations, the spaces F and $\tilde{F}$ are weakly equivalent to the corresponding homotopy fibres. Both F and $\tilde{F}$ are fibrant, since fibrations are stable under pullbacks.

We equip F with the action given by as follows:

$$\Omega B \times F \xrightarrow[\Omega g \times k]{\simeq} \Omega \tilde{B} \times \tilde{F} \longrightarrow \tilde{F} \xrightarrow[k^{-1}]{\simeq} F$$

The first map is a product of weak equivalences, the second is the action of $\Omega \tilde{B}$ on $\tilde{F}$ as given by Definition 2.1.8. As noted above, both F and $\tilde{F}$ are cofibrant and fibrant, so k has a homotopy inverse k^{-1} by the Whitehead theorem [GJ99, Theorem II.1.10].

This is well defined in $\text{Ho}\mathcal{C}$ and by definition k is Ωg -equivariant. Therefore, we get the following commutative diagram in $\text{Ho}\mathcal{C}$, in which the vertical arrows are isomorphisms and k is Ωg -equivariant.

$$\begin{array}{ccccc}
F & \longrightarrow & E & \xrightarrow{p} & B \\
\downarrow k \cong & & \downarrow h \cong & & \downarrow g \cong \\
\tilde{F} & \longrightarrow & \tilde{E} & \xrightarrow{\tilde{p}} & \tilde{B}
\end{array}$$

Thus $F \rightarrow E \rightarrow B$ is a fibre sequence. ■

REMARK 2.3.4 (i) *More generally it follows that for any morphism $f : X \rightarrow Y$, the sequence $\mathrm{hofib}(f) \rightarrow X \rightarrow Y$ is a fibre sequence. We see that the theory of fibre sequences and homotopy pullbacks is easier in proper model categories than in general model categories.*

(ii) *Note that the condition that all objects be cofibrant is satisfied for the Jardine model structure on $\Delta^{\mathrm{op}}\mathrm{Shv}(T)$. It is however not actually essential, since the definition of the action of ΩB on the fibre F of a fibration $p : E \rightarrow B$ in Definition 2.1.8 still works if we replace F by a cofibrant approximation $\tilde{F}$. Therefore $\tilde{F} \rightarrow E \rightarrow B$ is also a fibre sequence. This argument can be used to drop the assumption that all objects are cofibrant.*

On Connectivity Assumptions: In the thesis, we generalize several results on fibrations and localizations for simplicial sets to the setting of simplicial presheaves. Some of these results, such as the classification of fibre sequences, several results from [Dro96], or the characterization of nullifications in [BF03] have connectivity assumptions in their statements. Such connectivity assumptions can not be generalized verbatim to sheaves. One of the reasons is that the definition of connectedness via π_0 does not apply very well to simplicial presheaves. Another reason is that there are several characterizations of connectedness which agree for simplicial sets, but differ for simplicial presheaves. In the following, we will define and compare several notions of connectedness which are used in the thesis.

The most obvious way to define connectedness is the one used e.g. in [MV99, Corollary 2.3.22].

DEFINITION 2.3.5 *Let X be a simplicial presheaf on a Grothendieck site T . We say that X is π_0 -connected if $L^2\pi_0X = 0$, where L^2 denotes again sheafification. In other words, for any point x of the topos $\mathrm{Shv}(T)$, we require that the simplicial set $x^*(X)$ is connected.*

There are many simplicial presheaves, in particular the zero-dimensional ones, which are not π_0 -connected in the above sense. For the classification of fibre sequences, it is necessary to know that all the fibres have the same homotopy type, which is the reason for the restriction to connected base spaces, cf. [All66].

One way to lift this connectedness assumption, which we will use in Section 3.2 for the construction of classifying spaces, is the restriction to locally trivial fibre sequences. This basically has the same effect as requiring that the fibres over all the connected components have the same homotopy type.

A morphism $p : X \rightarrow Y$ of schemes is called *locally trivial* in the topology T if there exists a T -covering U_i of Y and a scheme F such that $U_i \times F \cong U_i \times_Y X$. It is called *locally trivial with fibre F* if there exists a morphism $F \rightarrow X$ inducing the

above isomorphisms. This implies that for any point x of the topos $\mathcal{S}hv(T)$, the fibres of $x^*(p)$ over the various components of $x^*(B)$ will be isomorphic.

We can generalize this to fibrations as follows: For a fibration $p : E \rightarrow B$ with fibre F , and a point x of the topos, we can look at $x^*(p)$. The homotopy fibre $\text{hofib } x^*(p)$ is weakly equivalent to $x^*(\text{hofib } p)$, [MV99, Lemma 2.1.20]. Then we say that a fibration $p : E \rightarrow B$ of simplicial sheaves on a site T is *locally trivial* if for any point x of the topos $\mathcal{S}hv(T)$, any two connected components of $x^*(B)$ have weakly equivalent fibres. This is the same as requiring that for any point x and any morphism $x \rightarrow B$ we have a splitting $x \times_B E \simeq x \times F$, which in turn is equivalent to the assertion that for every object $U \in T$ and morphism $U \rightarrow B$ there exists a covering $U_i \rightarrow U$ such that $U_i \times_B E \simeq U_i \times F$.

Note that a fibre sequence in $\Delta^{\text{op}}P\mathcal{S}hv(T)$ need not be locally trivial w.r.t. the topology of T . It however is obvious that over a π_0 -connected simplicial presheaf B , every fibration $p : E \rightarrow B$ is locally trivial. There is yet another notion of connectedness which is defined exactly via the latter statement, which was defined in [CS06]:

DEFINITION 2.3.6 *An object B of a model category $\mathcal{C}$ is called path-connected if any morphism $f : E \rightarrow B$ is a weak equivalence iff its homotopy fibre $\text{hofib}(p)$ is trivial.*

This definition works in arbitrary model categories, and is weaker than π_0 -connectedness, as the following example shows:

EXAMPLE 2.3.7 The constant simplicial sheaf $\mathbb{G}_m$ on Sm_S is path-connected:

We consider a point x of the Zariski topos, which is a local ring R over the base S . Let two morphisms $f, g : \text{Spec } R \rightarrow \mathbb{G}_m$ be given. These are units of R , hence there exists an isomorphism $fg^{-1} : R \rightarrow R$ making the obvious diagram commute. The morphisms f and g represent two different *non-homotopic* elements of $x^*(\mathbb{G}_m)$ but due to the existence of the above isomorphism, we will always get induced weak equivalences on the fibres of a fibration over $\mathbb{G}_m$:

$$\begin{array}{ccccc} F_g & \xrightarrow{\simeq} & F_f & \longrightarrow & E \\ \downarrow & & \downarrow & & \downarrow p \\ \text{Spec } R & \xrightarrow{\simeq} & \text{Spec } R & \longrightarrow & \mathbb{G}_m \end{array}$$

The above isomorphisms are “homotopies” connecting different units of the local ring R which are however not visible at the level of sheaf homotopy. Their existence can only be seen from the fact that any fibration over $\mathbb{G}_m$ is locally trivial. $\square$

Therefore π_0 -connectedness implies path-connectedness, but the implication is strict. Path-connectedness is usually used in the following implication: Let $p : E \rightarrow B$ be a morphism, with B path-connected, whose homotopy fibre is trivial. Then p is a weak equivalence. With a little more control on the other components, which do not contain the base point, one can lift the restriction of path-connectedness.

Finally there is also a cohomotopy notion of connectedness. Whereas the π_0 -version of connectedness requires that all morphisms $S^0 \rightarrow X$ are homotopic, the dual version requires that all morphisms $X \rightarrow S^0$ are homotopic.

DEFINITION 2.3.8 *Let X be a pointed simplicial presheaf on a Grothendieck site T . We say that X is π^0 -connected if all pointed morphisms $X \rightarrow S^0$ are homotopic. Let $f : X \rightarrow Y$ be a morphism of simplicial presheaves in $\Delta^{\text{op}}PShv(T)$. We say this morphism is π^0 -connected iff for every representable $U \in T$, we have a bijection of morphisms in $\mathcal{C} = \text{Ho}(\Delta^{\text{op}}PShv(T))$:*

$$\text{Hom}_{\mathcal{C}}(U \times Y, S^0) \rightarrow \text{Hom}_{\mathcal{C}}(U \times X, S^0).$$

Note that π^0 -connectedness of a morphism f implies that S^0 is f -local, by definition of f -locality. As we have seen earlier in this section, a sheaf of simplicial sets of dimension zero is fibrant in the simplicial model structure. Since S^0 is a constant *sheaf* and by π^0 -connectedness is also f -local, it is a fibrant object in the f -local model structure.

Finally, we find that π^0 -connectedness is the weakest of the various notions defined: The morphism $* \rightarrow \mathbb{A}^1$ in $\Delta^{\text{op}}PShv(\text{Sm}_S)$ is connected in the above sense, and therefore S^0 is $\mathbb{A}^1$ -local: The projection $U \times \mathbb{A}^1 \rightarrow U$ induces a bijection on connected components, and therefore a bijection $S^0(U) \rightarrow S^0(U \times \mathbb{A}^1)$ on sets of sections of the constant simplicial sheaf S^0 .

We also want to note that one can define a notion of $\mathbb{A}^1$ -connectedness, cf. [Mor06a], but we will not have occasion to use this version of connectedness.

Chapter 3

Homotopy Colimits and Fibrations

This chapter deals with the homotopical theory of fibrations in categories of simplicial presheaves on a Grothendieck site: We discuss the importance of canonical homotopy colimit decompositions and homotopy distributivity. These two properties lie at the heart of most of the results proven in the thesis: Several of the important properties of the model category of simplicial sets can be generalized to categories of simplicial presheaves by using these two properties. We also show how to decompose a fibration of simplicial presheaves using the canonical homotopy colimit decomposition and homotopy distributivity. However, such a decomposition is not as powerful as the one for simplicial sets since it is not in general possible to reconstruct a simplicial presheaf as a homotopy colimit of *contractible* presheaves.

In Section 3.1, we will recall in detail the definition and basic properties of homotopy colimits and limits. We also discuss homotopy distributivity, which allows to interchange homotopy colimits and homotopy limits of simplicial presheaves on a Grothendieck site. Following this, we will present some homotopical applications of homotopy distributivity: In Section 3.2, we will construct classifying spaces for fibre sequences by simply plugging homotopy distributivity into the Brown representability theorem. This produces not only classifying spaces which are unique up to homotopy equivalence, but also universal fibre sequences for a given fibre. Such constructions are at the base of any interaction between topology and geometry, appearing in the classical and equivariant surgery and orientation theory. However, a potential surgery theory for schemes just serves as a motivation to consider classifying spaces for fibre sequences. The application we will deal with concretely is the construction of fibrewise localizations in Chapter 4. Moreover, if the universal fibre sequence is local, we immediately obtain classifying spaces for fibre sequences in the $\mathbb{A}^1$ -local model category. However, our results will also show that the classification of fibre sequences is in general not possible in the $\mathbb{A}^1$ -local model category.

As another application of homotopy distributivity, we will give a couple of examples of fibre sequences of simplicial presheaves in Section 3.3. Among them are torsors, i.e. principal fibre bundles which are locally trivial in the topology of the site T . It is also possible to prove Ganea's theorem for simplicial presheaves, which allows to get information on the homotopy type of the projective line $\mathbb{P}^1$.

Finally, the last two sections discuss further work directly related to homotopy distributivity. The one application we will discuss concerns cellular inequalities in Section 3.4. These can be seen as generalizations of fibre sequences: Whereas the three spaces in a fibre sequence are related by a long exact homotopy sequence, the link for cellular inequalities is somewhat weaker. A cellular inequality $X \ll Y$ means that the space Y can be built from copies of the space X via pointed homotopy colimits. Properties like connectivity or vanishing of homology of the space X imply properties of Y . For simplicial sets, these cellular inequalities make it possible to compare spaces related by homotopy pushouts. This provides general formulations of statements like the Blakers-Massey theorem. We will not prove anything in this section, we only discuss the problems that appear when generalizing from simplicial sets to simplicial presheaves.

The final application of homotopy distributivity is a local-global type Serre spectral sequence for simplicial presheaves in Section 3.5. The main step in proving such a spectral sequence is the decomposition of the fibre sequence over the canonical homotopy colimit presentation of the base space. Due to space and time restrictions, we only give a rough outline of the construction and describe the technical difficulties which have to be overcome in the proof of the Serre spectral sequence.

3.1 Homotopy Colimits of Simplicial Presheaves

This section discusses the results on simplicial sets collected in [Dro96, Appendix HL] in the context of simplicial presheaves. Most important for the rest of the thesis will be the notion of homotopy distributivity and the canonical homotopy colimit decomposition for an arbitrary simplicial presheaf.

Homotopy Limits and Colimits: In this paragraph we will repeat the definition, some basic and some advanced properties of homotopy limits and colimits. We will see that many of the properties of homotopy limits and colimits that do hold in the category of simplicial sets also hold in model categories of simplicial presheaves. This is in particular true for certain commutation rules. Since it is easier to define homotopy colimits and limits in simplicial model categories rather than in general model categories, we will always assume (and sometimes forget to mention) that the model categories we are dealing with are simplicial. This perfectly suffices for what we want to do.

We begin with the various possible definitions of homotopy colimits and limits. Homotopy colimits and limits are homotopy-invariant versions of the ordinary colimits and limits for categories. Abstractly, one can define the ordinary colimit of a diagram $\mathcal{X} : \mathcal{I} \rightarrow \mathcal{C}$ as left adjoint of the diagonal functor $\Delta_{\mathcal{I}} : \mathcal{C} \rightarrow \text{hom}(\mathcal{I}, \mathcal{C})$, where $\text{hom}(\mathcal{I}, \mathcal{C})$ is the category of $\mathcal{I}$ -diagrams in $\mathcal{C}$. Similarly, the ordinary limit is the right adjoint of the diagonal, cf. [Mac98, Section X.1].

The abstract definition of homotopy colimits and limits thus proceeds via homotopical algebra and Quillen's total derived functor theorem 2.1.13: The homotopy colimit is the left derived functor of the ordinary colimit functor. This is explained shortly in [GJ99, Example 7.11].

DEFINITION 3.1.1 (HOMOTOPY COLIMIT, ABSTRACT) *Let $\mathcal{C}$ be a cofibrantly generated simplicial model category, and $\mathcal{I}$ be any small category. Then the category $\text{hom}(\mathcal{I}, \mathcal{C})$ of functors from $\mathcal{I}$ to $\mathcal{C}$ becomes a simplicial model category. Since the diagonal $\Delta_{\mathcal{I}} : \mathcal{C} \rightarrow \text{hom}(\mathcal{I}, \mathcal{C})$ preserves fibrations and weak equivalences, the ordinary colimit functor preserves weak equivalences of cofibrant diagrams, and we can define the homotopy colimit*

$$\text{hocolim}_{\mathcal{I}} = \mathbf{L} \text{colim}_{\mathcal{I}}.$$

Dually, the diagonal functor also preserves cofibrations and weak equivalences, therefore it is possible to define a right derived functor of the ordinary limit, and we define the homotopy limit

$$\text{holim}_{\mathcal{I}} = \mathbf{R} \text{lim}_{\mathcal{I}}.$$

Homotopy colimits will usually be denoted by $\text{hocolim}_{\mathcal{I}}$, the only exception being the case of a pushout, which is denoted by $A \cup_B^h C$. Similarly, homotopy limits are usually denoted by $\text{holim}_{\mathcal{I}}$, and the homotopy pullbacks by $A \times_B^h C$.

There is also a concrete description of homotopy colimits and limits in simplicial model categories, derived from the original work of Bousfield and Kan [BK72] and discussed in detail in [Hir03, Chapter 18]. This can be seen as an analogue of the decomposition of a general colimit into direct sums and coequalizers [Mac98, Dual of Theorem V.2.2].

DEFINITION 3.1.2 (HOMOTOPY COLIMIT, CONCRETE) *For a diagram $\mathcal{X} : \mathcal{I} \rightarrow \mathcal{C}$ and a cosimplicial resolution $\Gamma : \mathcal{I} \rightarrow \Delta \mathcal{C}$ the homotopy colimit of $\mathcal{X}$ is defined to be*

$$\text{hocolim}_{\mathcal{I}} \mathcal{X} = \text{coker} \left[\coprod_{n \rightarrow m} \Gamma(n) \otimes_{\Delta} B(m \downarrow \mathcal{I})^{\text{op}} \rightrightarrows \coprod_n \Gamma(n) \otimes_{\Delta} B(n \downarrow \mathcal{I})^{\text{op}} \right].$$

REMARK 3.1.3 (i) *Since we are not going to need the concrete definition of homotopy colimits, we will not give complete details, only explain the notation in the above definition.*

Cosimplicial resolution of objects in model categories are explained e.g. in [Dug01b, Definitions 3.1 and 3.2] or [Hir03, Chapter 16]. For a model category $\mathcal{C}$, a cosimplicial resolution of X is a cofibrant cosimplicial object in $\mathcal{C}$ weakly equivalent to X . Computing homotopy colimits is one of the situations where one first passes to a reasonable resolution of the object and then computes some point-set construction, e.g. an ordinary colimit. The standard statements apply: To compute a homotopy colimit one needs to fix a cosimplicial resolution of the diagram functor, but the homotopy type of the homotopy colimit does not depend on such a choice.

The symbol $\otimes_{\Delta}$ denotes one of the adjoint functors of the simplicial structure on the category of cosimplicial objects in $\mathcal{C}$, cf. Definition 2.1.7, and $B(n \downarrow \mathcal{I})$ is the classifying space of the category of n -sequences in the index category $\mathcal{I}$.

(ii) *There is also a concrete definition of homotopy limits, given by a construction dual to the one for homotopy colimits. We will however see later that their*

behaviour is completely different from the one of homotopy colimits. Since we will not need general homotopy limits, we omit the concrete definition and only discuss homotopy pullbacks in detail.

The fact that homotopy colimits resp. limits can be defined as left resp. right derived functors of colimits resp. limits implies that they are homotopy invariant [Hir03, Theorem 18.5.3].

PROPOSITION 3.1.4 (HOMOTOPY INVARIANCE) *Let $\mathcal{C}$ be a simplicial model category, and let $\mathcal{I}$ be a small category. If $f : \mathcal{X} \rightarrow \mathcal{Y}$ is a morphism of $\mathcal{I}$ -diagrams of cofibrant objects in $\mathcal{C}$ which is an objectwise equivalence, then*

$$\mathrm{hocolim}_{\mathcal{I}} f : \mathrm{hocolim}_{\mathcal{I}} \mathcal{X} \rightarrow \mathrm{hocolim}_{\mathcal{I}} \mathcal{Y}$$

is a weak equivalence of cofibrant objects.

Dually, if $f : \mathcal{X} \rightarrow \mathcal{Y}$ is a morphism of $\mathcal{I}$ -diagrams in $\mathcal{C}$ which is an objectwise equivalence of fibrant objects, then

$$\mathrm{holim}_{\mathcal{I}} f : \mathrm{holim}_{\mathcal{I}} \mathcal{X} \rightarrow \mathrm{holim}_{\mathcal{I}} \mathcal{Y}$$

is a weak equivalence of fibrant objects.

Moreover, homotopy colimits and limits interact nicely with the corresponding left resp. right Quillen functors. One example is commuting colimits with the hom-functors as stated in [MV99, Lemma 2.1.19].

PROPOSITION 3.1.5 *For any diagram $\mathcal{X} : \mathcal{I} \rightarrow \Delta^{\mathrm{op}}\mathcal{PShv}(T)$ and any simplicial presheaf Y there are canonical isomorphisms*

$$\mathbf{Hom}(\mathrm{hocolim}_{\mathcal{I}} \mathcal{X}, Y) \cong \mathrm{holim}_{\mathcal{I}^{\mathrm{op}}} \mathbf{Hom}(\mathcal{X}, Y)$$

and

$$\mathbf{Hom}(Y, \mathrm{holim}_{\mathcal{I}} \mathcal{X}) \cong \mathrm{holim}_{\mathcal{I}} \mathbf{Hom}(Y, \mathcal{X}).$$

Since the only homotopy limits we will frequently use are finite ones, we recall the definition and some standard facts: A homotopy pullback (or homotopy cartesian square) is a commutative square

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & & \downarrow f \\ B & \longrightarrow & Y, \end{array}$$

such that for any factorization of f into a trivial cofibration $i : X \rightarrow \tilde{X}$ and a fibration $p : \tilde{X} \rightarrow Y$, the induced morphism $A \rightarrow B \times_Y \tilde{X}$ is a weak equivalence. For proper model categories, it suffices to check one such factorization, cf. [GJ99, Lemma II.8.16]. Note also that a homotopy pullback with $B = \mathrm{pt}$ is basically the same thing as a fibre sequence.

There are some other general facts about colimits and limits, such as the pullback lemma [Mac98, Exercise III.4.8] or the fact that directed colimits commute with finite limits [Mac98, Theorem IX.2.1]. According to the general philosophy,

these results should also hold for homotopy colimits and limits in general model categories.

We will need the following analogue of the homotopy pullback lemma for model categories, whose proof can be found e.g. in [GJ99, Lemma II.8.22].

LEMMA 3.1.6 (HOMOTOPY PULLBACK LEMMA) *Let $\mathcal{C}$ be a proper model category, and let the following commutative diagram be given:*

$$\begin{array}{ccccc} X_1 & \longrightarrow & X_2 & \longrightarrow & X_3 \\ \downarrow & & \downarrow & & \downarrow \\ Y_1 & \longrightarrow & Y_2 & \longrightarrow & Y_3. \end{array}$$

If the inner squares are homotopy pullback squares, then so is the outer. Also, if the outer square and the inner right square are homotopy pullback squares, then so is the inner left square.

I do not know if directed homotopy colimits do always commute with finite homotopy limits. By homotopy distributivity it is certainly true for model categories of simplicial presheaves.

Note that in general model categories, homotopy colimits do not commute with finite homotopy limits. This already fails for the ordinary limits and colimits. One of the most important classes of categories that do satisfy this sort of distributivity for ordinary limits and colimits are topoi. We will see in the next paragraph that in categories of simplicial presheaves, homotopy colimits commute with finite homotopy limits.

Homotopy Distributivity: As already mentioned, homotopy distributivity is a homotopical generalization of the usual infinite distributivity law which holds for topoi. It is a statement about commutation of arbitrary small homotopy colimits with finite homotopy limits. Since any finite homotopy limit can be constructed via homotopy pullbacks, it suffices to check that homotopy pullbacks distribute over arbitrary homotopy colimits. Most of the work on homotopy distributivity seems to be due to Rezk [Rez98].

The situation is the following: Let $\mathcal{C}$ be a simplicial model category, let $\mathcal{I}$ be a small category, and let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a morphism of $\mathcal{I}$ -diagrams in $\mathcal{C}$. The diagrams we are most interested in are the following:

For any $i \in \mathcal{I}$, we have a commutative square

$$\begin{array}{ccc} \mathcal{X}(i) & \longrightarrow & \operatorname{colim}_{\mathcal{I}} \mathcal{X} \\ f(i) \downarrow & & \downarrow \\ \mathcal{Y}(i) & \longrightarrow & \operatorname{colim}_{\mathcal{I}} \mathcal{Y}. \end{array} \quad (3.1)$$

Moreover, for any $\alpha : i \rightarrow j$ in $\mathcal{I}$ we have a commutative square

$$\begin{array}{ccc} \mathcal{X}(i) & \xrightarrow{\mathcal{X}(\alpha)} & \mathcal{X}(j) \\ f(i) \downarrow & & \downarrow f(j) \\ \mathcal{Y}(i) & \xrightarrow{\mathcal{Y}(\alpha)} & \mathcal{Y}(j). \end{array} \quad (3.2)$$

Now we are ready to state the definition of homotopy distributivity, following [Rez98].

DEFINITION 3.1.7 (HOMOTOPY DISTRIBUTIVITY) *In the above situation, we say that $\mathcal{C}$ satisfies homotopy distributivity if for any morphism $f : \mathcal{X} \rightarrow \mathcal{Y}$ of $\mathcal{I}$ -diagrams in $\mathcal{C}$ for which $\mathcal{Y}$ is a homotopy colimit diagram, i.e. $\text{hocolim}_{\mathcal{I}} \mathcal{Y} \rightarrow \text{colim}_{\mathcal{I}} \mathcal{Y}$ is a weak equivalence, the following two properties hold:*

(HD i) *If each square of the form (3.1) is a homotopy pullback, then $\mathcal{X}$ is a homotopy colimit diagram.*

(HD ii) *If $\mathcal{X}$ is a homotopy colimit diagram, and each diagram of the form (3.2) is a homotopy pullback, then each diagram of the form (3.1) is also a homotopy pullback.*

REMARK 3.1.8 *Let T be a site, and assume we know homotopy distributivity for $\Delta^{\text{op}}\mathcal{Shv}(T)$. We show how to deduce the usual form of the distributive law from point-set distributivity and (HD i) and (HD ii) above in the following special case: Let $\mathcal{I}$ be a small category, and let $\mathcal{J}$ be the category*

$$\begin{array}{ccc} 3 & \longrightarrow & 2 \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & 0. \end{array}$$

Let an $\mathcal{I} \times \mathcal{J}$ -diagram $\mathcal{B}$ be given, and consider the following diagram:

$$\begin{array}{ccccc} \mathcal{B}(i, 3) & \longrightarrow & Z & \longrightarrow & \text{hocolim}_{\mathcal{I}} \mathcal{B}(i, 2) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{B}(i, 1) & \longrightarrow & \text{hocolim}_{\mathcal{I}} \mathcal{B}(i, 1) & \longrightarrow & \text{hocolim}_{\mathcal{I}} \mathcal{B}(i, 0), \end{array}$$

where $Z = \text{hocolim}_{\mathcal{I}} \mathcal{B}(i, 1) \times_{\text{hocolim}_{\mathcal{I}} \mathcal{B}(i, 0)}^h \text{hocolim}_{\mathcal{I}} \mathcal{B}(i, 2)$. Then by point-set distributivity of $\Delta^{\text{op}}\mathcal{Shv}(T)$, we have $Z = \text{colim}_{\mathcal{I}} \mathcal{B}(i, 3)$. To show that this is a homotopy colimit diagram, we need to show that all of the left-hand squares in the above diagram are homotopy pullbacks and then employ (HD i). We know that the right-hand square is a homotopy pullback by definition, so it suffices to show that the outer rectangle is a homotopy pullback, by the homotopy pullback lemma, cf. Lemma 3.1.6. This follows from (HD ii), since $\text{hocolim}_{\mathcal{I}} \mathcal{B}(i, 2)$ is a homotopy colimit diagram, and all the squares

$$\begin{array}{ccc} \mathcal{B}(i, 3) & \longrightarrow & \mathcal{B}(i, 2) \\ \downarrow & & \downarrow \\ \mathcal{B}(i, 1) & \longrightarrow & \mathcal{B}(i, 0) \end{array}$$

are homotopy pullbacks. Therefore we can also present the homotopy pullback of the homotopy colimits Z as the homotopy colimits of the homotopy pullbacks $\mathcal{B}(i, 3)$, which is what we usually would expect of a distributive law.

We see, that the axioms (HD i) and (HD ii) describe the two necessary parts of the above argument, modulo the assumption point-set distributivity. Note that

we can not really formulate homotopy distributivity the usual way, since it is complicated to construct a commutation map $\text{hocolim holim} \rightarrow \text{holim hocolim}$: The homotopy versions of limits and colimits do not satisfy the usual universality.

EXAMPLE 3.1.9 The category $\Delta^{\text{op}}\text{Set}$ of simplicial sets satisfies homotopy distributivity. This follows e.g. from the work of Puppe [Pup74] and Mather [Mat76]. $\square$

More generally, homotopy distributivity holds for all model categories of simplicial presheaves on a Grothendieck site and can be proven quite easily if the site has enough points. We give a short and simple proof that homotopy colimits and homotopy pullbacks are determined locally, which implies homotopy distributivity.

PROPOSITION 3.1.10 *Let T be a site with enough points, let $\mathcal{I}$ be a small category, and let $\mathcal{X} : \mathcal{I} \rightarrow \Delta^{\text{op}}\text{PShv}(T)$ be a diagram of simplicial presheaves. Then $\mathcal{X}$ is a homotopy colimit diagram iff for each point p of T , the corresponding diagram $p^*(\mathcal{X}) : \mathcal{I} \rightarrow \Delta^{\text{op}}\text{Set}$ is a homotopy colimit diagram.*

PROOF: Recall that $\mathcal{X}$ is a homotopy colimit diagram iff the induced morphism $\text{hocolim}_{\mathcal{I}} \mathcal{X} \rightarrow \text{colim}_{\mathcal{I}} \mathcal{X}$ is a weak equivalence. Since weak equivalences are local, i.e. can be checked at points, this is equivalent to requiring that for any point p of T , the morphisms

$$p^*(\text{hocolim}_{\mathcal{I}} \mathcal{X}) \rightarrow p^*(\text{colim}_{\mathcal{I}} \mathcal{X})$$

is a weak equivalence of simplicial sets. By definition of a point, the functor $p^* : \Delta^{\text{op}}\text{PShv}(T) \rightarrow \Delta^{\text{op}}\text{Set}$ has a right adjoint p_* and therefore preserves all colimits, i.e. $p^*(\text{colim}_{\mathcal{I}} \mathcal{X}) \cong \text{colim}_{\mathcal{I}} p^*\mathcal{X}$ is an isomorphism. The same holds for the homotopy colimits, by [MV99, Lemma 2.1.20]: $p^*(\text{hocolim}_{\mathcal{I}} \mathcal{X}) \simeq \text{hocolim}_{\mathcal{I}} p^*\mathcal{X}$ is a weak equivalence. Putting everything together, we find that $\mathcal{X}$ is a homotopy colimit diagram iff for any point $\text{hocolim}_{\mathcal{I}} p^*(\mathcal{X}) \rightarrow \text{colim}_{\mathcal{I}} p^*(\mathcal{X})$ is a weak equivalence, which is equivalent to $p^*(\mathcal{X})$ being a homotopy colimit diagram for any point p of T . $\blacksquare$

PROPOSITION 3.1.11 *Let T be a site with enough points, and let the following commutative diagram $\mathcal{X}$ of simplicial presheaves in $\Delta^{\text{op}}\text{PShv}(T)$ be given:*

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow f \\ C & \longrightarrow & D. \end{array}$$

This is a homotopy pullback diagram iff for each point p of T , the diagram $p^*(\mathcal{X})$ of simplicial sets is a homotopy pullback diagram.

PROOF: Recall that $\mathcal{X}$ is a homotopy pullback diagram if there exists a factorization of $f : B \rightarrow D$ into a trivial cofibration $i : B \rightarrow \tilde{B}$ and a fibration $g : \tilde{B} \rightarrow D$, such that the induced morphism $A \rightarrow C \times_D \tilde{B}$ is a weak equivalence. This is equivalent to saying that for each point p of T , the induced morphism $p^*(A) \rightarrow p^*(C \times_D \tilde{B}) \cong p^*(C) \times_{p^*(D)} p^*(\tilde{B})$ is a weak equivalence. Note that points preserve finite limits by definition, which explains the last isomorphism.

We also know that for any fibration h of simplicial presheaves and any point p of the topos, $p^*(h)$ is a Kan fibration of simplicial sets, cf. [MV99, Proposition 2.1.59].

From the above it follows that $p^*(g)$ is a fibration of simplicial sets and $p^*(i)$ is a weak equivalence. We can factor this into a trivial cofibration $j : p^*(B) \rightarrow E$ and a trivial fibration $h : E \rightarrow p^*(\tilde{B})$.

Assume now that for each point p of T the diagram $p^*(\mathcal{X})$ is a homotopy pullback diagram. This implies that for the above factorization, $p^*(A) \rightarrow p^*(C) \times_{p^*(D)} p^*(\tilde{B})$ is a weak equivalence. Consider the following diagram:

$$\begin{array}{ccc} p^*(C) \times_{p^*(D)} E & \xrightarrow{\quad} & E \\ \downarrow & & \downarrow \\ p^*(C) \times_{p^*(D)} p^*(\tilde{B}) & \xrightarrow{\quad} & p^*(\tilde{B}) \\ \downarrow & & \downarrow \\ p^*(C) & \xrightarrow{\quad} & p^*(D). \end{array}$$

Since the outer square and the lower square are pullback diagrams, so is the upper by the homotopy pullback lemma. Pullbacks of trivial fibrations are again trivial fibrations [GJ99, Corollary II.1.3]. Therefore, the morphism

$$p^*(C) \times_{p^*(D)} E \rightarrow p^*(C) \times_{p^*(D)} p^*(\tilde{B})$$

is a trivial fibration. By 2-out-of-3, the morphism $p^*(A) \rightarrow p^*(C) \times_{p^*(D)} p^*(\tilde{B})$ is a weak equivalence for any point, and therefore, the diagram $\mathcal{X}$ is a homotopy pullback diagram of simplicial presheaves.

For the converse, one can prove that homotopy pullbacks are preserved by points in a similar way, but we can also cite [MV99, Lemma 2.1.20]. ■

COROLLARY 3.1.12 *Let T be a site with enough points. Then homotopy distributivity holds for the injective model structure on $\Delta^{\text{op}}\text{PShv}(T)$.*

PROOF: By Propositions 3.1.10 and 3.1.11 the properties of homotopy colimit resp. homotopy pullback diagrams can be checked locally. The assertion then follows from homotopy distributivity for simplicial sets. ■

REMARK 3.1.13 (i) *Note that the above did not explicitly use the particular structure of fibrations and cofibrations in the injective model structure on $\Delta^{\text{op}}\text{PShv}(T)$. Therefore, the result holds for any model structure on simplicial presheaves that has local weak equivalences and for which the point functors form a Quillen pair.*

Similarly, there is no difference in taking such a model structure on the category of simplicial presheaves on T or on the category of simplicial sheaves on T .

(ii) *There is actually no difference in taking unpointed or pointed simplicial presheaves, since the corresponding categories of simplicial sets are Quillen equivalent and both satisfy homotopy distributivity.*

- (iii) More generally, homotopy distributivity is one of the basic properties of model topoi and plays a crucial part in the homotopy Giraud theorem [TV05, Theorem 4.9.2].

Although the above simple proof works only for topoi with enough points, homotopy distributivity holds for an arbitrary Grothendieck topos. This has been proven by Rezk [Rez98, Theorem 1.4, 1.5] using a Boolean localization argument. He first shows that homotopy colimits resp. homotopy pullbacks behave well under geometric morphisms of topoi. Therefore, it is possible to deduce homotopy distributivity for the injective model structure on simplicial objects in a general Grothendieck topos from homotopy distributivity for simplicial objects in a Boolean topos. The latter follows since global fibrations in a Boolean topos are easy to handle. For details of the proof we refer to [Rez98].

PROPOSITION 3.1.14 *Let T be a site, and consider the injective model structure on $\Delta^{\text{op}}\text{PShv}(T)$. Then homotopy distributivity holds for $\Delta^{\text{op}}\text{PShv}(T)$.*

Moreover, for any geometric morphism of Grothendieck topoi $p : \mathcal{E} \rightarrow \mathcal{E}'$, the induced inverse image functor $p^* : \Delta^{\text{op}}\mathcal{E}' \rightarrow \Delta^{\text{op}}\mathcal{E}$ preserves homotopy pullbacks.

We next discuss two important consequences of homotopy distributivity for model categories of simplicial presheaves. One is a generalization of Mather's cube theorem [Mat76]. The other generalizes a theorem of Puppe [Pup74] on commuting homotopy fibres and homotopy pushouts to simplicial presheaves.

COROLLARY 3.1.15 (MATHER'S CUBE THEOREM) *Let T be a Grothendieck site, and consider the following cubical diagram of simplicial presheaves on T :*

$$\begin{array}{ccccc}
 & & X_1 & \xrightarrow{\quad} & X_2 \\
 & \swarrow & \downarrow & & \swarrow \\
 X_3 & \xrightarrow{\quad} & X_4 & & \\
 \downarrow & & \downarrow & & \downarrow \\
 & \swarrow & Y_1 & \xrightarrow{\quad} & Y_2 \\
 & \downarrow & \downarrow & & \downarrow \\
 Y_3 & \xrightarrow{\quad} & Y_4 & &
 \end{array}$$

Assume that the bottom face, i.e. the one consisting of the spaces Y_i , is a homotopy pushout, and that all the vertical faces are homotopy pullbacks. Then the top face is a homotopy pushout.

In particular, taking the homotopy fibre commutes with homotopy pushouts: For a commutative diagram

$$\begin{array}{ccccc}
 E_2 & \xleftarrow{\quad} & E_0 & \xrightarrow{\quad} & E_1 \\
 p_2 \downarrow & & \downarrow p_0 & & \downarrow p_1 \\
 B_2 & \xleftarrow{\quad} & B_0 & \xrightarrow{\quad} & B_1
 \end{array}$$

in which the squares are homotopy pullbacks, we have weak equivalences

$$\text{hofib } p_i \xrightarrow{\cong} \text{hofib}(p : E_1 \cup_{E_0}^h E_2 \rightarrow B_1 \cup_{B_0}^h B_2).$$

PROOF: This is a consequence of homotopy distributivity, cf. Corollary 3.1.12 resp. Proposition 3.1.14, applied to homotopy pushout diagrams. The assumption in Part (HD ii) is that the bottom face is a homotopy colimit diagram, i.e. a homotopy pushout. Moreover all the vertical faces are homotopy pullbacks. Then by the homotopy pullback lemma 3.1.6, we obtain that the diagonal in the cube containing X_1, Y_1, X_4 and Y_4 is also a homotopy pullback. By (HD ii) we conclude that X_4 is a homotopy colimit diagram, i.e. it is weakly equivalent to the homotopy pushout $X_2 \cup_{X_1}^h X_3$.

The restriction in the definition of homotopy distributivity that X_4 be the point-set pushout of X_2 and X_3 along X_1 is not essential. Without loss of generality we can assume that the morphisms $X_1 \rightarrow X_3$ resp. $Y_1 \rightarrow Y_3$ are cofibrations, and that X_4 resp. Y_4 are point-set pushouts. If this is not the case, just replace the morphism by cofibrations, and obtain a cube which is weakly equivalent to the cube we started with.

For the second statement note that since the squares are homotopy pullbacks, we have $\text{hofib } p_0 \cong \text{hofib } p_1 \cong \text{hofib } p_2$. Factoring $E_0 \rightarrow E_1$ resp. $B_0 \rightarrow B_1$ as a cofibration followed by a trivial fibration, we can assume that these morphisms are cofibrations. Denote $E = E_1 \cup_{E_0}^h E_2$ and $B = B_1 \cup_{B_0}^h B_2$. Then we are in the situation to apply (HD ii). This implies that all the squares

$$\begin{array}{ccc} E_i & \longrightarrow & E \\ \downarrow & & \downarrow \\ B_i & \longrightarrow & B \end{array}$$

are homotopy pullback squares. In particular, we get the desired weak equivalences $\text{hofib } p_i \cong \text{hofib } p$. $\blacksquare$

The following is a version of Puppe's Theorem [Pup74] for simplicial presheaves:

PROPOSITION 3.1.16 (PUPPE'S THEOREM) *Let $\mathcal{E} : \mathcal{I} \rightarrow \Delta^{\text{op}}\text{PShv}(T)$ be a diagram of simplicial presheaves over a fixed base simplicial presheaf B , i.e. the following diagram commutes for every $\alpha : i \rightarrow j$ in $\mathcal{I}$:*

$$\begin{array}{ccc} \mathcal{E}(i) & \xrightarrow{\mathcal{E}(\alpha)} & \mathcal{E}(j) \\ & \searrow & \swarrow \\ & B & \end{array}$$

There is an associated diagram of homotopy fibres

$$\mathcal{F} : \mathcal{I} \rightarrow \Delta^{\text{op}}\text{PShv}(T) : i \mapsto \text{hofib}(\mathcal{E}(i) \rightarrow B)$$

Denoting $E = \text{hocolim}_{\mathcal{I}} \mathcal{E}$ and $F = \text{hocolim}_{\mathcal{I}} \mathcal{F}$, we have a weak equivalence $\text{hofib}(E \rightarrow B) \simeq F$.

PROOF: We construct a new morphism of diagrams $\mathcal{G} \rightarrow \mathcal{E}$, where the diagram $\mathcal{G}$ is defined by

$$\mathcal{G} : \mathcal{I} \rightarrow \Delta^{\text{op}}\text{PShv}(T) : i \mapsto \mathcal{E}(i) \times_E^h \text{hofib}(E \rightarrow B).$$

Without loss of generality we can assume $\text{hofib}(E \rightarrow B) \rightarrow E$ is a fibration. Then the homotopy pullbacks above are ordinary pullbacks, and $\text{colim } \mathcal{G} \cong \text{hofib}(E \rightarrow B)$. We apply the homotopy pullback lemma to the following diagram:

$$\begin{array}{ccc}
\mathcal{E}(i) \times_E^h \operatorname{hofib}(E \rightarrow B) & \longrightarrow & \mathcal{E}(i) \\
\downarrow & & \downarrow \\
\operatorname{hofib}(E \rightarrow B) & \longrightarrow & E \\
\downarrow & & \downarrow \\
\operatorname{pt} & \longrightarrow & B
\end{array}$$

This implies the following weak equivalence

$$\mathcal{E}(i) \times_E^h \operatorname{hofib}(E \rightarrow B) \simeq \operatorname{hofib}(\mathcal{E}(i) \rightarrow B) = \mathcal{F}(i).$$

Invariance of homotopy colimits under weak equivalence, cf. Proposition 3.1.4, implies that $\operatorname{hocolim} \mathcal{G} \cong \operatorname{hocolim} \mathcal{F}$. Homotopy distributivity applied to the projection morphism $\mathcal{G} \rightarrow \mathcal{E}$ implies that $\mathcal{G}$ is a homotopy colimit diagram. Putting everything together we obtain weak equivalences $\operatorname{hofib}(E \rightarrow B) \cong \operatorname{colim} \mathcal{G} \simeq \operatorname{hocolim} \mathcal{G} \simeq \operatorname{hocolim} \mathcal{F}$, whence the desired statement follows. $\blacksquare$

Canonical Homotopy Colimit Decomposition: Given a (fibrant) simplicial set X , it is always possible to decompose it into its simplices. One obtains a diagram of simplicial sets, whose index category is the category of simplices $\Delta \downarrow X$. Every object in $\Delta \downarrow X$ is of the form $\Delta^n \rightarrow X$, and is mapped to the standard simplex Δ^n . It is clear that the colimit of this diagram is again X . Even more important is the fact that the diagram is a homotopy colimit diagram, so X is the homotopy colimit of the above diagram. Therefore, any simplicial set can be canonically decomposed into a homotopy colimit of contractible simplicial sets.

Similarly, any sheaf in a topos can be written as a canonical colimit of representable objects [MM92, Proposition I.5.1], a statement that generalizes to locally presentable categories.

The right generalization of the two situations above is the canonical homotopy colimit decomposition for objects in a combinatorial model category, which was described in detail in [Dug01b, Dug01a].

Let $\mathcal{M}$ be a combinatorial model category, $\mathcal{C}$ be a small category. For any functor $I : \mathcal{C} \rightarrow \mathcal{M}$ and a fixed cosimplicial resolution $\Gamma_I : \mathcal{C} \rightarrow \Delta \mathcal{M}$, we obtain a functor $\mathcal{C} \times \Delta \rightarrow \mathcal{M} : (U, [n]) \mapsto \Gamma(n)(U)$. For any object X , we can consider the over-category (resp. comma category in Mac Lane's terminology [Mac98, Section II.6]) $(\mathcal{C} \times \Delta \downarrow X)$ and the canonical diagram $(\mathcal{C} \times \Delta \downarrow X) \rightarrow \mathcal{M} : \Gamma(n)(U) \mapsto U \times \Delta^n$.

EXAMPLE 3.1.17 The standard cosimplicial object $\Delta^\bullet$ in $\Delta^{\operatorname{op}} \operatorname{Set}$ is a cosimplicial resolution of the functor $\operatorname{pt} : * \rightarrow \Delta^{\operatorname{op}} \operatorname{Set}$. The over-category above is then the category of simplices $\Delta \downarrow X$. $\square$

EXAMPLE 3.1.18 Let T be any Grothendieck site, and $\Delta^{\operatorname{op}} \operatorname{PShv}(T)$ the corresponding topos of simplicial presheaves. The standard cosimplicial objects Δ_U^n for a representable $U \in T$ assemble into a functor

$$\Delta_T^n : T \mapsto \Delta^{\operatorname{op}} \operatorname{PShv}(T) : U \mapsto \Delta_U^n = \left\{ V \mapsto \coprod_{\phi: V \rightarrow U} \Delta^n \right\}.$$

These functors assemble into a cosimplicial object $\Delta_T^\bullet : T \rightarrow \Delta\Delta^{\text{op}}PShv(T)$, which is a cosimplicial resolution of the Yoneda functor $y : T \rightarrow Shv(T)$. $\square$

DEFINITION 3.1.19 (CANONICAL HOMOTOPY COLIMIT) *Let $\mathcal{M}$ be a combinatorial model category, let $\mathcal{C}$ be the full subcategory $\mathcal{C}$ given by a small set of generators of $\mathcal{M}$, and denote by $I : \mathcal{C} \rightarrow \mathcal{M}$ the corresponding inclusion of categories. For an object $X \in \mathcal{M}$ and a chosen cosimplicial resolution Γ_I of I as above, we define the canonical homotopy colimit of X w.r.t. Γ_I to be the homotopy colimit of the functor $\Gamma_I : (\mathcal{C} \times \Delta \downarrow X) \rightarrow \mathcal{M}$ and denote it by*

$$\text{hocolim}(\mathcal{C} \times \Delta \downarrow X).$$

In [Dug01a], this is shown to be equivalent to the realization of the singular complex associated to a presentation of the combinatorial model category. This implies the following properties of canonical homotopy colimits:

PROPOSITION 3.1.20 *Let $\mathcal{M}$ be a combinatorial model category, and let $\mathcal{C}$ be the full subcategory given by a small set of generators of $\mathcal{M}$. Fix a cosimplicial resolution Γ of the inclusion $I : \mathcal{C} \hookrightarrow \mathcal{M}$. Then the following invariance properties can be derived:*

- (i) *If $X \rightarrow Y$ is a weak equivalence between fibrant objects, then the induced map $\text{hocolim}(\mathcal{C} \times \Delta \downarrow X) \rightarrow \text{hocolim}(\mathcal{C} \times \Delta \downarrow Y)$ is a weak equivalence.*
- (ii) *For two cosimplicial resolutions Γ_I and Γ'_I and fibrant X , we obtain weakly equivalent canonical homotopy colimits*

$$\text{hocolim}_{\Gamma_I}(\mathcal{C} \times \Delta \downarrow X) \simeq \text{hocolim}_{\Gamma'_I}(\mathcal{C} \times \Delta \downarrow X)$$

- (iii) *For any fibrant object X , the canonical morphism*

$$\text{hocolim}(\mathcal{C} \times \Delta \downarrow X) \rightarrow \text{colim}(\mathcal{C} \times \Delta \downarrow X) \rightarrow X$$

is a weak equivalence.

REMARK 3.1.21 (i) *Recall that in a topos $Shv(T)$, every object is canonically the colimit of representable objects $\text{Hom}_T(-, C)$. Assertion (iii) in the above proposition is the “homotopy topos” analogue of this: Any simplicial sheaf is the homotopy colimit of representable homotopy types, in a rather canonical way. This decomposition is not completely canonical, since it depends on the cosimplicial resolution.*

- (ii) *Let T be a Grothendieck site. We have shown in Proposition 2.1.19 that the injective model structure on $\Delta^{\text{op}}PShv(T)$ is combinatorial. We will usually use the cosimplicial resolution described in Example 3.1.18, where the set of small generators is the set of representable objects $U \in T$. We will therefore denote by $(T \times \Delta \downarrow X)$ the canonical homotopy colimit decomposition of a fibrant simplicial presheaf X .*

The canonical homotopy colimit decomposition together with homotopy distributivity imply that the decomposition of fibrations of simplicial sets can be generalized to simplicial presheaves, where it unfortunately loses some of its usefulness.

Let $p : E \rightarrow B$ be a fibration of fibrant simplicial sets. Then the canonical homotopy colimit decomposition of B allows to write B as homotopy colimit of standard simplices $\Delta^n \rightarrow B$. Then we can pull back the fibration p to these simplices and obtain the homotopy fibres. By homotopy distributivity, E can be written as the homotopy colimit over the simplex category $\Delta \downarrow B$ of the homotopy fibres.

The same statement works for simplicial presheaves:

LEMMA 3.1.22 *For any fibration $p : E \rightarrow B$ of fibrant simplicial presheaves, p is weakly equivalent to the morphism of simplicial presheaves*

$$\mathrm{hocolim} \mathcal{F} \rightarrow \mathrm{hocolim}(T \times \Delta \downarrow B),$$

where the diagram $\mathcal{F}$ is the diagram of homotopy fibres: The index category is still $(T \times \Delta \downarrow B)$, but an object $U \times \Delta^n \rightarrow B$ is mapped to the pullback $(U \times \Delta^n) \times_B E$, which is the fibre of p over U .

This is not as useful as the same construction for simplicial sets, since the homotopy types of the various $U \in T$ are different, which is the same as saying that a simplicial presheaf is not locally contractible. Therefore, not all of the simplicial presheaves $(U \times \Delta^n) \times_B E$ are weakly equivalent.

3.2 Classifying Spaces for Fibre Sequences

One of the basic theorems in the theory of fibrations for CW-complexes is the existence of classifying spaces. Not only do such classifying spaces exist for fibre spaces with a given fibre, it is also possible to construct such spaces for all sorts of structures on these fibre sequences, like orientations etc. Indeed, many results relating to surgery and manifold structures on homotopy types are expressed and solved with the help of classifying spaces, as the following quote from Chapter 1 of [MM79] illustrates:

A basic technique in topology is to reduce a geometric classification problem to a homotopy classification of maps into an associated “classifying space” or universal object.

The geometric part of surgery or topology in general is the abovementioned reduction of a geometric problem like classifying manifold structures or bundle structures to an associated homotopy classification problem. The homotopical part of surgery is then purely concerned with computing the homotopy groups or other invariants of classifying spaces resp. of homotopy fibres of maps between classifying spaces. The importance of classifying spaces to cobordism theory has been described e.g. in [Rud98]. In essence we can say that the theory of classifying spaces for fibrations has had a great deal of applications in the algebraic topology of CW-complexes. It is therefore obviously interesting to study how far the techniques

for the construction of classifying spaces generalize to the setting of simplicial presheaves.

There exist basically two approaches to construct classifying spaces. The bar construction approach starts with the work of Milnor [Mil56] who constructed the classifying space BG and in fact the universal principal G -bundle for a topological group G . This work was generalized to classifying spaces of monoids by Stasheff [Sta63]. The most general way of using bar constructions to define universal fibre sequences was given in the monograph [May75]. Of course, many more papers have been written on the construction of universal bundles via bar constructions, but the above three treatments seem to be the most influential ones.

The second approach we want to mention is the abstract construction of classifying spaces using the Brown representability theorem. For classifying spaces, this has been done by Allaud [All66]. Similar work on the problem of constructing classifying spaces via Brown representability has been done by Dold [Dol66] and Schön [Sch82]. A textbook treatment of this approach can be found in [Rud98, Chapter IV]. This is the approach that we will follow for our construction of classifying spaces in the setting of simplicial presheaves.

There are also generalizations of the above, which do not only classify fibre sequences, but fibre sequences with extra structures. Using the general bar construction approach of May [May75], it is possible to classify all sorts of structures, e.g. orientations, on fibre sequences, as well as fibre sequences with different fibres. Such classification theorems are also available for equivariant topology, mostly due to the work of Waner [Wan80].

Classifying spaces for principal G -bundles have also been constructed in the setting of homotopy theory of simplicial presheaves. The first work on classifying spaces of sheaves of groupoids I know of is [JT93], followed by rather recent work by Jardine and Luo [JL06]. Morel and Voevodsky [MV99, Section 4] use the classical construction of Milnor to provide models for classifying spaces for sheaves of monoids. It seems, however, that classifying spaces for fibre sequences have not yet been considered in full generality.

In this section, we want to show how the topological proof of existence of classifying spaces for fibre sequences can be modified to work for certain well-behaved categories of simplicial sheaves. We will mimick the construction of classifying spaces via the Brown representability theorem, following the original work by Allaud [All66] and the book of Rudyak [Rud98, Chapter IV.1]. The steps we take are basically the same as outlined there, with some necessary modifications. In the first paragraph, we will define the functor that assigns to each simplicial presheaf X the set of fibre sequences over X with fibre a given simplicial presheaf F . Then we discuss the Brown representability theorem of Jardine [Jar06c], which works in the setting of compactly generated model categories. The model category of simplicial sheaves on a Grothendieck site is not in general compactly generated, which is why we do not get classifying spaces for all Grothendieck topoi. However, the model categories of interest for $\mathbb{A}^1$ -homotopy are in fact compactly generated, cf. Proposition 2.2.10. Since the Brown representability theorem also works for morphisms, we also get a universal fibre sequence. Finally, we show that the fibre-sequence functor satisfies the conditions of the Brown representability theorem. At this point we make crucial use of the homotopy distributivity discussed in Section 3.1.

Possible future work includes combining the Brown representability approach with a bar construction, as done in [Boo98] for topological spaces. Using a bar construction one could extend the construction of classifying spaces for fibre sequences to other topoi, as well as to unpointed settings. Moreover, one could try to classify more elaborate structures like orientable and oriented fibrations. However, it seems that the relevant pieces of a motivic orientation theory are not yet in place. We therefore content ourselves with the basic classifying spaces described in the following section.

Definition of the Fibre-Sequence Functor: This paragraph describes the functor mapping a simplicial presheaf to the *set* of fibre sequences with fixed fibre over this simplicial presheaf. We will work in the injective model category of *pointed* simplicial presheaves on some site T . This is due to the fact that fibre sequences as in Definition 2.1.9 are only defined in pointed model categories. Moreover, the Brown representability theorem also requires pointed model categories. There are examples in [Hel81] showing that Brown representability might fail already for unpointed topological spaces.

DEFINITION 3.2.1 *Recall from Definition 2.1.9 that a fibre sequence over X with fibre F is a diagram $F \rightarrow E \xrightarrow{p} X$ with an ΩX -action on F which is isomorphic in the homotopy category to the fibre sequence associated to a fibration $p : E' \rightarrow X'$ of fibrant objects. Up to isomorphism in the homotopy category, we will usually assume that our fibre sequence $F \rightarrow E \xrightarrow{p} X$ is represented by some actual fibration over some fibrant replacement $\tilde{X}$ of X .*

A morphism of fibre sequences is a diagram in $\Delta^{\text{op}}\mathcal{PShv}(T)$

$$\begin{array}{ccccc} F_1 & \longrightarrow & E_1 & \xrightarrow{p_1} & B_1 \\ f \downarrow & & g \downarrow & & \downarrow h \\ F_2 & \longrightarrow & E_2 & \xrightarrow{p_2} & B_2, \end{array}$$

such that the left square commutes up to homotopy, and the right square is commutative, and f is Ωh -equivariant, i.e. the following diagram is homotopy commutative:

$$\begin{array}{ccc} \Omega B_1 \times F_1 & \longrightarrow & F_1 \\ \Omega h \times f \downarrow & & \downarrow f \\ \Omega B_2 \times F_2 & \longrightarrow & F_2. \end{array}$$

This in particular allows to define what an equivalence of fibre sequences over X is: Two fibre sequences over X with fibre F are equivalent if there is an isomorphism of fibre sequences

$$\begin{array}{ccccc} F & \longrightarrow & E_1 & \longrightarrow & X \\ f \downarrow & & \downarrow & & \downarrow g \\ F & \longrightarrow & E_2 & \longrightarrow & X, \end{array}$$

in the homotopy category $\text{Ho } \Delta^{\text{op}}\mathcal{PShv}(T)$, for which f is Ωg -equivariant. We denote this by $E_1 \sim E_2$.

REMARK 3.2.2 (i) *The following can be assumed without loss of generality: We can assume that the base B is fibrant, that the morphism p is a fibration, and that F is the point-set fibre of p over $* \hookrightarrow B$. This basically follows from Proposition 2.3.3.*

(ii) *Note that in the definition of a morphism of fibre sequences we can always arrange for the right square to be commutative on the nose: We just lift the morphism $h \circ p_1$ along the fibration p_2 . This makes the right square commutative, and leaves the left square commutative up to homotopy.*

(iii) *If the base B is not path-connected as discussed in Section 2.3, we have to restrict to fibre sequences such that for all points of the topos $\mathcal{Shv}(T)$, the induced morphism $p^*(E) \rightarrow p^*(B)$ has the homotopy fibre $p^*(F)$ for all possible choices of base points in $p^*(B)$. This is the usual restriction that also appears in [All66]. Note that a constant simplicial presheaf is usually not path-connected, so such restrictions are even more important in the presheaf setting than in the topology of CW-complexes.*

In the following we replace the connectivity assumption by restricting to fibre sequences that are locally trivial in some topology, as explained in Section 2.3. Such restrictions can partially be lifted by working with categories of fibres, cf. Remark 3.2.7. However, some restrictions are necessary to get a set-valued functor, and to compare fibre sequences with torsors under groups of homotopy-autoequivalences. We henceforth assume that the fibre sequences we consider are locally trivial for some (possibly finer) topology on T .

LEMMA 3.2.3 *Equivalence of fibre sequences is an equivalence relation.*

PROOF: This is clear since equivalence was defined by isomorphism in the homotopy category, which implies reflexivity, symmetry and transitivity. $\blacksquare$

If we restrict the definition of the fibre sequences to fibrations of fibrant objects over some fibrant replacement of the base B , we could also define equivalence of fibre sequences by morphisms of fibre sequences in the *model category* which admit an inverse. This would basically yield the same functor. However, since we allow arbitrary simplicial presheaves in the fibre sequences, we have to use isomorphisms in the *homotopy category*.

DEFINITION 3.2.4 (PULLBACK OF FIBRE SEQUENCES) *Let $f : B_1 \rightarrow B_2$ be a pointed map, and let $F \rightarrow E_2 \rightarrow B_2$ be a fibre sequence. We define a fibre sequence with fibre F over B_1 as follows:*

$$\begin{array}{ccccc}
 & & F & & \\
 & \swarrow a & \downarrow & \searrow & \\
 E_1 & \xrightarrow{\quad} & E_2 & & \\
 \downarrow p_1 & & \downarrow & & \downarrow p_2 \\
 & \swarrow & * & \searrow & \\
 B_1 & \xrightarrow{\quad f \quad} & B_2 & &
 \end{array}$$

We assume that p_2 is a fibration, and define E_1 as the pullback $E_2 \times_{B_2} B_1$ of p_2 along f . Note that E_1 is therefore also the homotopy pullback of p_2 along f , and p_1 is a fibration. By the universal property of pullbacks, we have a morphism $a : F \rightarrow E_1$. Moreover, by the pullback lemma we have $p_1^{-1}(*) = F$, and since p_1 is a fibration, this is also the homotopy fibre.

Let T be a Grothendieck site. For given pointed simplicial presheaves X and F on T , let $\mathcal{H}(X, F)$ denote the collection of equivalence classes of fibre sequences over X with fibre F modulo the equivalence $\sim$. We want to show that this is a set. Since many simplicial presheaves, especially those of simplicial dimension zero, are not path-connected in the sense of Definition 2.3.6, we require that there is a (possibly finer) topology T' on the site T such that the fibre sequences are all locally trivial in the topology T' , cf. Remark 3.2.2.

PROPOSITION 3.2.5 *For any $X, F \in \Delta^{\text{op}}\mathcal{S}hv(T)_*$, the collection $\mathcal{H}(X, F)$ is a set. Hence, with the pullbacks as in Definition 3.2.4, we have a functor*

$$\mathcal{H}(-, F) : \Delta^{\text{op}}\mathcal{S}hv(T)_* \rightarrow \mathcal{S}et_*.$$

The natural base point of $\mathcal{H}(X, F)$ is given by the trivial fibre sequence $F \rightarrow X \times F \rightarrow X$, where the first map is inclusion via the base point $* \rightarrow X$, and the second is the product projection.

PROOF: We first show that for every simplicial presheaf X there is only a set of equivalence classes of fibre sequences $F \rightarrow E \rightarrow X$. We follow the lines of [Rud98, Theorem IV.1.55].

We start with the case of fibre sequences $F \rightarrow E \rightarrow U$ for $U \in T$, i.e. the base is a zero-dimensional object. For the above fibre sequence, there exists a T' -covering U_i of U such that $F \rightarrow E \times_U U_i \rightarrow_U U_i$ is a trivial fibre sequence with given trivializations and transition morphisms $E \times_U U_i \times_U U_j \rightarrow E \times_U U_i \times U_j$ which are weak equivalences. Denoting by $\text{Aut}_\bullet(F)$ the simplicial group of sheaves of homotopy automorphisms of F , a fibre sequence $F \rightarrow E \rightarrow U$ as above arises from a T' -torsor under $\text{Aut}_\bullet(F)$. For definitions and results on torsors under sheaves of simplicial groups, cf. [JT93] resp. [JL06]. Therefore, the collection of fibre sequences over U with fibre F which are T' -locally trivial is contained in the set

$$\check{H}^1(U, \text{Aut}_\bullet(F)) = \varinjlim \check{H}^1(\mathcal{U}, \text{Aut}_\bullet(F))$$

of T' -torsors over U under $\text{Aut}_\bullet(F)$, hence is itself a set.

Next, we extend this result to simplicial presheaves of the form $U \times \Delta^n$ for $U \in T$. The argument in Proposition 3.2.14 is independent of the $\mathcal{H}(-, F)$ being sets. It therefore shows that for a weak equivalence $f : X \rightarrow Y$, if $\mathcal{H}(X, F)$ is a set, then so is $\mathcal{H}(Y, F)$. We find that for any simplicial presheaf of the form $U \times \Delta^n$ with $U \in T$, $\mathcal{H}(U \times \Delta^n, F)$ is a set.

Finally, we use the decomposition of fibrations over the canonical homotopy colimit presentation of the base simplicial presheaf, cf. Lemma 3.1.22. We consider F -fibre sequences over B , and decompose B as a homotopy colimit over the category of simplices $(T \times \Delta \downarrow B)$. The simplicial presheaves indexed by this diagram are of the form $U \times \Delta^n$ for $U \in T$, and we have already shown that

fibre sequences over these form a set. Moreover, the site T is (essentially) small, therefore the diagram is set-indexed.

We therefore have to show that for a set-indexed homotopy colimit $\operatorname{hocolim}_\alpha X_\alpha$ of spaces X_α for which $\mathcal{H}(X_\alpha, F)$ is a set, the collection $\mathcal{H}(\operatorname{hocolim}_\alpha X_\alpha, F)$ is also a set. Since all homotopy colimits can be decomposed into homotopy pushouts and wedges, it suffices to show this assertion for these special homotopy colimits.

The proof of the wedge axiom in Proposition 3.2.16 shows that if $\mathcal{H}(X_\alpha, F)$ is a set for a set-indexed collection X_α , then $\mathcal{H}(\bigvee_\alpha X_\alpha, F)$ is also a set.

For the homotopy pushouts, we use the proof of Proposition 3.2.15. We get a surjective morphism of classes

$$d : \mathcal{H}(B_1 \cup_{B_0} B_2, F) \rightarrow \mathcal{H}(B_1, F) \times_{\mathcal{H}(B_0, F)} \mathcal{H}(B_2, F).$$

By assumption, $\mathcal{H}(B_1, F) \times_{\mathcal{H}(B_0, F)} \mathcal{H}(B_2, F)$ is a set. The morphism d decomposes a fibre sequence E over $B_1 \cup_{B_0} B_2$ into the pullbacks of the fibre sequence E to B_1 resp. B_2 . These fibre sequences are remembered in the element in $\mathcal{H}(B_1, F) \times_{\mathcal{H}(B_0, F)} \mathcal{H}(B_2, F)$. What is forgotten, and what constitutes the kernel of d is the isomorphism between the pullbacks of E to B_1 resp. B_2 . Since there is only a set of automorphisms for any given fibre sequence, the kernel of d is also a set. This implies that $\mathcal{H}(B_1 \cup_{B_0} B_2, F)$ is also a set.

Therefore, $\mathcal{H}(B, F)$ is a set for any simplicial presheaf B .

We still need to check that the pullback is really well-defined. This is a simple diagram check, using the cogluing lemma and therefore needing properness.

Assume we have two fibre sequences E_1 and E_2 over B , which are isomorphic in the homotopy category. We may assume that $p_i : E_i \rightarrow B$ are fibrations. If not, we choose factorizations. The independence of the choice of such fibrant replacements is proven in Proposition 3.2.6. We consider the pullback $E_i \times_B A$ of the fibre sequence E_i along the morphism $f : A \rightarrow B$. The isomorphism in the homotopy category lifts to a zig-zag of weak equivalences, so it suffices to show that a weak equivalence $g : E_1 \rightarrow E_2$ pulls back to a weak equivalence $f : E_1 \times_B A \rightarrow E_2 \times_B A$. This follows from the cogluing lemma 2.3.2.

Finally, for any fibre sequence $F \rightarrow E \rightarrow B$, which is locally trivial in the T -topology the pullback $F \rightarrow E \times_B B' \rightarrow B'$ along any morphism $B' \rightarrow B$ is again locally trivial. This follows by a simple argument from the pullback lemma: The pullback of $E' = E \times_B B'$ along any morphism $U \rightarrow B'$ for $U \in T$ is also the pullback of E along $U \rightarrow B' \rightarrow B$. $\blacksquare$

PROPOSITION 3.2.6 (i) For any commutative diagram

$$\begin{array}{ccc} E_1 & \xrightarrow{\cong} & E_2 \\ p_1 \searrow & & \swarrow p_2 \\ & B & \end{array}$$

with p_1 and p_2 fibrations, the induced weak equivalence on the fibres is equivariant for the ΩB -action.

(ii) Let $p : E \rightarrow B$ be any morphism with homotopy fibre $F = \operatorname{hofib} p$. Then the class $[\tilde{p}] \in \mathcal{H}(B, F)$ of a fibrant replacement $\tilde{p} : \tilde{E} \rightarrow B$ of p is independent of the choice of fibrant replacement.

(iii) For any homotopy pullback

$$\begin{array}{ccc} E_1 & \longrightarrow & E_2 \\ p \downarrow & & \downarrow q \\ B_1 & \xrightarrow{f} & B_2, \end{array}$$

we have $f^*[q] = [p]$.

PROOF: (i) This follows since the action as in Definition 2.1.8 is given by liftings in a diagram:

$$\begin{array}{ccc} A & \xrightarrow{i \circ u} & E_1 \\ i_0 \downarrow & \nearrow \theta & \downarrow \cong \\ A \times I & \xrightarrow{h} & B \\ & & \downarrow p_2 \\ & & E_2 \end{array}$$

The action of on u is given by the lift θ which factors through the fibre. Lifting to E_1 and composing with the weak equivalence $E_1 \rightarrow E_2$ yields a lift for E_2 . Therefore, the induced weak equivalence of the fibres is equivariant for the ΩB -action.

(ii) Note that the injective model structure on simplicial presheaves is a proper model category, see Proposition 2.1.19. By Proposition 2.3.3, $F \rightarrow E \rightarrow B$ is a fibre sequence for p a fibration. Now for an arbitrary map $p : E \rightarrow B$, we define $[p]$ as the fibre sequence associated to the fibration in a factorization

$$\begin{array}{ccc} E & \xrightarrow{\simeq} & \tilde{E} \\ & \searrow p & \downarrow \tilde{p} \\ & & B. \end{array}$$

We need to prove that this is independent of the factorization. We consider two replacements of p by a fibration: $p_1 : E_1 \rightarrow B$ and $p_2 : E_2 \rightarrow B$. Note that by construction we have trivial cofibrations $E_1 \xrightarrow{\cong} E$ and $E_2 \xrightarrow{\cong} E$. We then consider the lift in the following diagram:

$$\begin{array}{ccc} E & \xrightarrow{\simeq} & E_2 \\ \simeq \downarrow & & \downarrow p_2 \\ E_1 & \xrightarrow{p_1} & B. \end{array}$$

This is a weak equivalence by 2-out-of-3. As in the proof of Proposition 2.3.3 we obtain a weak equivalence between the fibres F_1 and F_2 . By (i), this morphism is equivariant for the ΩB -action, so $[p_1] = [p_2]$.

(iii) Consider the homotopy pullback square in the statement of the proposition. By [GJ99, Lemma II.8.16], there exists a factorization of q into a trivial cofibration

$i : E_2 \rightarrow \tilde{E}_2$ and fibration $\tilde{q} : \tilde{E}_2 \rightarrow B_2$ such that the induced morphism $E_1 \rightarrow B_1 \times_{B_2} \tilde{E}_2$ is a weak equivalence. Note that $f^*[q]$ is given by $B_1 \times_{B_2} \tilde{E}_2$. By (ii) we can take any fibration $\tilde{E}_1 \rightarrow B_1$ with $E_1 \rightarrow \tilde{E}_1$ a weak equivalence, and by (i), the homotopy fibres of $\tilde{E}_1$ and $B_1 \times_{B_2} \tilde{E}_2$ are weakly equivalent and the weak equivalence is ΩB -equivariant. Therefore $f^*[q] = [p]$. ■

REMARK 3.2.7 (CATEGORIES OF FIBRES) *In the previous paragraph we considered the situation with one given fibre. This in particular necessitated the restriction to fibrations all whose fibres have the same homotopy type. There are generalizations of this one could discuss for simplicial presheaves: In [May75] a notion of categories of fibres is developed, which allows to classify fibrations such that for each $x \in B$, the fibre of p over x is an object of a given category of fibres. This has also been done in the setting of equivariant topology [Fre03], which is quite close to the setting of simplicial presheaves.*

Brown Representability: The Brown representability theorem is not really a single theorem, but rather a class of results stating conditions under which a set-valued functor on a model or homotopy category is representable. The first appearance is in the article of Brown [Bro62] in which it is proven that a contravariant homotopy-continuous functor on the category of topological spaces is representable, with main application to the construction of spaces representing generalized cohomology theories. A more detailed analysis of why contravariant functors and pointed model categories are necessary assumptions was done in [Hel81]. Nowadays any reasonable textbook on algebraic topology contains a section on Brown representability for topological spaces. There has developed a large amount of literature on the Brown representability theorem for triangulated categories, for a survey see [Nee01].

There are not so many results on Brown representability for general, in particular unstable model categories. For a general model category, Brown representability usually fails, and at least some smallness assumptions are necessary. The paper [Ros05] proves Brown representability for combinatorial model categories, but for functors preserving weak colimits instead of homotopy colimits. The Brown representability theorem for compactly generated model categories, which is proven in [Jar06c], concerns functors preserving homotopy colimits.

There are several names for the condition on the functors: Functors that satisfy the conditions for the representability were called half-exact in [Bro62], but we use the term homotopy-continuous. The terminology homotopy-continuous is reminiscent of Mac Lane's usage of the term *continuous* for a functor which preserves limits [Mac98, Section V.4]. Homotopy-continuous functors are the model category analogues of such continuous functors, as the Brown representability is a version of the adjoint functor theorem for model categories.

DEFINITION 3.2.8 (HOMOTOPY-CONTINUOUS FUNCTOR) *A functor $F : \mathcal{C}^{\text{op}} \rightarrow \text{Set}_*$ on a pointed model category $\mathcal{C}$ is called homotopy-continuous if it satisfies the following assumptions:*

(HC i) F takes weak equivalences to bijections.

(HC ii) $F(\text{pt}) = \{*\}$.

(HC iii) For any coproduct $\bigvee_{\alpha} X_{\alpha}$ of a set $\{X_{\alpha}\}$ of objects of $\mathcal{C}$ the following wedge axiom is satisfied:

$$F\left(\bigvee_{\alpha} X_{\alpha}\right) = \prod_{\alpha} F(X_{\alpha}).$$

(HC iv) For any homotopy pushout

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & & \downarrow \\ B & \longrightarrow & Y, \end{array}$$

the induced morphism is surjective:

$$F(Y) \twoheadrightarrow F(B) \times_{F(A)} F(X).$$

This is called the Mayer-Vietoris axiom.

We will use Jardine's version of the Brown representability [Jar06c, Theorem 8] as this formulation is closest to our problems.

THEOREM 3.2.9 (BROWN REPRESENTABILITY) *For a pointed, left proper, compactly generated model category $\mathcal{C}$ and a homotopy-continuous functor $F : \mathcal{C}^{\text{op}} \rightarrow \text{Set}_*$, there exists an object Y of $\mathcal{C}$, a universal element $u \in F(Y)$, and a natural isomorphism*

$$T_u : \text{Hom}_{\text{Ho}(\mathcal{C})}(X, Y) \xrightarrow{\cong} F(X) : f \mapsto f^*(u)$$

for any object X of $\mathcal{C}$.

COROLLARY 3.2.10 *Let $\mathcal{C}$ be any left proper, compactly generated model category, let $F, G : \mathcal{C}^{\text{op}} \rightarrow \text{Set}_*$ be homotopy-continuous functors with classifying spaces Y_F resp. Y_G and universal elements u_F resp. u_G . For any natural transformation $T : F \rightarrow G$ there exists a morphism $f : Y_F \rightarrow Y_G$, unique up to homotopy, such that the following diagram commutes for all $X \in \mathcal{C}$:*

$$\begin{array}{ccc} \text{Hom}_{\text{Ho}(\mathcal{C})}(X, Y_F) & \xrightarrow{f_*} & \text{Hom}_{\text{Ho}(\mathcal{C})}(X, Y_G) \\ T_{u_F}(X) \downarrow & & \downarrow T_{u_G}(X) \\ F(X) & \xrightarrow{T(X)} & G(X) \end{array}$$

PROOF: We set $X = Y_F$. Then $T(Y_F) \circ T_{u_F}(Y_F)(\text{id})$ yields an element of $G(X)$. By representability, we have that $T_{u_G}(Y_F)$ is an isomorphism and hence the element above is of the form $T_{u_G}(Y_F)(f)$ for a morphism $Y_F \rightarrow Y_G$, which is unique up to homotopy. $\blacksquare$

REMARK 3.2.11 *This in particular implies that the classifying spaces constructed above are unique up to weak equivalence. Note that we see from Corollary 3.2.10 that there are no problems with phantom maps in model categories of presheaves. Since we have seen that they do not obstruct Brown representability for morphisms, they probably form a square zero ideal as in the topological situation.*

From Proposition 2.2.10 we know that all the model categories of simplicial presheaves that are interesting for us are indeed compactly generated.

COROLLARY 3.2.12 *The injective and projective model structures associated to the Zariski or Nisnevich topology on the category $\mathbf{Sm}_S$ of smooth schemes over a finite type scheme S do satisfy Brown representability.*

If S is a field admitting resolution of singularities, then the cdh-topology on $\mathbf{Sch}_S$ and the scdh-topology on $\mathbf{Sm}_S$ satisfy Brown representability.

REMARK 3.2.13 *It may be possible that some version of the Brown representability theorems also holds for sites like the étale site. Maybe it already follows from Rosický's work [Ros05]. The advantage of using this version of the Brown representability is that it applies to all model categories of simplicial presheaves, since these are always combinatorial, cf. Proposition 2.1.19. However, Rosický works with weak colimits, and an investigation of the applicability of his results would involve comparing weak colimits and homotopy colimits. We will therefore not discuss such issues here and content ourselves with the Brown representability for cd-topologies.*

Proof of Homotopy-Continuity: In this paragraph we will prove that the functor $\mathcal{H}(-, F)$ from Definition 3.2.1 is homotopy-continuous. Then, by applying the Brown representability theorem discussed above, we get classifying spaces for fibre sequences and universal fibrations.

First note that $\mathcal{H}(*, F)$ is the singleton set consisting of the fibre sequence $F \rightarrow F \rightarrow *$, settling (HC ii).

The next serious thing to do is to show (HC i), i.e. that the functor $\mathcal{H}(-, F)$ is homotopically meaningful, in the sense that it carries weak equivalences between simplicial presheaves to isomorphisms of (pointed) sets. This implies in particular that there is a right derived functor $\mathbf{R}\mathcal{H}(-, F) : \mathbf{Ho}C^{op} \rightarrow \mathbf{Set}_*$.

PROPOSITION 3.2.14 *The functor $\mathcal{H}(-, F)$ sends weak equivalences of simplicial presheaves to bijections of pointed sets.*

PROOF: Let $f : X \rightarrow Y$ be a weak equivalence, and consider $f^* : \mathcal{H}(Y, F) \rightarrow \mathcal{H}(X, F)$.

To show f^* is surjective, let $F \rightarrow E \xrightarrow{p} X$ be a fibre sequence with p a fibration. Consider the following diagram:

$$\begin{array}{ccc} E & \xrightarrow{i} & \tilde{E} \\ p \downarrow & \simeq & \downarrow \tilde{p} \\ X & \xrightarrow{f} & Y. \end{array}$$

Therein, $\tilde{E}$ is obtained by factoring $f \circ p$ into a trivial cofibration i and a fibration $\tilde{p}$. Since both i and f are weak equivalences, this square is a homotopy pullback. Hence by applying Proposition 3.2.6 we get $f^*\tilde{p} = p$.

To see that f^* is injective, let $p_1 : F \rightarrow E_1 \rightarrow Y$ and $p_2 : F \rightarrow E_2 \rightarrow Y$ be fibre sequences whose pullbacks are equivalent, i.e. $f^*p_1 = f^*p_2 \in \mathcal{H}(X, F)$. We assume that p_1 and p_2 are actually fibrations. By properness and the fact that f is a

weak equivalence, we obtain weak equivalences $E_i \times_Y X \simeq f^*E_i \rightarrow E_i$. Therefore, the fibre sequences $F \rightarrow f^*E_i \rightarrow X$ and $F \rightarrow E_i \rightarrow Y$ are isomorphic in the homotopy category. Since we also assumed that $f^*p_1 = f^*p_2$, we have isomorphisms of fibre sequences in the homotopy category, which are also equivariant:

$$E_1 \xrightarrow{\simeq} f^*E_1 \xrightarrow{\simeq} f^*E_2 \xleftarrow{\simeq} E_2.$$

Thus p_1 and p_2 are equivalent fibre sequences. ■

In the following propositions we will prove the two main parts of homotopy-continuity of the functor $\mathcal{H}(-, F)$, namely the Mayer-Vietoris and the wedge property. This is the point where we make essential use of the homotopy distributivity. This remarkable property allows to glue together fibrations defined on a covering of the base. The outcome will not be a fibration, but we still can determine the homotopy fibre, and therefore, using the work from Section 2.3 and Proposition 3.2.6, we know what the associated fibre sequence looks like. This is also the key argument in the work of Allaud [All66], although there is a lot more to do if one wants to work with homotopy equivalences of CW-complexes.

PROPOSITION 3.2.15 (MAYER-VIETORIS AXIOM) *Let $B_1 \xleftarrow{\iota_1} B_0 \xrightarrow{\iota_2} B_2$ be a diagram of simplicial presheaves. We assume without loss of generality that $\iota_1 : B_0 \hookrightarrow B_1$ is in fact a cofibration, so that the homotopy pushout is given by the point-set pushout: $B := B_1 \cup_{B_0}^h B_2 = B_1 \cup_{B_0} B_2$.*

Then the induced morphism

$$\mathcal{H}(B_1 \cup_{B_0} B_2, F) \twoheadrightarrow \mathcal{H}(B_1, F) \times_{\mathcal{H}(B_0, F)} \mathcal{H}(B_2, F)$$

is surjective.

PROOF: What we have to show is the following: Assume given two fibre sequences $F \rightarrow E_1 \rightarrow B_1$ and $F \rightarrow E_2 \rightarrow B_2$, such that the corresponding pullbacks $\iota_1^*E_1$ and $\iota_2^*E_2$ are isomorphic fibre sequences via a given isomorphism $\rho : \iota_1^*E_1 \xrightarrow{\cong} \iota_2^*E_2$. Then we have to show that there is a fibre sequence over B inducing them compatibly.

We will apply Mather's cube theorem from Corollary 3.1.15. Since we know that the pullbacks of the fibre sequences are equivalent, the squares in the diagram are homotopy pullback squares:

$$\begin{array}{ccccc} E_1 & \xleftarrow{\iota_1^*E_1 \cong \iota_2^*E_2} & & \xrightarrow{\quad} & E_2 \\ p_1 \downarrow & & \downarrow & & \downarrow p_2 \\ B_1 & \xleftarrow{\iota_1} & B_0 & \xrightarrow{\iota_2} & B_2. \end{array}$$

The homotopy fibres of the vertical arrows are all weakly equivalent to F . Then Mather's cube theorem produces the following fibre sequence:

$$F \rightarrow E_1 \cup_{E_0}^h E_2 \xrightarrow{p} B \cong B_1 \cup_{B_0}^h B_2.$$

Actually, we only get the morphism p , and know that its homotopy fibre is F . Then we still have to do some kind of rectification as in Section 2.3 to really get a fibre sequence with total space $E := E_1 \cup_{E_0}^h E_2 \in \mathcal{H}(B, F)$.

What is left to show is that pulling back E to B_i yields equivalences $\phi : E_1 \xrightarrow{\cong} E$ and $\psi : E_2 \xrightarrow{\cong} E$ such that over B_0 we have $\phi = \psi \circ \rho$. This also follows from homotopy distributivity: We assume p has been rectified to a fibration. Since the squares

$$\begin{array}{ccc} E_i & \longrightarrow & E \\ \downarrow p_i & & \downarrow p \\ B_i & \longrightarrow & B \end{array}$$

are homotopy pullback squares, the map $E_i \rightarrow E$ factors through a unique weak equivalence from E_i to the point-set pullback of E along $B_i \rightarrow B$. This provides the equivalences ϕ and ψ . All we need to show is that the following diagram is commutative:

$$\begin{array}{ccccc} & & \iota_1^* E_1 & & \\ & \swarrow & \downarrow \rho & \searrow & \\ B_0 & \longleftarrow & \iota_2^* E_2 & \longrightarrow & E \\ & \swarrow & \downarrow \psi & \searrow & \\ & & E \times_B B_0 & & \end{array}$$

By the universal property of the pullback, this implies that $\psi \circ \rho = \phi$, since the morphism $\iota_1^* E_1 \rightarrow E \times_B B_0$ is by definition ϕ .

The upper left triangle commutes, since ρ was defined over B_0 . The lower triangles commute because ψ was defined using the universal property of the pullback $E \times_B B_0$. The upper right triangle commutes because E was defined as the glueing of $\iota_1^* E_1$ and $\iota_2^* E_2$ along ρ . Therefore, we get that $\phi = \psi \circ \rho$. In particular, the image of $E \in \mathcal{H}(B, F)$ in $\mathcal{H}(B_1, F) \times_{\mathcal{H}(B_0, F)} \mathcal{H}(B_2, F)$ is exactly the class of (E_1, E_2, ρ) we started with. $\blacksquare$

PROPOSITION 3.2.16 (WEDGE AXIOM) *Let X_α with $\alpha \in I$ be a set of pointed simplicial presheaves. Then*

$$\mathcal{H}\left(\bigvee_{\alpha} X_{\alpha}, F\right) \rightarrow \prod_{\alpha} \mathcal{H}(X_{\alpha}, F) : E \mapsto \iota_{\alpha}^*(E)$$

is a bijection, with $\iota_{\alpha} : X_{\alpha} \hookrightarrow \bigvee_{\alpha} X_{\alpha}$ the inclusion. In particular, $\mathcal{H}(\bigvee_{\alpha} X_{\alpha}, F)$ is a set if $\mathcal{H}(X_{\alpha}, F)$ is a set for every $\alpha \in I$.

PROOF: By homotopy distributivity applied to the homotopy colimit diagrams $\bigvee_{\alpha} p_{\alpha} : \bigvee_{\alpha} E_{\alpha} \rightarrow \bigvee_{\alpha} X_{\alpha}$, we find that all the following squares are homotopy pullbacks:

$$\begin{array}{ccc} E_{\alpha} & \longrightarrow & \bigvee_{\alpha} E_{\alpha} \\ \downarrow & & \downarrow \\ B_{\alpha} & \longrightarrow & \bigvee_{\alpha} B_{\alpha}. \end{array}$$

Therefore, the homotopy fibre of $\bigvee_{\alpha} p_{\alpha}$ is also F . Rectifying it to a fibre sequence, we get $\bigvee_{\alpha} E_{\alpha} \in \mathcal{H}(\bigvee_{\alpha} X_{\alpha}, F)$. This proves surjectivity.

Now assume given two fibre sequences E_1 and E_2 over $\bigvee_{\alpha} X_{\alpha}$. We first prove the weak equivalence $\bigvee_{\alpha} \iota_{\alpha}^* E \simeq E$ for any fibre sequence $E \rightarrow \bigvee_{\alpha} X_{\alpha}$. This follows from distributivity in categories of simplicial presheaves, since E is isomorphic to the colimit of $\iota_{\alpha}^* E_{\alpha}$. Then one can either use that the wedge is already the homotopy direct product, or again appeal to the homotopy distributivity. Then we have the sequence of weak equivalences:

$$E_1 \simeq \bigvee_{\alpha} \iota_{\alpha}^* E_1 \simeq \bigvee_{\alpha} \iota_{\alpha}^* E_2 \simeq E_2.$$

The middle weak equivalence follows from the fact [Jar96, Lemma 13.(3)] that for a set-indexed collection of weak equivalences $f_{\alpha} : X_{\alpha} \rightarrow Y_{\alpha}$, the morphism $\bigvee_{\alpha} f_{\alpha}$ is also a weak equivalence and the assumption that $\prod \iota_{\alpha}^* E_1 = \prod \iota_{\alpha}^* E_2$ in $\prod_{\alpha} \mathcal{H}(X_{\alpha}, F)$.

The set theory statement is then clear, since a set-indexed product of sets is again a set. $\blacksquare$

THEOREM 3.2.17 *The functor $\mathcal{H}(-, F)$ is homotopy continuous and therefore representable by a space $B^f F$. The space $B^f F$ is unique up to weak equivalence.*

The universal element $u_F \in \mathcal{H}(B^f F, F)$ corresponds to the universal fibre sequence of simplicial presheaves with fibre F :

$$F \rightarrow E^f F \rightarrow B^f F$$

We use the notation $B^f F$ to distinguish from other possible classifying spaces we will discuss later on.

Using homotopy distributivity once again, we can construct change-of-fibre natural transformations, which via Brown representability for morphisms give rise to morphisms between the corresponding classifying spaces.

PROPOSITION 3.2.18 *For any morphism $f : F \rightarrow F'$ of simplicial presheaves, there is a natural transformation $\mathcal{H}(-, F) \rightarrow \mathcal{H}(-, F')$. By Brown representability this is representable by a morphism $B^f F \rightarrow B^f F'$.*

PROOF: We start with a fibre sequence $F \rightarrow E \rightarrow B$. For any morphism $F \rightarrow F'$ we can consider the new total space $E' = E \cup_F^h F'$. We note that since the morphism $F \rightarrow B$ is trivial, the morphism $F \rightarrow F'$ is a B -morphism for F and F' with the trivial structure morphisms $*$: $F \rightarrow B$. By the universal property of the pushout, we have a morphism $E' \rightarrow B$.

Homotopy distributivity implies that the homotopy fibre of $E' \rightarrow B$ is F' : We write E' as the homotopy colimit

$$\begin{array}{ccc} F & \longrightarrow & E \\ \downarrow & & \downarrow \\ F' & \longrightarrow & E' \end{array}$$

The diagram of homotopy fibres is then given by

$$\begin{array}{ccc}
F \times \Omega B & \longrightarrow & F \\
\downarrow & & \downarrow \\
F' \times \Omega B & \longrightarrow & F'.
\end{array}$$

This is easily seen to be a homotopy pushout and by Proposition 3.1.16 the homotopy fibre of $E' \rightarrow B$ is F' . Now we just rectify it to a fibre sequence over B , using Proposition 3.2.6. This is natural, since it is constructed using universal constructions, and the class in $\mathcal{H}(-, F')$ does not depend on choice of rectification. Therefore, we have a natural transformation $\mathcal{H}(-, F) \rightarrow \mathcal{H}(-, F')$ for any morphism of simplicial presheaves $F \rightarrow F'$.

Using Brown representability from Corollary 3.2.10 we have a morphism of classifying spaces $B^f F \rightarrow B^f F'$. $\blacksquare$

In the following we will describe some interesting operations on fibre sequences generalizing well-known operations on bundles. For topological spaces a discussion of such operations can be found in [Rud98, Proposition 1.43] or [May80].

Let $F_1 \rightarrow E_1 \rightarrow B_1$ and $F_2 \rightarrow E_2 \rightarrow B_2$ be two fibre sequences. Then the product of these is given by $F_1 \times F_2 \rightarrow E_1 \times E_2 \rightarrow B_1 \times B_2$. This product operation yields a natural transformation $\mathcal{H}(B_1, F_1) \times \mathcal{H}(B_2, F_2) \rightarrow \mathcal{H}(B_1 \times B_2, F_1 \times F_2)$.

There are external operations starting from fibre sequences over B_1 and B_2 , applying a change-of-fibre to the product of the fibre sequences to get a fibre sequence over $B_1 \times B_2$. For $B_1 = B_2$ one can also define internal operations by base change along the diagonal. We first define the external operations.

The fibrewise smash product is given by applying Proposition 3.2.18 to the change-of-fibre morphism $F_1 \times F_2 \rightarrow F_1 \wedge F_2$. Composing with the product, we get an external operation $\mathcal{H}(B_1, F_1) \times \mathcal{H}(B_2, F_2) \rightarrow \mathcal{H}(B_1 \times B_2, F_1 \wedge F_2)$.

Since suspension is defined by $\Sigma X = S_s^1 \wedge X$, we can also define fibrewise suspension by first taking the product $S^1 \times F \rightarrow B \times S^1 \times E \rightarrow B \times B$, where the S^1 -fibre sequence is trivial, and then changing the fibre along $S^1 \times X \rightarrow \Sigma X$. Base change along the diagonal yields the fibrewise suspension $\mathcal{H}(B, F) \rightarrow \mathcal{H}(B, \Sigma F)$.

Similarly, there is an external fibrewise join $\mathcal{H}(B_1, F_1) \times \mathcal{H}(B_2, F_2) \rightarrow \mathcal{H}(B_1 \times B_2, F_1 * F_2)$, where the join of simplicial presheaves is defined by $F_1 * F_2 = \Sigma(F_1 \wedge F_2)$.

The corresponding internal operations are defined by base change along the diagonal $\Delta : B \rightarrow B \times B$, the fibrewise smash product over the base B being a natural transformation $\oplus : \mathcal{H}(B, F_1) \times \mathcal{H}(B, F_2) \rightarrow \mathcal{H}(B, F_1 \wedge F_2)$. The notation $\oplus$ should emphasize the similarity of the fibrewise smash with the Whitney sum of bundles.

COROLLARY 3.2.19 *The operations defined above induce morphisms of classifying spaces, i.e. there are morphisms of classifying spaces associated to fibrewise smash*

$$B^f(\wedge) : B^f F_1 \times B^f F_2 \rightarrow B^f(F_1 \wedge F_2),$$

and fibrewise suspension $B^f F \rightarrow B^f \Sigma F$. In particular, for any H -group structure $\circ : F \wedge F \rightarrow F$ on F , we obtain an H -group structure on the classifying space $B^f F$ by composing $B^f(\wedge)$ with the change-of-fibre along the operation $\circ$.

Similarly, there are morphisms from usual classifying spaces of sheaves of groups to classifying spaces for fibre sequences. For a group G acting on F via $\phi : G \times F \rightarrow F$, take a principal fibration with structure group G and a given trivialization. Then replace the group fibres by F , and use the transition maps induced via ϕ by the transition maps of the principal G -bundle. This produces a locally trivial fibre sequence with fibre F . In the following we use the change-of-fibre morphisms to make this precise.

PROPOSITION 3.2.20 *Let F be any simplicial presheaf, let G be any sheaf of groups and let an action $\phi : G \times F \rightarrow F$ be given. Then there exists a morphism of classifying spaces $BG \rightarrow B^f F$.*

Note that the action above is via homotopy self-equivalences, and the group of homotopy self-equivalences contains the group of automorphisms of F in the category of sheaves.

PROOF: The idea for the construction is similar to what we did for the fibrewise smash products: We want to assign to any principal G -fibration a fibre sequence with fibre F in a functorial way to obtain a natural transformation $\mathrm{Hom}(-, BG) \rightarrow \mathcal{H}(-, F)$.

Let a principal G -fibration $G \rightarrow E \rightarrow B$ be given. We take the product with the trivial F -fibre sequence and obtain $G \times F \rightarrow E \times F \times B \rightarrow B \times B$. Then we apply the change-of-fibre along the action $G \times F \rightarrow F$. Finally, we pull back the resulting F -fibre sequence over $B \times B$ along the diagonal $\Delta : B \rightarrow B \times B$.

Now we just apply Proposition 3.2.18 to obtain the morphisms of the classifying spaces. ■

Comparison with Related Work: In this paragraph, we want to compare the classifying spaces we have constructed in Theorem 3.2.17 to other classifying spaces that have been described previously.

We first want to note that classifying spaces for fibre sequences of simplicial presheaves have not yet been considered in the literature. There is of course work on torsors under simplicial presheaves of groupoids and classifying spaces for such objects. The most important references seem to be [JT93], [JL06] and [MV99].

The work of Joyal and Tierney [JT93] considers classifying spaces for simplicial groupoids. This allows to construct classifying spaces for torsors under the simplicial groupoid of homotopy automorphisms of a simplicial presheaf X .

Similar results are proven in [JL06] resp. [Jar06a]. This provides an identification of the set $H^1(*, G)$ of G -torsors over the final object, and the set of homotopy classes of morphisms $* \rightarrow BG$. One would however like this identification to hold for an arbitrary simplicial presheaf X instead of only the final object $*$.

On the other hand, Morel and Voevodsky in [MV99] provide the identification of torsors in $H^1(X, G)$ and homotopy classes of morphisms $X \rightarrow BG$, but only discuss the construction of classifying spaces for a simplicial group G of simplicial dimension zero.

Finally, Waner constructed classifying spaces for fibre sequences of G -CW-complexes in [Wan80]. This setting is similar to the simplicial presheaves setting, since the G -CW-complexes are interpreted as presheaves on the orbit category of the group G , the difference being that there is no Grothendieck topology on the

orbit category involved. Still, the equivariant setting can always be used as a source of examples for statements on simplicial presheaves.

Restricting to fibre sequences which are locally trivial in some Grothendieck topology on the site T , it is possible to compare torsors under the simplicial group sheaf $\mathrm{Aut}_\bullet(F)$ of homotopy automorphisms of F to fibre sequences with fibre F as follows: Using Proposition 3.2.20, it is possible to construct a morphism $B\mathrm{Aut}_\bullet(F) \rightarrow B^f F$ which associates to each torsor a fibre sequence. This morphism is usually not a weak equivalence, as the examples below will illustrate. Note also that we have to require that the fibre is identifiable, in the sense that there is a global morphism $F \rightarrow E$. This might fail for étale coverings with finite structure group, cf. Example 3.2.21. If we restrict to fibre sequences which are locally trivial in the topology T , as in the proof of Proposition 3.2.5, and every $\mathrm{Aut}_\bullet(F)$ -torsor also yields a fibre sequence, then the above morphism will be a weak equivalence. To prove this, it would suffice to check the identification of morphisms $U \rightarrow B\mathrm{Aut}_\bullet(F)$ with morphisms $U \rightarrow B^f F$ for representable schemes U , since the behaviour of the above objects under homotopy colimits is the same and therefore this identification extends to general simplicial presheaves. The general description of locally trivial fibre sequences in terms of torsors under the homotopy automorphism group sheaves $\mathrm{Aut}_\bullet(F)$ is left to future work.

We now want to compare the classifying spaces we constructed above to the constructions of classifying spaces in [MV99] in some interesting zero-dimensional cases. Let G be a sheaf of groups. We denote by BG the classifying space of G -torsors, and by $B^f G$ the classifying space of G -fibre sequences. It is interesting to compare the two: First, the existence of a reduction of a fibre sequence to a G -torsors depends on the topology in which we can trivialize the fibre sequence. No étale locally trivial G -fibre sequences for a finite group G yields Nisnevich torsors under G unless it is trivial, cf. Example 3.2.21. Secondly, the group sheaf G can have outer automorphisms, which also yield fibre sequences, which do not come from torsors. Finally, even if there is a reduction of the fibre sequence to a torsor, the uniqueness is not guaranteed. (Thanks to Denis-Charles Cisinski for insisting on this point, thinking about his questions made me understand a little more the subtle interplay of topology and geometry in the classifying spaces constructed above.)

We will deal with two major examples, namely finite groups and $\mathbb{G}_m$.

EXAMPLE 3.2.21 We first discuss the case of classifying spaces for a finite group scheme G , comparing the three objects: BG , $B^f G$ and $B_{\mathrm{\acute{e}t}} G$.

The first of these classifies G -torsors in the Nisnevich topology. Since all of these are trivial, the homotopy groups of BG are

$$\pi_i(BG) = \begin{cases} G & i = 1 \\ 0 & \text{otherwise,} \end{cases}$$

where G is the sheaf represented by the group scheme G . For a finite group G , the group $H_{\mathrm{Nis}}^1(X, G)$ is trivial. This is basically due to the fact that no finite étale morphism is rationally trivial unless it is trivial. This implies that the classifying space BG of Nisnevich G -torsors for a finite G has trivial π_0 .

The space $B_{\text{ét}}G$ classifies étale coverings with Galois group G . Therefore its homotopy groups are given by

$$\pi_i(B_{\text{ét}}G) = \begin{cases} H_{\text{ét}}^1(-, G) & i = 0 \\ G & i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

This is proven in [MV99, Corollary 4.3.2].

Now it is not exactly clear what the group of connected components of $B^f G$ is. We can require that all the fibre sequences are locally trivial in the étale topology, i.e. that for any morphism $U \rightarrow B$ into the base of the fibre sequence $G \rightarrow E \rightarrow B$, the pullback $G \rightarrow E \times_B U \rightarrow U$ is an étale morphism. Then any fibre sequence will come from an étale torsor resp. would be strictifiable to an étale torsor, whence we would get a morphism $B^f G \rightarrow B_{\text{ét}}G$.

On the other hand, some of the étale G -torsors are obviously not fibre sequences with fibre G . Consider for example a finite field extension with Galois group G . This is classified by a morphism into $B_{\text{ét}}G$, but it is not a fibre sequence with fibre $\mathbb{Z}/2$. The reason is simply that there is no morphism $\mathbb{Z}/2 = \text{Spec } k \oplus k \rightarrow \text{Spec } L$ for L/k a finite field extension, not even in the homotopy category. The morphism $\text{Spec } L \rightarrow \text{Spec } k$ is just a trivial fibre sequence with fibre $\text{Spec } L$.

Any Nisnevich torsor over a smooth scheme under a finite group is trivial as noted above, and we obtain obvious morphisms $BG \rightarrow B_{\text{ét}}G$ and $BG \rightarrow B^f G$.

So the spaces $B^f G$, BG and $B_{\text{ét}}G$ are pairwise distinct for finite G .

We finally give an example of a geometric fibre sequence with fibre $\mathbb{Z}/2$ which is classified by $B^f G$: The Enriques surface yields one example of a fibre sequence with $\mathbb{Z}/2$ -fibre: We consider any rational point $x \in X$, which lifts to two rational points on Y , the Kummer surface covering X . Then we have an explicit morphism $\mathbb{Z}/2 \rightarrow Y$, and we only need to check it is really the inclusion of the fibre. Since $Y \rightarrow X$ is a $\mathbb{Z}/2\mathbb{Z}$ -torsor, it is étale-locally a projection away from $\mathbb{Z}/2$, which proves the claim. The only problem is that this argument has to be done in the étale topology, and we have to transfer it to the Nisnevich topology as follows: Since it is a torsor in the étale topology, it is classified by a map to $B_{\text{ét}}G$, which implies the lifting property exactly as in the work of Morel [Mor06b]. Then we just need to show that the inclusion of $\mathbb{Z}/2$ is really the fibre, which we check locally. Another way to see that this fibre sequence in the étale topology pushes forward to a fibre sequence in the Nisnevich topology is the paragraph on change-of-site in Section 3.3.

One could imagine many more examples of such geometric fibre sequences with finite fibres, and we find that there are morphisms with fibre a finite group which are not locally trivial in the Nisnevich topology, but still are fibrations with fibre this finite group. We will see more examples of this type in the next section. $\square$

EXAMPLE 3.2.22 We now turn to the classifying spaces for the group scheme $\mathbb{G}_m$: In this case, we know by homotopy invariance of the group of units and the Picard group that $B\mathbb{G}_m$ and $B_{\text{ét}}\mathbb{G}_m$ are weakly equivalent and the homotopy groups are given by

$$\pi_i(B\mathbb{G}_m) = \pi_i(B_{\text{ét}}\mathbb{G}_m) = \begin{cases} \text{Pic}(-) & i = 0 \\ \mathcal{O}^*(-) & i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

This is described in [MV99, Proposition 4.3.8] and is due to the fact that any étale $\mathbb{G}_m$ -torsor has a rational section and therefore is also a torsor under $\mathbb{G}_m$ in the Nisnevich topology.

Any $\mathbb{G}_m$ -torsor induces a fibre sequence with fibre $\mathbb{G}_m$, therefore we have a morphism $B_{\text{ét}}\mathbb{G}_m \cong B\mathbb{G}_m \rightarrow B^f\mathbb{G}_m$.

However, not every fibre sequence with fibre $\mathbb{G}_m$ is a torsor. There is an outer automorphism

$$\mathbb{G}_m \rightarrow \mathbb{G}_m : x \mapsto x^{-1},$$

and we can use this to construct fibre sequences with fibre $\mathbb{G}_m$ which do not come from $\mathbb{G}_m$ -torsors: We take an étale covering of X of degree 2, trivialize it in the étale topology and glue in $\mathbb{G}_m$ as a fibre with the action of $\mathbb{Z}/2\mathbb{Z}$ given by the above automorphism. This induces a $\mathbb{G}_m$ -fibre sequence by an argument similar to the previous example. One should note however that the fibre sequence is not locally trivial in the Nisnevich topology, since the $\mathbb{Z}/2\mathbb{Z}$ -covering is not locally trivial in the Nisnevich topology.

We see that there are several fibre sequences with fibre $\mathbb{G}_m$ not coming from torsors, but rather being induced from outer automorphisms. If we restrict to fibre sequences which are étale locally trivial, then there should be a decomposition

$$B^f\mathbb{G}_m \simeq B\mathbb{G}_m \times B_{\text{ét}}(\mathbb{Z}/2\mathbb{Z}).$$

The interesting thing to note is that the first factor is $\mathbb{P}^\infty$, which upon complex realization yields $\mathbb{C}P^\infty$, whereas the second is the real projective space $\mathbb{R}P^\infty$. $\square$

The question of strictification can conveniently be formulated in the framework of classifying spaces. As we saw earlier, there is a morphism $BG \rightarrow B^fF$ from the classifying space of G -torsors in the Nisnevich topology to the space classifying fibre sequences with fibre F , where F is any simplicial presheaf with a G -action. The obstructions and classification invariants needed to decide if a given G -fibre sequence comes from a G -torsor are then all contained in the homotopy type of the homotopy fibre of the morphism $BG \rightarrow B^fF$. One example of this situation is discussed in the next paragraph.

Note that there are many interesting classifying spaces we have not yet discussed as the examples above only referred to zero-dimensional fibres.

Remarks on Applications: One application of classifying spaces for fibre sequences is discussed at length in Chapter 4. There we will describe how the existence of classifying spaces for fibre sequences implies the possibility to construct fibrewise localizations of such fibre sequences.

Another possible application we want to mention is a yet to be developed surgery theory for schemes. This idea is due to Fabien Morel, mentioned e.g. in [Mor06a] and aims at a version of the surgery program to algebraic varieties at least over a field. We recall how important classifying spaces of various structures are in the surgery theory and classification of manifolds, for details see e.g. [MM79] and references therein. The first obstruction for a homotopy type X to be the homotopy type of a closed topological, PL or smooth manifold is *finiteness* and

the existence of a fundamental class $[X] \in H_n(X, \mathbb{Z})$ inducing the Poincaré duality isomorphism

$$- \cap [X] : H^i(X, \mathbb{Z}) \rightarrow H_{n-i}(X, \mathbb{Z}).$$

Such finite CW-complexes are called Poincaré duality spaces. Note that we restrict to simply-connected CW-complexes for simplicity. To such a space one can associate a spherical fibration, i.e. a fibre sequence $S^k \rightarrow E \rightarrow X$ with a sphere as fibre. This is called the *Spivak normal fibration*. For a smooth manifold, this normal fibration is given by the normal sphere bundle coming out of the Thom construction applied to the tangent bundle. A necessary condition for X to have a PL or smooth manifold structure is then the existence of a reduction of the Spivak normal fibration to a vector bundle with a smooth or PL structure.

Denoting as in [MM79] by BG the classifying space of spherical fibrations, the Spivak normal fibration is classified by a morphism $f : X \rightarrow BG$. There are also classifying spaces for topological, smooth or PL structures on such fibrations, which are denoted by $BTOP$, BO resp. BPL . The existence of smooth manifold structures on X is then related to the existence of liftings of the morphism $f : X \rightarrow BG$ along the morphism $\phi : BO \rightarrow BG$. The latter comes out of the Thom construction applied to an orthogonal vector bundle. Denoting the homotopy fibre of ϕ by G/O , the normal cobordism classes of smooth structures on X are then classified by the different possible lifts of f along ϕ , which are exactly the morphisms $X \rightarrow G/PL$. The same procedure works for topological resp. PL manifold structures on X .

Therefore, a manifold structure can be seen as a homotopy type having Poincaré duality together with some tangent data [Sul05]. The problem of classifying manifold structures on a given CW-complex X then reduces to describing the various classifying spaces, and the homotopy fibres of morphisms $BPL \rightarrow BG$ or $BO \rightarrow BG$. Indeed, most of the work in [MM79] is devoted to computing cohomology and homotopy groups of G/PL , G/TOP and other related spaces. A very famous result of Kirby and Siebenmann can be written as $TOP/PL \simeq K(\mathbb{Z}/2, 3)$. This gives a precise obstruction lying in $H^4(X, \mathbb{Z}/2)$ to the existence of PL structures on a topological manifold X , the different PL structures are then classified by $H^3(X, \mathbb{Z}/2)$. This enabled the construction of counterexamples to the triangulation conjecture and the Hauptvermutung of combinatorial topology.

As another famous application of classifying spaces of spherical fibrations, we want to mention Sullivan's proof of the Adams conjecture. This proof proceeds via fibrewise localization and completion of spherical fibrations, which is most naturally expressed in the setting of classifying spaces of such spherical fibrations. This was pointed out and investigated by May in [May80].

It is still rather unclear if there is any way to make a surgery theory work for schemes over a field, let alone over a more general base scheme S . Nevertheless, we can try to develop the homotopical side, looking at the analogues of spherical fibrations and their classifying spaces in the homotopy theory of simplicial sheaves on Sm_S resp. in $\mathbb{A}^1$ -homotopy theory. Note however that the geometric significance of these constructions remains yet to be determined.

Now in $\mathbb{A}^1$ -homotopy there are more spheres than in topology. First, there are not just the simplicial spheres S_s^n which are the constant simplicial presheaves $U \mapsto S^n = \Delta^n / \partial \Delta^n$. There is at least one other interesting sphere, namely the

representable sheaf $S_t^1 = \mathbb{G}_m$. These give rise to a bi-indexed collection of spheres $S^{p,q} = S_s^{p-q} \wedge S_t^q$. Up to $\mathbb{A}^1$ -homotopy equivalence, there are geometric models for some of these spheres, e.g. $S^{2,1} = \Sigma_s \mathbb{G}_m \simeq \mathbb{P}^1$, $S^{2n,n} \simeq \mathbb{A}^n / (\mathbb{A}^n \setminus \{0\}) \simeq \mathbb{P}^n / \mathbb{P}^{n-1}$ and $S^{2n-1,n} \simeq \mathbb{A}^n \setminus \{0\}$.

The problem with these many spheres is that they can only be identified after localizing $\mathbb{A}^1$, and these various different models of these spheres give rise to possibly different theories of fibre sequences and different classifying spaces in the simplicial model structure before localization.

We therefore chose to work with a fibrant and $\mathbb{A}^1$ -local model of the spheres. The fibrewise localization which we will discuss in Chapter 4 then allows to construct localization morphisms $B^f S^{p,q} \rightarrow B^f L_{\mathbb{A}^1} S^{p,q}$ on the classifying spaces via Brown representability for morphisms. A spherical fibration is therefore a fibre sequence of simplicial presheaves

$$L_{\mathbb{A}^1} S^{p,q} \rightarrow E \rightarrow B.$$

Using Theorem 3.2.17, these spherical fibrations are classified by a space which we denote $B^f L_{\mathbb{A}^1} S^{p,q}$. Note that in particular we do not require that all the simplicial presheaves are $\mathbb{A}^1$ -local: We need to work in the model category of simplicial sheaves, since only there we know homotopy distributivity, which is the crucial point in our construction of classifying spaces.

This however gives rise to a very interesting question that does not appear in topology: Is the corresponding space $B^f L_{\mathbb{A}^1} S^{p,q}$ actually $\mathbb{A}^1$ -local? Is every spherical fibration as above actually an $\mathbb{A}^1$ -local fibre sequence? A positive answer would of course imply that the classifying space we constructed here is a classifying space for spherical fibrations in the $\mathbb{A}^1$ -local model structure, whereas a negative answer would exhibit a quite surprising twist in the theory of motivic spherical fibrations.

Another interesting question to ask is whether there are always nice “geometric” models for spherical fibrations, e.g. is any spherical fibration $L_{\mathbb{A}^1}(\mathbb{A}^n \setminus \{0\}) \rightarrow E \rightarrow B$ represented by a principal $\text{Aut}(\mathbb{A}^n \setminus \{0\})$ -bundle? This is also a question how much the spaces $\mathbb{A}^n \setminus \{0\}$ and $\text{Sing}_{\bullet}^{\mathbb{A}^1}(\mathbb{A}^n \setminus \{0\})$ have in common, and whether the latter space is quasi-fibrant etc. All such questions are questions about homotopy fibres of morphisms between the classifying spaces described above.

Although what we have described above yields unstable spherical fibrations for all the motivic spheres, those on the diagonal $B^f L_{\mathbb{A}^1} S^{2n,n}$ seem to be closest to the topological situations. The other spheres have a tendency not to behave like one would expect from topology.

Finally, the classifying spaces for stable spherical fibrations can also be constructed. We construct the space $B^f L_{\mathbb{A}^1} S^{2\bullet,\bullet}$ of stable spherical fibrations with fibre any sphere $S^{2n,n}$. These spaces are related via smash products

$$S^{p,q} \wedge S^{p',q'} \rightarrow S^{p+p',q+q'}.$$

These induce fibrewise smash product operations $B^f S^{2,1} \times B^f S^{p,q} \rightarrow B^f S^{p+2,q+1}$ by Corollary 3.2.19. Taking as first argument the trivial $S^{2,1}$ -bundle yields the fibrewise $S^{2,1}$ -suspension we are interested in. We then obtain a directed colimit diagram

$$B^f L_{\mathbb{A}^1} \mathbb{P}^1 \longrightarrow \cdots \longrightarrow B^f L_{\mathbb{A}^1} S^{2n,n} \xrightarrow{-\wedge_{\Sigma_S} \mathbb{P}^1} B^f L_{\mathbb{A}^1} S^{2n+2,n+1} \longrightarrow \cdots$$

Then we just take the homotopy colimit, and obtain the classifying space of stable spherical fibrations, which we denote $B^f L_{\mathbb{A}^1} S^{2\bullet,\bullet}$. Note that this classifying space for stable spherical fibration has the structure of an H -space due to the fibrewise smash products.

Looking at the topological situation [MM79, Corollary 3.8], we should expect that the homotopy groups of the above classifying spaces are actually the stable homotopy groups of the sphere spectrum. In lack of a good orientation theory, we can not really talk about oriented spherical fibrations. However, with the right notion of orientation, we should have the splitting of $B^f L_{\mathbb{A}^1} S^{2\bullet,\bullet}$ into one factor $K(\pi_1(B^f L_{\mathbb{A}^1} S^{2\bullet,\bullet}), 1)$ describing the possible orientations of the spheres, and the classifying space for oriented spherical fibrations, whose homotopy groups π_n for $n \geq 2$ should be the stable homotopy groups of the sphere spectrum.

There is however one construction that we can do, which is the motivic analogue of the topological morphism $BO \rightarrow BG$. Let Gr_∞ denote an $\mathbb{A}^1$ -local fibrant model of the infinite Grassmannian. It is proven in [MV99, Proposition 4.3.14] that this object represents K-theory in the simplicial model structure in the sense that there is an isomorphism for any regular scheme X over a noetherian base S of finite dimension

$$K_n(X) = \mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(\Sigma_S^n X_+, \mathbf{L}p_S^*(\mathrm{Ex}_{\mathbb{A}^1}^\infty Gr_\infty \times \mathbb{Z}, *)),$$

where $\mathcal{C} = \Delta^{\mathrm{op}} PShv(\mathrm{Sm}_S, \mathrm{Nis})$ and $\mathrm{Ex}_{\mathbb{A}^1}^\infty Gr_\infty$ is a fibrant model of the infinite Grassmannian over $\mathrm{Spec} \mathbb{Z}$. Using Brown representability for morphisms, one can construct a morphism from this space representing K-theory to the classifying space of spherical fibrations. The following argument explains how to construct a natural transformation $K_0(-) \rightarrow \mathcal{H}(-, L_{\mathbb{A}^1} S^{2\bullet,\bullet})$. Let therefore X be a smooth scheme over S , and let $p : E \rightarrow X$ be a vector bundle representing a class $[E] \in K_0(X)$. We apply the Thom-space construction of [MV99, Definition 3.2.16] to this vector bundle, which is defined to be the pointed sheaf

$$\mathrm{Th}(E) = E/(E \setminus z(X)),$$

where $z : X \rightarrow E$ is the zero section of E . This is Nisnevich-locally trivial, and the fibre is $\mathbb{A}^n/(\mathbb{A}^n \setminus \{0\})$. Then we do a fibrewise localization or just a change-of-fibre along the localization morphism $\mathbb{A}^n/(\mathbb{A}^n \setminus \{0\}) \rightarrow L_{\mathbb{A}^1} S^{2n,n}$. This yields a spherical fibration with fibre $L_{\mathbb{A}^1} S^{2n,n}$ for a vector bundle of rank n and therefore a stable spherical fibration in general. This construction is natural since all its steps are natural. What is left to show is that for two stably isomorphic vector bundles E_1 and E_2 , the induced spherical fibrations are equivalent. This follows since the trivial vector bundle $\mathbb{A}^n \times X \rightarrow X$ induces the trivial spherical fibration $* \wedge \mathrm{id} : L_{\mathbb{A}^1} S^{2n,n} \wedge X \rightarrow * \wedge X$, and isomorphic vector bundles are mapped to equivalent spherical fibrations. What we have shown is that there is a natural transformation of functors $K_0(-) \rightarrow \mathcal{H}(-, L_{\mathbb{A}^1} S^{2\bullet,\bullet})$ on Sm_S which gives rise to a morphism on classifying spaces $BGL_\infty \times \mathbb{Z} \rightarrow B^f L_{\mathbb{A}^1} S^{2\bullet,\bullet}$.

It is interesting to look at the homotopy fibre of this morphism, which is the algebraic analogue of the space G/O . This is the space that should classify the normal cobordism classes of vector bundle reductions of a given spherical bundle.

The geometric significance of such a construction remains yet to be determined. In topology, this space classifies normal cobordism classes of smooth structures on a Poincaré duality space. It is not clear if one should expect scheme structures to reduce to motivic homotopy types with Poincaré duality together with tangent data. Note also that already in topology, the determination of the homotopy type of G/O requires several deep results including the solution of the Adams conjecture and the exotic smooth structures on spheres. The algebraic situation tends to be even more complicated, since we can describe neither algebraic K-theory nor the stable homotopy of the motivic spheres except in rather special situations.

Using the above strategy, one can also construct other interesting morphisms from $BGL_\infty \times \mathbb{Z}$ corresponding to other constructions one can apply to vector bundles. One possibility is to simply delete the zero-section of a vector bundle of rank n to obtain a fibre sequence with fibre $\mathbb{A}^n \setminus \{0\}$, which yields a morphism $BGL_\infty \times \mathbb{Z} \rightarrow B^f L_{\mathbb{A}^1} S^{2\bullet-1, \bullet}$. It is also possible to apply a change-of-fibre along $\mathbb{A}^n \setminus \{0\} \rightarrow \mathbb{P}^{n-1}$. This yields a morphism $BGL_n \rightarrow B^f \mathbb{P}^{n-1}$, which factors through $BPGL_n$. The homotopy fibre of this morphism describes if a fibre sequence with fibre the projective space $\mathbb{P}^{n-1}$ is equivalent to the projective space associated to some vector bundle. Furthermore, we have the fibrewise localization morphism $B^f \mathbb{P}^n \rightarrow B^f L_{\mathbb{A}^1} \mathbb{P}^n$ whose homotopy fibre should say something about fake projective spaces and fake projective bundles, since such can only appear after $\mathbb{A}^1$ -localization. Already in classical algebraic topology, there are fake projective spaces homotopy equivalent to $\mathbb{C}P^n$ but not diffeomorphic to it, so the algebraic situation is probably worse and depends on the arithmetic of the base scheme. Probably, there is a difference between homotopy automorphisms of $L_{\mathbb{A}^1} \mathbb{P}^n$ and projective linear transformations in PGL_{n+1} , and the homotopy fibre of the morphism $BPGL_{n+1} \rightarrow B^f L_{\mathbb{A}^1} \mathbb{P}^n$ describes these.

One final remark we want to make is that the techniques described above only work for topologies given by cd-structures. Since we have proven homotopy-continuity for simplicial presheaves on arbitrary Grothendieck sites, the construction of classifying spaces could be extended to the general case either by using a description of classifying spaces by a general bar construction or an extension of the Brown representability theorem to model categories of simplicial presheaves.

3.3 Examples of Fibre Sequences

In this section, we explore possibilities to construct fibre sequences of simplicial presheaves.

One way is to use Proposition 3.1.11: It is possible to check if $F \rightarrow E \rightarrow B$ is a fibre sequence by checking if $p^*(F) \rightarrow p^*(E) \rightarrow p^*(B)$ is a fibre sequence for a set of jointly surjective points of the topos $\mathcal{S}hv(T)$.

This of course works only in case there are enough points. In the general case of an arbitrary Grothendieck topos, one has to use Rezk's notion of sharp maps: Sharp maps can be glued nicely, and their point-set fibres and homotopy fibres are weakly equivalent. They behave like the quasi-fibrations of Dold and Thom [DT58].

It is also possible to use the change-of-site functors. For a geometric morphism of topoi $f : \mathcal{E} \rightarrow \mathcal{E}'$, the functor f_* is a right Quillen functor, and its associated right derived functor $\mathbf{R}f_*$ takes fibre sequences to fibre sequences. This allows to get fibrations of representable objects which are fibrant, cf. Section 2.3.

Finally, there are some “topological” fibre sequences coming out of a presheaf analogue of Ganea's theorem, which describes the relation between wedge sums and products. These directly follow from homotopy distributivity.

Torsors and Fibre Sequences: We first prove the statement that fibre sequences can be checked locally.

PROPOSITION 3.3.1 *Let T be a site such that the topos $\mathcal{S}hv(T)$ has enough points. Let $F \rightarrow E \rightarrow B$ be a sequence of morphisms of pointed simplicial presheaves in $\Delta^{\text{op}}PShv(T)$, with F fibrant. This is a fibre sequence iff $p^*(F) \rightarrow p^*(E) \rightarrow p^*(B)$ is a fibre sequence for each point.*

PROOF: This follows from Proposition 3.1.11 since $F \rightarrow E \rightarrow B$ is a fibre sequence in a proper model category iff the left square in the following diagram is a homotopy pullback:

$$\begin{array}{ccccc} F & \longrightarrow & E & \xrightarrow[\cong]{i} & \tilde{E} \\ \downarrow & & \downarrow & & \downarrow p \\ * & \longrightarrow & B & \xrightarrow[\cong]{j} & \tilde{B} \end{array}$$

To see this, we consider the right square, which is constructed by first taking a fibrant replacement $j : B \rightarrow \tilde{B}$, and then factoring the composition $E \rightarrow \tilde{B}$ as a trivial cofibration followed by the fibration p . The right square is a homotopy pullback, so by the homotopy pullback lemma the left square is a homotopy pullback iff the composition of the two squares is. The outer square is a homotopy pullback iff $F \cong p^{-1}(*)$ iff $F \rightarrow E \rightarrow B$ is a fibre sequence. ■

Sharp Maps: For a general site, it is quite difficult to figure out what the fibrations are. Even for topological spaces, it is sometimes difficult to handle fibrations, in particular when it comes to glueing. The work-around Dold and Thom used in

their work was the notion of quasi-fibration [DT58]. Quasi-fibrations do not necessarily satisfy the lifting property, but still give rise to long-exact sequences for the homotopy groups, i.e. the point-set fibre is weakly equivalent to the homotopy fibre.

A similar notion that works for general model categories of simplicial sets is the notion of *sharp maps*, due to Rezk [Rez98, Section 2]. The stability properties these morphisms satisfy make them quite useful for constructing fibre sequences, as we will see in this paragraph. For a topos with enough points, the results presented here could also be proved using Proposition 3.3.1, but the theory of sharp maps makes it possible to settle the general case. We first recall the definition of sharp maps.

DEFINITION 3.3.2 *Let $\mathcal{C}$ be a model category. A morphism $f : X \rightarrow Y$ in $\mathcal{C}$ is called sharp if for each diagram of the form*

$$\begin{array}{ccccc} X'' & \xrightarrow{j} & X' & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow f \\ Y'' & \xrightarrow{i} & Y' & \longrightarrow & Y, \end{array}$$

in which i is a weak equivalence and each square is a pullback square, the map j is also a weak equivalence.

As was the case for the theory of homotopy pullback squares, it is more convenient to assume the model category $\mathcal{C}$ is actually (right) proper. This e.g. implies that any fibration is sharp. However, it also implies that any pullback square with a sharp map is already a homotopy pullback square. Therefore sharp maps give rise to fibre sequences. The first part of the following proposition is [Rez98, Proposition 2.7].

PROPOSITION 3.3.3 *Let $\mathcal{C}$ be a proper model category. If $p : E \rightarrow B$ is a sharp map then any pullback of p along any morphism $f : X \rightarrow B$ is a homotopy pullback.*

Choosing in particular the morphism $*$ $\rightarrow B$, we obtain a weak equivalence between the point set fibre $p^{-1}(*)$ and the homotopy fibre $\text{hofib}(p)$. Therefore, $F = p^{-1}(*) \rightarrow E \rightarrow B$ is a fibre sequence in $\mathcal{C}$.

PROOF: Let $p : E \rightarrow B$ be sharp. Choose a factorization of f into a trivial cofibration $i : X \rightarrow \tilde{X}$ and a fibration $\tilde{f} : \tilde{X} \rightarrow B$. We get the following diagram:

$$\begin{array}{ccccc} E \times_B X & \xrightarrow{j} & E \times_B \tilde{X} & \longrightarrow & E \\ \downarrow & & \downarrow & & \downarrow p \\ X & \xrightarrow{i} & \tilde{X} & \xrightarrow{\tilde{f}} & B. \end{array}$$

Since p is sharp and i is a weak equivalence, the morphism j is also a weak equivalence. Therefore the composition of the two squares is a homotopy pullback square by definition.

For the second claim note that the point-set fibre is the pullback $E \times_B *$. The homotopy fibre is $E \times_B \tilde{*}$ in the above notation, in other words it is the point-set fibre over the path-space of B . The fibre sequence statement then follows from an argument similar to the one in the proof of Proposition 3.3.1. ■

There are some nice stability properties satisfied by sharp maps in the model category $\Delta^{\text{op}}\mathcal{PShv}(T)$ of simplicial presheaves on any Grothendieck site T . These imply that locally trivial morphisms are sharp maps. The following proposition lists the stability properties of sharp maps described in [Rez98, Theorem 5.1].

PROPOSITION 3.3.4 *Let T be any Grothendieck site. All the morphisms considered in the following are morphisms of simplicial presheaves on T . Then the following hold:*

- (i) *Morphisms, which sheafify to local fibrations are sharp.*
- (ii) *For any object $X \in \Delta^{\text{op}}\mathcal{PShv}(T)$ the projection $X \rightarrow *$ is sharp.*
- (iii) *Sharp maps are closed under base change.*
- (iv) *If f is a map such that the base-change of f along some epimorphism is sharp, then f is sharp.*
- (v) *Coproducts of sharp maps are sharp.*

Using these stability properties, we can show that locally trivial morphisms in a model category of simplicial presheaves are sharp.

COROLLARY 3.3.5 *Let T be a site, and $f : E \rightarrow B$ a morphism of T which is locally trivial with fibre $F \rightarrow E$ for the topology of the site. Using the Yoneda embedding $y : U \mapsto \text{Hom}_T(-, U) : T \rightarrow \mathcal{Shv}(T)$ followed by sheafification L^2 and the functor $c : \mathcal{Shv}(T) \rightarrow \Delta^{\text{op}}\mathcal{Shv}(T)$ taking every sheaf F on T to the sheaf of constant simplicial sets associated to F , the morphism $c \circ y(f)$ of simplicial sheaves is sharp.*

PROOF: By assumption, there is a covering $U_i \rightarrow B$ such that the pullback $E \times_B U_i \rightarrow U_i$ is isomorphic to $U_i \times F \rightarrow U_i$. In other words, the following square is a pullback:

$$\begin{array}{ccc} \coprod_{\alpha} U_{\alpha} \times F & \longrightarrow & E \\ \downarrow & & \downarrow p \\ \coprod_{\alpha} U_{\alpha} & \longrightarrow & B \end{array}$$

Since the Yoneda embedding commutes with finite limits, the above diagram is a pullback also if we view it as diagram of representable presheaves. Point-set distributivity in topoi implies that the following diagram is commutative:

$$\begin{array}{ccc} \coprod_{\alpha} (U_{\alpha} \times F) & \xrightarrow{\cong} & (\coprod_{\alpha} U_{\alpha}) \times F \\ & \searrow & \swarrow \\ & \coprod_{\alpha} U_{\alpha} & \end{array}$$

Almost by definition, the map

$$\coprod_{\alpha} L^2 \circ y(U_{\alpha}) \rightarrow L^2 \circ y(B)$$

is an epimorphism in the category of sheaves iff $U_{\alpha} \rightarrow Y$ is a cover.

The closure properties from Proposition 3.3.4 then imply that f is sharp: By (iv), it suffices to show that the morphism $\coprod_{\alpha} U_{\alpha} \times F \rightarrow \coprod_{\alpha} U_{\alpha}$ is sharp, since this is the base change of p along the epimorphism coming from the covering $U_{\alpha} \rightarrow B$. This morphism is a coproduct, so by (v) it suffices to check if $U_{\alpha} \times F \rightarrow U_{\alpha}$ is sharp for any α . This follows by (ii) applied to $F \rightarrow *$ and (iii) base change $F \rightarrow *$ along $U_{\alpha} \rightarrow *$. $\blacksquare$

REMARK 3.3.6 • *Note that there are Quillen equivalences relating the injective model structure to other model structures (projective and several intermediate ones) on simplicial presheaves. The corresponding functors are the identity functors, and basically only “compare” the sets of cofibrations resp. fibrations. Moreover, the injective model structure is the one with the least fibrations, hence the right Quillen functor of the Quillen equivalence goes from the injective model structure to e.g. the projective. Thus it preserves fibration sequences, and the above corollary holds for all those model structures having more fibrations than the injective model structure.*

- *The above argument only works for the given embedding as a constant simplicial presheaf. Embedding the smooth schemes via the $\mathbb{A}^1$ -resolution $X \mapsto \mathrm{Hom}_{\mathrm{Sm}/S}(- \times \mathbb{A}^1, X)$ will not produce a surjective map of simplicial sheaves for the covering $U_{\alpha} \rightarrow X$. The simplest (but general enough) counterexample to this can be described as follows: Consider the covering $\mathbb{G}_m \sqcup \mathbb{G}_m \rightarrow \mathbb{A}^1$, given by the complements of 0 resp. 1. We consider the sheaves of 1-simplices, where the morphism corresponding to the cover is*

$$\mathrm{Hom}(- \times \mathbb{A}^1, \mathbb{G}_m \sqcup \mathbb{G}_m) \rightarrow \mathrm{Hom}(- \times \mathbb{A}^1, \mathbb{A}^1),$$

given by simple composition. If this was surjective, it would also be surjective on global sections, i.e. on sections over k . But that is false, as any map $\mathrm{Hom}(\mathbb{A}^1, \mathbb{G}_m \sqcup \mathbb{G}_m)$ must be the constant map into one of the components, whereas the endomorphism group $\mathrm{Hom}(\mathbb{A}^1, \mathbb{A}^1)$ contains many more nontrivial elements.

- *The same proof (modulo minor modifications) shows that locally trivial fibrations induce fibre sequences of sheaves of spectra. We have not included the precise statement and proof, as the result may only have limited use: In stable model categories, the notions of cofibre sequence and fibre sequence coincide, and cofibre sequences might be easier to construct. Note however, that precisely for this reason fibre sequences are preserved by any localization of a stable model category. Thus for any locally trivial fibration $E \rightarrow B$, we have an $\mathbb{A}^1$ -local fibre sequence of suspension spectra*

$$\Sigma_{\mathbb{P}^1}^{\infty} F \rightarrow \Sigma_{\mathbb{P}^1}^{\infty} E \rightarrow \Sigma_{\mathbb{P}^1}^{\infty} B$$

Most of the results of this thesis are only interesting in the unstable setting anyway.

Change of Site: One can also use different sites to produce fibre sequences. The key idea is to note that for a morphism of sites $\pi : T_1 \rightarrow T_2$, there is an induced geometric morphism of topoi $\pi : \mathcal{S}hv(T_1) \rightarrow \mathcal{S}hv(T_2)$ which yields a Quillen pair. The total right derived functor $\mathbf{R}\pi_* : \Delta^{\text{op}}P\mathcal{S}hv(T_1) \rightarrow \Delta^{\text{op}}P\mathcal{S}hv(T_2)$ preserves the right part of the model structure, in particular fibrations, homotopy limits and fibre sequences. Therefore a fibration $p : E \rightarrow B$ of fibrant simplicial presheaves on T_1 yields a fibration $\pi_*(p) : \pi_*(E) \rightarrow \pi_*(B)$ of fibrant simplicial presheaves on T_2 . This allows to produce fibre sequences on T_2 which are not locally trivial in the topology of T_2 .

As an example, consider $\text{Sm}_{S,\text{ét}} \rightarrow \text{Sm}_{S,\text{Nis}}$ and let $p : E \rightarrow B$ be an étale locally trivial morphism $p : E \rightarrow B$. Then the fibrant replacement $\tilde{p} : \tilde{E} \rightarrow \tilde{B}$ is a fibration of fibrant simplicial presheaves in the model structure for the *Nisnevich* topology. As we have seen in Section 2.3, simplicial presheaves of simplicial dimension zero are fibrant if they are sheaves. We have therefore proven the following proposition:

PROPOSITION 3.3.7 *Let $F \rightarrow E \rightarrow B$ be a morphism of sheaves on Sm_S equipped with the étale topology, such that there exists an étale covering U_i of B and isomorphisms $F \times U_i \cong E \times_B U_i$. Then $F \rightarrow E \rightarrow B$ is a fibre sequence of simplicial sheaves in $\Delta^{\text{op}}P\mathcal{S}hv(\text{Sm}_{S,\text{Nis}})$.*

Therefore also families of schemes which are not rationally trivial, i.e. trivial in the Nisnevich topology, can still be fibre sequences for the Nisnevich topology as long as there is an inclusion of fibre morphism $F \rightarrow E$. As already remarked in the previous section, this does not imply that the homotopy fibre of a Galois extension $\text{Spec } L \rightarrow \text{Spec } k$ is the Galois group $\text{Gal}(L/k)$: There is no morphism $\text{Gal}(L/k) \rightarrow L$, although after base changing to L both are isomorphic. However, for an étale covering $p : Y \rightarrow X$ such that there is a morphism $\text{Gal}(Y/X) \rightarrow Y$, there also is a fibre sequence $\text{Gal}(Y/X) \rightarrow Y \rightarrow X$, as was already noted in [Mor06b, Lemma 4.5, Example 4.6].

In general, the above proposition only works nicely for simplicial dimension zero since in this case no fibrant replacement is necessary. The fibre sequences obtained are however only interesting in the $\mathbb{A}^1$ -local model structure, since before localization, there is only a dimension zero-component.

Geometric Fibre Sequences: The following give some examples of fibre sequences of sheaves of simplicial dimension zero. These are rather simple things. In fact, any morphism $f : X \rightarrow Y$ of simplicial dimension zero sheaves is a quasi-fibration: The base Y is fibrant, and replacing f by a fibration will not change the homotopy type of the sets of sections since X is also fibrant, cf. Section 2.3. It does not seem like there would be an identification of the “right fibre” for a general morphism $f : E \rightarrow B$ of sheaves. Therefore, although such a morphism is a quasi-fibration, there is no obvious fibre: For any choice of a rational base point $x \in E$, i.e. a section $x \in E(S)$ over the base scheme S , there is a corresponding fibre sequence of sheaves of constant simplicial sets

$$f^{-1}(f(x)) \rightarrow E \rightarrow B.$$

In general these fibres will vary wildly for different choices of the base point. This is due to the fact that there is no higher homotopy in simplicial presheaves of dimension zero. Such fibre sequences only become interesting when localizing $\mathbb{A}^1$.

In particular, any morphism $f : X \rightarrow B$ with B an abelian variety induces an $\mathbb{A}^1$ -local fibre sequence $f^{-1}(b) \rightarrow X \rightarrow B$ for any choice of base point $b \in B$. An example of such a morphism is the Albanese morphism $X \rightarrow \text{Alb}(X)$.

In the following, we will describe some morphisms of simplicial dimension zero spaces where the variation of the fibre can be brought under control. Only for such spaces there is a hope that we can obtain fibre sequences which also pass to the $\mathbb{A}^1$ -local model structure and yield interesting fibre sequences with higher homotopy.

EXAMPLE 3.3.8 For simplicity we work over a field k . Any torsor under a torus is locally trivial in the Zariski topology, therefore gives rise to a fibre sequence. These fibre sequences are also $\mathbb{A}^1$ -local fibre sequences.

Particularly interesting are the universal torsors of [CTS87]: For a smooth, proper and geometrically connected variety X over k , whose geometric Picard group $\text{Pic}(X \times_k \bar{k})$ is finitely generated and torsion free, the *Néron-Severi torus* of X is the k -torus associated to the discrete $\text{Gal}(\bar{k}/k)$ -module $\text{Pic}(X \times_k \bar{k})$. For every torsor under a torus S over X , there is an associated morphism from the character group to the geometric Picard group:

$$\chi(S) : \widehat{S} = \text{Hom}(S \times_k \bar{k}, \mathbb{G}_m \times_k \bar{k}) \rightarrow \text{Pic}(\bar{X}).$$

This morphism is called its type, and a torsor under the Néron-Severi torus is called universal if its type is an isomorphism. These torsors play an important role in the arithmetic of rational varieties, cf. [CTS87].

These torsors induce fibre sequences, which are $\mathbb{A}^1$ -local as we will see later. One example of the situation is the fibre sequence with fibre a torus over a del Pezzo surface of degree 5, whose total space is an open subset of the Grassmannian $G(3, 5)$, as described in [Sko01].

Another example of this type is the homogeneous coordinate ring for a toric variety as defined in [Cox95]. This produces for any toric variety X a graded ring R and an ideal $I \subseteq R$ depending only on the fan $\Delta(X)$, such that the open complement Y of the ideal I in $\text{Spec } R$ has a torus action by the Néron-Severi torus of X , making X the quotient of Y by this torus action. This generalizes the $\mathbb{G}_m$ -covering $\mathbb{A}^{n+1} \setminus \{0\} \rightarrow \mathbb{P}^n$. The homogeneous coordinate ring of $\mathbb{P}^n$ is $k[x_0, \dots, x_n]$, and the ideal is $(x_0, \dots, x_n)$. The open complement of this is exactly $\mathbb{A}^{n+1} \setminus \{0\}$ with the usual torus action. For more details see [Cox95].

For other examples of universal torsors, see [Sko01]. □

EXAMPLE 3.3.9 Étale coverings are fibre sequences, if the inclusion of the fibre is defined over the base.

Any finite étale Galois covering $f : Y \rightarrow X$ is a sharp map for the étale topology: Indeed, one has

$$Y \times_X Y \cong \coprod_{\sigma \in \text{Gal}(Y/X)} Y$$

and therefore f is locally trivial in the étale topology. This is explained e.g. in Remark 5.4 of Milne's book on étale cohomology. We still need that there is a morphism $\mathrm{Gal}(Y/X) \rightarrow Y$ defined over the base scheme such that $\mathrm{Gal}(Y/X) = p^{-1}(*)$. This is the case for a morphism of varieties $Y \rightarrow X$ over a field k such that Y has a k -rational point $y \in Y$. The $\mathrm{Gal}(Y/X)$ -orbit of y consists of rational points, and these are mapped to a rational point $f(y) \in X$. Choosing these as base points, we get a fibre sequence of simplicial presheaves $\mathrm{Gal}(Y/X) \rightarrow Y \rightarrow X$.

Examples of the above are n -coverings of abelian varieties, as explained in [Sko01]. Let $f : X \rightarrow B$ be an n -covering of the abelian variety B , assuming that n is coprime to the characteristic of k . Then $f^{-1}(0)$ is a torsor under the finite group scheme $B[n]$ of n -torsion points of B , and we have a fibre sequence $f^{-1}(0) \rightarrow X \rightarrow B$.

The same statement holds for the K3-cover of an Enriques surface: Let Y be a K3-surface with a rational point as base point. There is an involution, i.e. a $\mathbb{Z}/2$ -action defined on Y . The quotient X of Y by the involution is an Enriques surface. With the rational points as base points, we obtain a fibre sequence $\mathbb{Z}/2 \rightarrow Y \rightarrow X$.

Of course these are rather arbitrary examples, one could give many more. $\square$

EXAMPLE 3.3.10 There are some general local triviality results for homogeneous spaces. Let X be a principal homogeneous space over a base scheme S , i.e. X is an S -scheme, G is S -group scheme which operates on X , and the corresponding morphism $G \times_S X \rightarrow X \times_S X$ is an isomorphism. We assume that there exists a geometric quotient Y for the G -action on X . In this situation we speak of a principal fibre bundle with structure group G . It is called locally isotrivial if the quotient morphism $\pi : X \rightarrow Y$ is locally trivial in the étale topology.

Grothendieck and Seshadri have proven that any principal fibre bundle with a linear algebraic group as structure group is locally isotrivial, cf. [Gro59].

This applies to homogeneous spaces G/H , with G a linear algebraic group over S and H a closed subgroup of G defined over the base scheme. In this case there is a natural choice of base point for G , H and G/H , namely the identity. For this choice of base point, we obtain a fibre sequence

$$H \rightarrow G \rightarrow G/H.$$

It is not clear if this is always an $\mathbb{A}^1$ -local fibre sequence for any homogeneous space. We will investigate some cases in Chapter 5. Contrary to the case of an arbitrary morphism, the variation of the point-set fibre in such a locally trivial morphism is rather restricted: For other choices of base points, the fibre is going to be a torsor over the base scheme S under the group scheme H .

One example of the above is the universal covering of a general semisimple group G : There is a fibre sequence

$$A \rightarrow \tilde{G} \rightarrow G,$$

where $\tilde{G}$ is simply connected in the sense that any connected étale covering of $\tilde{G}$ is the identity, and A is a finite central S -subgroup of $\tilde{G}$. This is again an example of a fibre sequence with fibre a finite group scheme.

Other examples where we do not need the étale topology for trivializations are the general linear groups. Torsors under the general linear group are already

locally trivial in the Zariski topology. One can give quite explicit trivializations for fibre sequences like $GL_n \rightarrow GL_{n+1} \rightarrow \mathbb{P}^n$ or the universal GL_n -bundle $GL_n \rightarrow V_n \rightarrow Gr_n$. Such trivializations are given e.g. in [DI05], where they are used to prove cellularity of the varieties involved in the above fibre sequences.

One could look at more general quotients constructions coming out of geometric invariant theory. Most interesting would be the case of quotients which only exist as algebraic spaces. An algebraic space can be seen as a sheaf of groupoids, and therefore is not simplicial dimension zero any more. This implies that it is not necessarily fibrant. It is probably fibrant if the usual stack conditions are satisfied, see the homotopy theory of stacks. $\square$

Ganea's Theorem: There are some fibre sequences of simplicial sets which are direct consequences of homotopy distributivity, which we can prove for simplicial presheaves. These fibre sequences are all more or less consequences of Ganea's work. We start out with a theorem describing the homotopy fibre of the fold map. The proof is essentially the one given in [Dro96, Appendix HL], which simply applies homotopy distributivity to one of the simplest situations possible:

PROPOSITION 3.3.11 (GANEA'S THEOREM) *Let T be a Grothendieck site, and X be a simplicial presheaf on T . The sequence $\Sigma\Omega X \rightarrow X \vee X \rightarrow X$ is a fibre sequence in $\Delta^{\text{op}}\text{PShv}(T)$.*

PROOF: This is an instance of Proposition 3.1.16 applied to the diagram:

$$\begin{array}{ccccc} X & \xleftarrow{\quad} & \text{pt} & \xrightarrow{\quad} & X \\ \downarrow & & \downarrow & & \downarrow \\ X & \xleftarrow{=} & X & \xrightarrow{=} & X \end{array}$$

We are taking the homotopy colimit of the diagram over the fixed base space X , and the homotopy colimit of the upper line yields $X \vee X$. The map to X is the fold map $\vee : X \vee X \rightarrow X$. Then Proposition 3.1.16 shows that the fibre is given by the homotopy colimit of the diagram of fibres:

$$\text{pt} \xleftarrow{\quad} \Omega X \xrightarrow{\quad} \text{pt}.$$

This is by definition $\Sigma\Omega X$. $\blacksquare$

EXAMPLE 3.3.12 A particular topological instance of the above is the fibre sequence

$$S^2 \rightarrow \mathbb{C}P^\infty \vee \mathbb{C}P^\infty \rightarrow \mathbb{C}P^\infty.$$

A similar fibre sequence exists in $\Delta^{\text{op}}\text{PShv}(\text{Sm}_S)$ with any of the usual topologies. This implies that there is a fibre sequence

$$\Sigma_s^1 \mathbb{G}_m \rightarrow B\mathbb{G}_m \vee B\mathbb{G}_m \rightarrow B\mathbb{G}_m.$$

Then we can use the identification $B\mathbb{G}_m \cong \mathbb{P}^\infty$. We will see later in Chapter 4 that this is indeed an $\mathbb{A}^1$ -local fibre sequence.

Using the van-Kampen theorem, this allows to determine the fundamental group of the simplicial presheaf $\Sigma_s^1 \mathbb{G}_m$ as follows:

$$\pi_1(\Sigma_s^1 \mathbb{G}_m) = \ker(\vee : \mathbb{G}_m * \mathbb{G}_m \rightarrow \mathbb{G}_m).$$

In the above, $\mathbb{G}_m * \mathbb{G}_m$ is the free product of the sheaf of groups $\mathbb{G}_m$ with itself. This can be described by a point-wise free product. We need the van-Kampen theorem to determine $\pi_1(X \vee Y) \cong \pi_1(X) * \pi_1(Y)$. The morphism $\vee : \mathbb{G}_m * \mathbb{G}_m \rightarrow \mathbb{G}_m$ induced by the fold map can be described on sections as follows: Any section of $(\mathbb{G}_m * \mathbb{G}_m)(U)$ is given by a word $a_1 b_1 \cdots a_n b_n$ of units of U , alternating between the two copies of $\mathbb{G}_m$. The morphism $\vee$ takes this word to the product $a_1 b_1 \cdots a_n b_n \in \mathbb{G}_m(U)$. That $\pi_1(\Sigma_s^1 \mathbb{G}_m)$ is the kernel of the morphism $\vee$ follows since $B\mathbb{G}_m$ is the classifying space of $\mathbb{G}_m$ -torsors and by [MV99, Proposition 4.1.16] we have $\pi_2(B\mathbb{G}_m) = 0$. $\square$

There are also other fibre sequences one can obtain: By considering similar diagrams as in [Dro96, Appendix HL] we get the following fibration sequences in any model category of simplicial sheaves.

PROPOSITION 3.3.13 *Let T be a Grothendieck site, and X be a simplicial presheaf on T . The sequence $\Omega X_0 * \Omega X_1 \rightarrow X_0 \vee X_1 \rightarrow X_0 \times X_1$ is a fibre sequence in $\Delta^{\text{op}}\text{PShv}(T)$.*

PROOF: Apply Puppe's theorem 3.1.16 to the following diagram, the horizontal lines are the pushout diagrams and the vertical lines are fibre sequences:

$$\begin{array}{ccccc} \Omega X_0 \times * & \xleftarrow{\quad} & \Omega X_0 \times \Omega X_1 & \xrightarrow{\quad} & * \times \Omega X_1 \\ \downarrow & & \downarrow & & \downarrow \\ * \times X_1 & \xleftarrow{\quad} & * & \xrightarrow{\quad} & X_0 \times * \\ \downarrow & & \downarrow & & \downarrow \\ X_0 \times X_1 & \xleftarrow{\quad} & X_0 \times X_1 & \xrightarrow{\quad} & X_0 \times X_1. \end{array}$$

■

EXAMPLE 3.3.14 An instantiation of the above fibre sequence similar to the one given in Example 3.3.12 is the following fibre sequence in $\Delta^{\text{op}}\text{PShv}(\text{Sm}_S)$:

$$\mathbb{G}_m * \mathbb{G}_m \rightarrow B\mathbb{G}_m \vee B\mathbb{G}_m \rightarrow B\mathbb{G}_m \times B\mathbb{G}_m.$$

This fibre sequence will also turn out to be an $\mathbb{A}^1$ -local fibre sequence. $\square$

The higher homotopy of a wedge sum can then be split following [Hat02, Example 4.52], the proofs are actually the same:

PROPOSITION 3.3.15 *There are split short exact sequences*

$$0 \rightarrow \pi_{n+1}(\Omega X_0 * \Omega X_1) \rightarrow \pi_n(X_0 \vee X_1) \rightarrow \pi_n(X_0 \times X_1) \rightarrow 0.$$

PROOF: The sheaves of homotopy groups of a fibrant simplicial presheaf are computed sectionwise. Therefore we have

$$\pi_n\left(\prod_{\alpha} X_{\alpha}\right) \cong \bigoplus_{\alpha} \pi_n(X_{\alpha}).$$

Assume that the site T has enough points. Then surjectivity of the morphism

$$f : \pi_n(X_0 \vee X_1) \rightarrow \pi_n(X_0 \times X_1)$$

can be checked on the points, where it is clear: The corresponding morphism of groups is $\pi_n(p^*(X_0) \vee p^*(X_1)) \rightarrow \pi_n(p^*(X_0) \times p^*(X_1))$, which is surjective [Hat02, Example 4.52]. This implies that f is surjective, hence the long exact homotopy sequence associated to the fibre sequence from Ganea's theorem splits into the short exact sequences described above.

The last statement follows since we have a concrete identification of the homotopy fibre of f , and the relative homotopy groups are defined to be the homotopy groups of a homotopy fibre. $\blacksquare$

There is a more general version of this, which uses a construction called the *thick wedge*. Our treatment basically follows [Cha97].

DEFINITION 3.3.16 *Let $(X_0, X_1, \dots, X_n)$ be a finite family of pointed simplicial presheaves. Denote by i_k the following inclusion:*

$$i_k : X_0 \times \dots \times X_{k-1} \times * \times X_{k+1} \times \dots \times X_n \hookrightarrow \prod_{0 \leq l \leq n} X_l,$$

which are the identity except at the k -th factor, where it is just inclusion of the base point $* \hookrightarrow X_k$. The thick wedge of the spaces $X_0, \dots, X_n$ is denoted by $W(X_0, \dots, X_n)$ and is defined to be the union of the images of the i_k .

There is an alternative inductive definition. Denote by $V = W(X_0, \dots, X_{n-1})$. Then define

$$W(X_0, \dots, X_n) = \operatorname{colim} \left(\prod_{0 \leq l \leq n-1} X_l \hookrightarrow V \hookrightarrow V \times X_n \right).$$

Since the maps in this diagrams are cofibrations, this is also the homotopy colimit.

PROPOSITION 3.3.17 *For any finite set $X_0, \dots, X_n$ of simplicial presheaves on a Grothendieck site T , the following is a fibre sequence in $\Delta^{\text{op}}\text{PShv}(T)$:*

$$\Omega X_0 * \dots * \Omega X_n \longrightarrow W(X_0, \dots, X_n) \longrightarrow X_0 \times \dots \times X_n.$$

PROOF: This is proven by induction, the case $n = 1$ being Proposition 3.3.13. The induction step applies Puppe's theorem to the diagram

$$\begin{array}{ccccc} \Omega X_n & \longleftarrow & \Omega X_n \times (\Omega X_0 * \dots * \Omega X_{n-1}) & \longrightarrow & \Omega X_0 * \dots * \Omega X_{n-1} \\ \downarrow & & \downarrow & & \downarrow \\ \prod_{0 \leq l \leq n-1} X_l & \longleftarrow & W(X_0, \dots, X_{n-1}) & \longrightarrow & W(X_0, \dots, X_{n-1}) \times X_n \\ \downarrow & & \downarrow & & \downarrow \\ \prod_{0 \leq l \leq n} X_l & \longleftarrow & \prod_{0 \leq l \leq n} X_l & \longrightarrow & \prod_{0 \leq l \leq n} X_l, \end{array}$$

assuming that the case $n - 1$ fibre sequence has already been established:

$$\Omega X_0 * \cdots * \Omega X_{n-1} \longrightarrow W(X_0, \dots, X_{n-1}) \longrightarrow X_0 \times \cdots \times X_{n-1}.$$

■

This in particular allows to prove the standard theorems about the James construction and the homotopy fibre of $X \rightarrow \Omega \Sigma X$. We will not go into details here, and hope the reader is convinced that one can construct quite interesting fibre sequences using only homotopy distributivity.

As a final example, we reprove yet another theorem of Ganea [Gan65], which had great applications in generalizing the EHP-sequence and defining Hopf invariants. In [Sel96], Selick used a similar theorem to construct homotopy decompositions of H-spaces. It should by now be obvious, which diagram to apply Puppe's theorem to.

PROPOSITION 3.3.18 *Let T be a Grothendieck site, and let be any fibre sequence $F \rightarrow E \rightarrow B$ of simplicial presheaves. Then there is another fibre sequence*

$$F * \Omega B \longrightarrow E \cup CF = E/F \longrightarrow B.$$

3.4 Remarks on Cellular Inequalities

In this section, we want to discuss cellular inequalities. Although these cellular inequalities are not directly related to fibre sequences, we will explain below that some inequalities can be viewed as generalizations of fibre sequences. We are not going to prove anything, only discuss how useful cellular inequalities are for simplicial sets, and why it is more difficult to generalize them to simplicial presheaves.

As explained in [Dro96, Chapter 9], a cellular inequality is written $X \ll Y$ and means that Y can be built from X using (possibly infinitely many) pointed homotopy colimits. There are then cellularization functors CW_A , which are dual to localization, and which generalize the CW-approximation, cf. [Dro96, Chapter 2]. Many useful cellular inequalities are derived in [Cha97].

In some sense cellular inequalities can be viewed as a generalization of fibre sequences. Consider the following cellular inequality known for simplicial sets [Dro96, Corollary 9.A.10]:

For any cofibre sequence $A \rightarrow X \rightarrow X/A$, we have a cellular inequality $A \ll \text{hofib}(X \rightarrow X/A)$.

Although we do not know what the homotopy fibre is, we can at least say that there is a way of constructing it using only pointed homotopy colimits of the space A . This implies that certain connectivity results or cohomological statements concerning $\text{hofib}(X \rightarrow X/A)$ can be derived from the corresponding statements about A : If two spaces are related via a cellular inequality $Y \ll Z$, then if Y is n -connected, so is Z . Similarly, if the homology of Y vanishes below degree n , then the same holds for Z . This is how we can get some information on the homotopy fibre of $X \rightarrow X/A$ above.

More generally, there are cellular inequalities due to Chachólski [Cha97] *cite triad paper*, which compare homotopy pushouts and homotopy pullbacks. With

these cellular inequalities, it is possible to measure how far a given homotopy pushout is from being a homotopy pullback. This generalizes the above inequality, where the homotopy fibre is cellular over A .

There are lots of interesting applications of such inequalities. In his papers, Chachólski derives as consequences the Blakers-Massey theorem on excision for homotopy groups as well as a more precise version of the Freudenthal suspension theorem

$$\mathrm{hofib}(X \rightarrow \Omega \Sigma X) \gg \Omega X * \Omega X,$$

from which the usual connectivity of the morphism $X \rightarrow \Omega^n \Sigma^n X$ can be derived. Moreover, several results on vanishing of homology groups can also be proven using this method.

As can be seen, this is also an inherently unstable technique. In the category of spectra, homotopy pushouts are the same thing as homotopy pullbacks, and the above results trivialize. However, cellular inequalities give a way to measure the failures of standard stable results precisely in terms of explicit simplicial sets. A standard example is excision, which is one of the Eilenberg-Steenrod axioms for a cohomology theory. Excision does not hold unstably, but the Blakers-Massey theorem [BM51] states that in favourable circumstances, excision holds for the homotopy groups in some range.

The reason why we are discussing such matters here, is that the proof of the most interesting cellular inequalities uses almost exclusively homotopy distributivity and the canonical homotopy colimit decomposition for simplicial sets. The only problem is that also one very special thing about simplicial sets is used: Any simplicial set is a homotopy colimit of *contractible* simplicial sets. Although we have seen in Section 3.1 that homotopy distributivity and the canonical homotopy colimit decomposition work for model categories of simplicial sheaves, we can not directly generalize the proofs of the cellular inequalities, since simplicial presheaves are not locally contractible in the above sense.

We nevertheless want to mention that there are already some attempts to general cellularization for simplicial presheaves, and also for motivic homotopy theory. The general construction of the cellularization functors and model structures on cellular objects is due to Nofech [Nof99]. Some discussion of cellularization in the context of $\mathbb{A}^1$ -homotopy theory can be found in [DI05]. Some of their results can be interpreted as cellular inequalities, like $\mathbb{G}_m \ll GL_n$ or $\mathbb{G}_m \ll X$ for X a toric variety. Note however that the homotopy colimits for these cellular inequalities are taken in the $\mathbb{A}^1$ -local model structure. These need not coincide with the homotopy colimits taken in the category of simplicial sheaves, as the example $\Sigma \mathbb{G}_m$ illustrates. This only becomes weakly equivalent to $\mathbb{P}^1$ after $\mathbb{A}^1$ -localization.

Although we can not prove much about cellular inequalities at the moment, it is still worthwhile thinking about applications such a theory could have. The most important computational tools for motivic cohomology or K-theory come from homotopy pushouts. One example of this is the localization sequence. Of course we can not expect such exact sequences to exist for unstable $\mathbb{A}^1$ -homotopy, since the relevant homotopy pushouts will most certainly not be homotopy pullbacks. However, it would be interesting to know if it is possible to measure the unstable failure of excision or the long exact localization sequence. These would all be instances of a “motivic Blakers-Massey” theorem. The interesting homotopy pushouts to

apply such theorems to would be the localization pushout [MV99, Theorem 3.2.21] generalizing the sequence

$$0 \rightarrow j_! j^* F \rightarrow F \rightarrow i_* i^* F \rightarrow 0$$

of sheaves of abelian groups on a scheme S with closed subscheme $i : Z \rightarrow S$ and open complement $j : U \rightarrow S$ which gives rise to the usual localization sequences and specialization morphisms. Other possible applications would be the blow-up squares of [MV99, Proposition 3.2.29]. An obvious problem with such applications is that in the Blakers-Massey theorem as well as in the cellular inequalities of Chachólski, there are always connectivity assumptions. However, the typical simplicial presheaf we would apply such theorems to is usually not connected, and definitely not simply connected.

Still the cellular inequalities seem to be an interesting subject for further research, providing a global approach to the unstable homotopy theory of simplicial presheaves, and maybe also to unstable $\mathbb{A}^1$ -homotopy theory.

3.5 Remarks on the Serre Spectral Sequence

A particularly important and helpful result in the theory of fibre sequences of simplicial sets is the Serre spectral sequence. This sequence starts from the cohomology of the base “with coefficients in the fibre” and converges to the cohomology of the total space of the fibration. In other words, for any fibration $p : E \rightarrow B$ with fibre F and any generalized homology theory h_* , there is a spectral sequence

$$E_2^{p,q} = H_p(B, h_q(F)) \Rightarrow h_{p+q}(E).$$

There are two special cases, applying the above spectral sequence to an identity morphism and to a trivial fibration. The former special case is the Atiyah-Hirzebruch spectral sequence

$$E_2^{p,q} = H_p(X, h_q(*)) \Rightarrow h_{p+q}(X),$$

which computes the generalized homology $h_*(X)$ of a simplicial set X by starting from the ordinary homology of X with coefficients in the graded abelian group $h_*(*)$. The latter special case is the Künneth formula, which computes the homology of a product of simplicial sets.

A conceptually clear way which lends itself to a generalization to fibrations of simplicial presheaves is given in [GJ99, Section IV.5.1]. The key tools used in this construction are homotopy distributivity and the canonical homotopy colimit decomposition. Using these, it is possible to decompose the fibration $p : E \rightarrow B$ into a morphism of homotopy colimits. The base is decomposed into the homotopy colimits of its category of simplices $\Delta \downarrow B$, and the total space is decomposed into the homotopy colimit of the fibres over these simplices, cf. Lemma 3.1.22. The same procedure has been used in [MS93] to construct a Serre spectral sequence for homology of diagrams of simplicial sets, which then yields a Serre spectral sequence for equivariant homology.

We give a short outline of how to construct a Serre spectral sequence for a fibration of simplicial presheaves. Following [MS93], the key idea is to produce for

a given fibration $p : E \rightarrow B$ of fibrant simplicial presheaves corresponding to computing the cohomology of E resp. B . This uses the canonical homotopy colimit decomposition and homotopy distributivity. Then one applies the sheaf hypercohomology Cartan-Leray spectral sequence. The technical issues to be dealt with include defining what cohomology of a simplicial presheaf with local coefficients is, and how one can provide a multiplicative structure for such a spectral sequence. We only review the necessary results from the literature.

Sheaf Hypercohomology: The definition of sheaf hypercohomology using a Godement resolution is discussed in [Tho85, Nis89]. The definition of sheaf hypercohomology as given in [Tho85, Definition 1.33] needs the existence of enough points for the topos.

DEFINITION 3.5.1 (SHEAF HYPERCOHOMOLOGY) *Let T be a Grothendieck site with enough points, and choose a conservative family I of points, which is small. Then there is an endofunctor*

$$G : \mathcal{Shv}(T) \rightarrow \mathcal{Shv}(T) : X \mapsto \prod_{\pi \in I} \pi_* \pi^* X.$$

The unit and counit of the adjunction induce the structure of a triple on G , cf. [MM92, Section IV.4]. Therefore the iterates G^n of the functor G applied to a sheaf X produce a cosimplicial object $G^\bullet X$ in the topos. For a presheaf of fibrant spectra $\mathcal{F}$, this produces a cosimplicial presheaf of fibrant spectra $G^\bullet \mathcal{F}$. The sheaf hypercohomology of $\mathcal{F}$ is then defined to be

$$\mathbb{H}^\bullet(T, \mathcal{F}) = \operatorname{holim}_{\Delta} G^\bullet \mathcal{F}(*),$$

where $*$ is the terminal object of the topos $\mathcal{Shv}(T)$.

There is a local-global spectral sequence which allows to compute sheaf hypercohomology. We refer to [Tho85, Proposition 1.36] for more details.

PROPOSITION 3.5.2 (SHEAF HYPERCOHOMOLOGY SPECTRAL SEQUENCE) *Let T be a site with enough points, and let $\mathcal{F}$ be a presheaf of fibrant spectra on T .*

Then there is a hypercohomology spectral sequence

$$E_2^{p,q} = H^p(T, \tilde{\pi}_q \mathcal{F}) \implies \pi_{q-p} \mathbb{H}^\bullet(T, \mathcal{F}).$$

The differentials in this spectral sequence are

$$d_r : E_r^{p,q} \rightarrow E_r^{p+r, q+r-1}.$$

REMARK 3.5.3 (i) *This is basically equivalent to computing the right derived functor to the global sections $\mathcal{SptShv}(T) \rightarrow \mathcal{Spt}$, where the category $\mathcal{Spt}$ is a suitable model for the category of spectra. The same procedure works for sheaves with values in any suitable stable model category.*

- (ii) For the spectral sequence to be useful, it is better to have convergence assertions. These depend on the cohomological dimension of the site. For all the sites of interest to motivic cohomology such cohomological dimension assertions are known. [Tho85, Corollaries 1.47, 1.48] as well as [Nis89, Theorems 1.22, 2.22] provide the statements that the spectral sequences converge strongly for noetherian schemes of finite Krull dimension with the Zariski or Nisnevich topology. For the étale topology one has restrictions on the cohomological dimensions of the fields involved, see the discussion in [AGV73, Exposé X].

The above formulae define sheaf hypercohomology for the final object of the topos $\mathbb{H}_T^\bullet(\mathrm{pt}, \mathcal{F}) = \mathbb{H}(T, \mathcal{F})$. This definition can be extended to define hypercohomology spectra $\mathbb{H}_T^\bullet(X, \mathcal{F})$ for any simplicial sheaf X with coefficients in a presheaf of fibrant spectra $\mathcal{F}$. This can be found in [Voe03, Appendix A].

Grothendieck-Leray Spectral Sequence: For any morphism of sites $f : C \rightarrow D$, there is a spectral sequence computing the cohomology of sheaves on C using the cohomology of sheaves on D and the right derived functor of f_* . Thomason's version of this Grothendieck-Leray spectral sequence is the following [Tho85, Theorem 1.56]:

THEOREM 3.5.4 (GROTHENDIECK-LERAY) *Let $f : C \rightarrow D$ be a morphism of Grothendieck sites, and assume that the corresponding topoi $\mathrm{Shv}(C)$ and $\mathrm{Shv}(D)$ have enough points. Furthermore, let $\mathcal{F}$ be a fibrant presheaf of spectra on C , and assume that there exists an N such that for all $U \in D$, we have $H_C^p(f^*U, \pi_q \mathcal{F}) = 0$ for $p > N$.*

Then the following map is a weak equivalence

$$\mathbb{H}^\bullet(C, \mathcal{F}) \xrightarrow{\cong} \mathbb{H}^\bullet(D, \mathbb{R}^\bullet f_*(\mathcal{F})).$$

Using the sheaf hypercohomology spectral sequence from Proposition 3.5.2 we then have a spectral sequence

$$E_2^{p,q} = H^p(D, \tilde{\pi}_q \mathbb{R}^\bullet f_*(\mathcal{F})) \implies \pi_{q-p} \mathbb{H}^\bullet(C, \mathcal{F}).$$

A similar result is also proved in a paper of Jardine [Jar89]. Note that the cohomological boundedness assumption is satisfied for the Nisnevich sites over schemes noetherian scheme of finite Krull dimension.

Twisted Coefficients: As remarked earlier the key idea is to reduce the Serre spectral sequence for a fibration of simplicial presheaves to the above Grothendieck-Leray spectral sequence. For simplicial sets, one introduces the category of simplices $\Delta \downarrow B$, and in [MS93] the same thing is done producing a category of simplices fibred over the diagram category. As we are dealing with a sheaf situation, we replace the simplicial presheaves by their sites of simplices.

DEFINITION 3.5.5 *Let T be a Grothendieck site. For a given simplicial presheaf X in $\Delta^{\mathrm{op}} \mathrm{PShv}(T)$, we define the site of simplices of X , which is denoted by $(T \times \Delta \downarrow X)$. The underlying category is the canonical homotopy colimit decomposition,*

which is a Grothendieck construction for the category of simplices. Then we define the Grothendieck topology on $(T \times \Delta \downarrow X)$ by giving the following basis: To each object $U \times \Delta^n$ in $(T \times \Delta \downarrow X)$ we associated the family of coverings

$$K_{(T \times \Delta \downarrow X)}(U \times \Delta^n) =$$

$$\left\{ \{f_i : U_i \times \Delta^{n_i} \rightarrow U \times \Delta^n\} \mid \{f_{i1} : U_i \rightarrow U\} \in K_T(U) \text{ and } f_{i2} = \text{id} : \Delta^n \rightarrow \Delta^n \right\}$$

In a similar way, we define the edge-path site $\pi_T(X)$ of X , by only taking into account the 0- and 1-simplices of each simplicial set $X(U)$ with $U \in T$.

We then have a diagram of Grothendieck sites

$$\begin{array}{ccc} (T \times \Delta \downarrow X) & \xrightarrow{v_x} & \pi_T(X) \\ & \searrow p_X \quad \swarrow q_X & \\ & T & \end{array}$$

There are two remarks we want to make. First, it may be that the site of simplices does not have all finite limits, whence it is difficult to define the topology by giving a basis, cf. [MM92, Definition III.2.2]. There is however also a definition working in the general case, using sieves. The second remark concerns size issues again: We assume that T is a small site. By combinatoriality, any simplicial presheaf on T is α -bounded for some suitably chosen cardinal α , and therefore we can assume the site of simplices is also small.

Next, we need some results generalizing the steps in the paper of Moerdijk and Svensson. The following proposition generalizes [MS93, Proposition 2.2].

PROPOSITION 3.5.6 *For any Grothendieck site T , any fibrant simplicial presheaf X on T , and any presheaf of fibrant spectra $\mathcal{F}$, we have an isomorphism of presheaves of fibrant spectra:*

$$p_X^* \mathbb{H}_T^\bullet(X, \mathcal{F}) \cong \mathbb{H}_{(T \times \Delta \downarrow X)}^\bullet(\text{pt}, p_X^* \mathcal{F}),$$

This proposition allows to identify the sheaf hypercohomology of an arbitrary (but fibrant) simplicial presheaf with the sheaf hypercohomology of the final object in a suitable topos, namely the topos of sheaves on the site of simplices of X . The reason for this is the canonical homotopy colimit decomposition, which states that the homotopy colimit of the diagram

$$(T \times \Delta \downarrow X) \rightarrow \Delta^{\text{op}} \text{PShv}(T) : (\sigma : U \times \Delta^n \rightarrow X) \mapsto U \times \Delta^n$$

is weakly equivalent to X again, cf. Proposition 3.1.20. For a fibration $f : X \rightarrow Y$ of fibrant simplicial presheaves, we then obtain a morphism of sites

$$(T \times \Delta \downarrow f) : (T \times \Delta \downarrow X) \rightarrow (T \times \Delta \downarrow Y).$$

To this morphism we can then apply the Grothendieck-Leray spectral sequence.

With all the definitions of sites of simplices and the edge-path site in place, it is possible to define what a local coefficient system should be:

DEFINITION 3.5.7 *A fibrant sheaf $\mathcal{F}$ of spectra on the site of simplices $(T \times \Delta \downarrow X)$ is called a local coefficient system if it factors through the edge path-site $\pi_T(X)$. Similarly, $\mathcal{F}$ is said to be a constant local coefficient system if it factors through the site T .*

Note that local coefficient system in the above definition does not refer at all to the topology of the site T . Rather, all the sheaves that are pulled back from the site T are *constant* local coefficient systems. It is rather related to the nontriviality of the fundamental groupoid sheaf of the given simplicial presheaf X .

Having local coefficient systems, it is then possible to define cohomology of X with local coefficients as the sheaf hypercohomology of the (terminal object of the) site $(T \times \Delta \downarrow X)$, where the coefficient sheaf is a local coefficient system. It then follows by the standard argument that a weak equivalence of fibrant simplicial presheaves induces isomorphisms in cohomology also with local coefficients.

PROPOSITION 3.5.8 *Let $f : Y \rightarrow X$ be a weak equivalence of simplicial presheaves. Then for any local system $\mathcal{F}$ of presheaves of fibrant spectra on X , the map f induces an isomorphism*

$$\mathbb{H}_T^\bullet(X, \mathcal{F}) \cong \mathbb{H}_T^\bullet(Y, f^* \mathcal{F}).$$

Then we define for a given fibration of fibrant simplicial presheaves a local coefficient system, which is given by the cohomology of the fibres. This is again a standard procedure.

DEFINITION 3.5.9 *Let T be a Grothendieck site, and let $f : Y \rightarrow X$ be a fibration of fibrant simplicial presheaves in $\Delta^{\text{op}}\text{PShv}(T)$. Furthermore, let $\mathcal{F}$ be a presheaf of fibrant spectra on Y . For an object $\sigma : U \times \Delta^n \rightarrow X$ of $(T \times \Delta \downarrow X)$, let $\sigma^*(Y)$ be the simplicial sheaf obtained by pulling back f along σ :*

$$\begin{array}{ccc} \sigma^*(Y) & \xrightarrow{\sigma'} & Y \\ \downarrow & & \downarrow f \\ U \times \Delta^n & \xrightarrow{\sigma} & X. \end{array}$$

We then define the local coefficient system on X associated to f as the presheaf $\mathbb{H}_T^\bullet(f, \mathcal{F})$ given by

$$\sigma \mapsto \mathbb{H}_T^\bullet(\sigma^*(Y), \sigma'^*(\mathcal{F})).$$

LEMMA 3.5.10 *The local coefficient system associated to a fibration of simplicial sheaves defined in Definition 3.5.9 is indeed a local coefficient system as defined in Definition 3.5.7.*

We give some short examples of local coefficient systems, which shows that computing the E_2 -term of the Serre spectral sequence to be constructed can be a quite non-trivial question.

EXAMPLE 3.5.11 The simplest example is given by a fibration over a constant simplicial sheaf, e.g. the constant simplicial sheaf associated to an abelian variety

A/k on the Nisnevich site Sm_k for a field k . As remarked in Section 2.3, such a simplicial sheaf is fibrant and therefore homotopy discrete.

Thus the site of simplices as well as the edge-path site are equal to the site of points of the abelian variety, whose objects are morphisms $f : U \rightarrow A$ with U a smooth scheme over k . The topology is given by the Nisnevich topology on Sm_A .

Any étale locally trivial morphism $g : X \rightarrow A$ produces a locally constant sheaf

$$U \mapsto H_{\mathrm{Nis}}^*(X \times_A U, \mathbb{Z}),$$

which is the local coefficient system associated to the fibration $\tilde{g} : \tilde{X} \rightarrow A$. Here $\tilde{g}$ is a fibrant replacement of g . The E_2 -term of the spectral sequence is then the cohomology of a locally constant sheaf on the abelian variety A . For étale locally constant, such an object is basically a representation of the étale fundamental group of A . $\square$

EXAMPLE 3.5.12 As another example of how complicated it can be to determine cohomology with local coefficients, we consider $\mathbb{A}^1$ -local models of Chevalley groups. As we will see in Chapter 5, the $\mathbb{A}^1$ -fundamental groupoid of the Chevalley groups with root system Φ is given by the corresponding unstable Karoubi-Villamayor K-groups $KV_1(\Phi, -)$ resp. $KV_2(\Phi, -)$. Moreover, the crossed module structure relating KV_1 and KV_2 can also be rather non-trivial.

Now a local coefficient system on a Chevalley group $G(\Phi)$ is then basically given by a representation of the 1-type crossed module of $G(\Phi)$, whose homotopy groups are $KV_1(\Phi, -)$ resp. $KV_2(\Phi, -)$. We therefore see that the edge-path site of a fibrant $\mathbb{A}^1$ -local model of the Chevalley group $G(\Phi)$ contains much information about the arithmetic of the base scheme, as it describes the lower unstable K-groups. It is not clear, how many non-constant local coefficient systems actually exist, and what the twisted cohomology of the Chevalley groups looks like. $\square$

EXAMPLE 3.5.13 Finally, we want to recall that the universal torsors resp. the Cox rings, cf. Example 3.3.8, also yield examples of coverings and therefore of local coefficient systems of simplicial presheaves. Let $\mathcal{X}$ be a simplicial presheaf which is a fibrant and $\mathbb{A}^1$ -local model for a toric variety X_Δ . Then there is a torus bundle over X_Δ with fibre a torus T , whose total space is $\mathbb{A}^n \setminus Z$ for some closed subscheme $Z \subseteq \mathbb{A}^n$.

The above covering is an $\mathbb{A}^1$ -covering which makes the torus T a part of the $\mathbb{A}^1$ -fundamental group of X_Δ . In particular, the edge-path site of $\mathcal{X}$ factors through the morphism of sites $\mathrm{Sm}_{\mathbb{A}^n \setminus Z} \rightarrow \mathrm{Sm}_{X_\Delta}$ given by $(f : U \rightarrow \mathbb{A}^n \setminus Z) \mapsto \pi \circ f$ with $\pi : \mathbb{A}^n \setminus Z \rightarrow X_\Delta$. Therefore, any sheaf $\mathcal{F}$ over $\mathbb{A}^n \setminus Z$ gives rise to a local coefficient system of X_Δ by pulling back $\mathcal{F}$ to $\pi_T(X)$. In particular, a general sheaf $\mathcal{F}$ on $\mathbb{A}^n \setminus Z$ does not come from a sheaf on X_Δ , which provides many non-constant local coefficient systems. $\square$

Motivic Cohomology with Local Coefficients: What we have used above were general sheaves of fibrant spectra on the edge-path sites of a given simplicial presheaf. If we now want to define what *motivic cohomology with local coefficients* should be, we have to restrict the coefficient systems: Motivic cohomology is not the Nisnevich cohomology of a sheaf of abelian groups, but rather is itself Nisnevich

hypercohomology with coefficients in some complex or spectrum usually abusively denoted by $\mathbb{Z}(n)$.

These sheaves of fibrant spectra are defined on the site Sm_k of smooth varieties over a field k , cf. [MVW06], and therefore, we get fibrant sheaves of spectra on the sites $(\mathrm{Sm}_k \times \Delta \downarrow X)$ for any fibrant simplicial presheaf X on Sm_k by pulling back $\mathbb{Z}(n)$ along the morphism of sites $(\mathrm{Sm}_k \times \Delta \downarrow X) \rightarrow \mathrm{Sm}_k$. This yields a constant coefficient system, which computes the motivic cohomology of X , cf. Proposition 3.5.6.

Then we consider the stable model category of sheaves of spectra over the site $(\mathrm{Sm}_k \times \Delta \downarrow X)$ resp. over the edge-path site $\pi_{\mathrm{Sm}_k}(X)$. We want some highly structured model of this situation, as discussed in [ØR06] resp. the full version of this paper. Note that again there are some technical issues in setting this up. Once we have such a highly structured model for the sheaves of spectra on $\pi_{\mathrm{Sm}_k}(X)$, we can also talk about modules over the sheaf of ring spectra $p^*\mathbb{Z}(n)$, where p is the morphism of sites $\pi_{\mathrm{Sm}_k}(X) \rightarrow \mathrm{Sm}_k$. We claim that the right definition of local coefficient system for motivic cohomology is the pullback to $(\mathrm{Sm}_k \times \Delta \downarrow X)$ of a *module over $p^*\mathbb{Z}(n)$ on $\pi_{\mathrm{Sm}_k}(X)$* . Motivic cohomology with local coefficient systems can then be computed via the local-global spectral sequence for sheaf hypercohomology on $(\mathrm{Sm}_k \times \Delta \downarrow X)$, cf. Proposition 3.5.2. Note that both the motive of a variety over k as well as the K-theory sheaves are modules over motivic cohomology, and therefore yield examples of motivic cohomology of the field k “with coefficients”.

The Local-Global Serre Spectral Sequence: Assuming the results from the previous paragraphs which we did not actually prove, it is possible to obtain the following version of the Serre spectral sequence for a fibration of simplicial presheaves using almost the same proof as [MS93, Theorem 3.2].

THEOREM 3.5.14 *For any fibration $p : E \rightarrow B$ of simplicial sheaves on the site T , and any presheaf $\mathcal{F}$ of fibrant spectra on T , there is a natural spectral sequence*

$$E_2^{p,q} = H_T^p(B, \tilde{\pi}_q \mathbb{H}_T^\bullet(f, \mathcal{F})) \implies \pi_{q-p} \mathbb{H}_T^\bullet(E, \mathcal{F}).$$

The argument runs something like this: We start with a fibration $f : X \rightarrow Y$ of fibrant simplicial presheaves, from which we construct a morphism of sites

$$(T \times \Delta \downarrow f) : (T \times \Delta \downarrow X) \rightarrow (T \times \Delta \downarrow Y).$$

Homotopy distributivity and the corresponding decomposition of fibrations, cf. Lemma 3.1.22, implies that the local coefficient system on $(T \times \Delta \downarrow Y)$ defined in Definition 3.5.9 really computes the cohomology of the total space, since the total space is the homotopy colimit of the fibres, indexed by the canonical homotopy colimit decomposition of the base space. Then we apply the Grothendieck-Leray spectral sequence from Theorem 3.5.4 to the above morphism of sites. All we need to prove is that the coefficient system $\mathbb{H}^\bullet(f, \mathcal{F})$ on $(T \times \Delta \downarrow Y)$ defined in Definition 3.5.9 is the same as the coefficient system $\mathbb{R}^\bullet f_*(\mathcal{F})$. This can be proven along the same lines of the proof of [MS93, Theorem 3.2].

One of the special cases of the above spectral sequence are the descent spectral sequences for algebraic K-theory, cf. [Tho85, Nis89].

It would be interesting to know if it is possible to prove some motivic version of the Whitehead theorem stating that any morphism of motivic spaces is a weak equivalence if and only if it induces an isomorphism on the fundamental groupoid and isomorphisms on ordinary cohomology with twisted coefficients. For diagrams of simplicial sets, such a statement has been obtained in [MS95]. A similar version for simplicial presheaves seems therefore plausible, but the $\mathbb{A}^1$ -localization process needed to obtain motivic spaces really poses serious difficulties.

Multiplicative Structures: The multiplicative structure on the Serre spectral sequence is a very useful thing for determining the ring structure e.g. of the cohomology of the Grassmannians. To obtain a multiplicative structure on the Serre spectral sequence for a fibration of simplicial presheaves, there are two technicalities to surmount.

On the one hand, one has to show that the presheaf of fibrant spectra used to define the hypercohomology already has a multiplicative structure. For motivic cohomology, this is done in [MVW06, Construction 3.9]. For K-theory, a construction of a multiplicative structure can be found e.g. in [Jar97]. A general discussion of product structures in hypercohomology can be found in [FS02, Appendix A].

On the other hand, the proof of the multiplicative structure on the Serre spectral sequence in [MS93] uses that there is a monoidal model structure on the category of diagrams of chain complexes of abelian groups. To generalize this proof to the setting of hypercohomology of presheaves of fibrant spectra, one would need a monoidal model structure on the category of presheaves of spectra. This is also a rather nontrivial technicality, since the usual injective model structure on the category of chain complexes on a site is not monoidal. One either has to use a version of the *flat model structure* as developed in the papers [Hov01a, Gil06] or a highly structured model for the category of sheaves of spectra as e.g. in [ØR06]. The latter work also allows to talk about modules over motivic cohomology, which we need to say what motivic cohomology with local coefficients is. Mastering the technicalities of such highly structured models for stable motivic homotopy is yet another complication in a complete proof of the Serre spectral sequence.

Motivic Serre Spectral Sequence: One particular case of interest is the local-global type Serre spectral sequence we obtain by plugging in the presheaf of fibrant spectra corresponding to the motivic complexes $\mathbb{Z}(n)$, cf. [MVW06]. However, the E_2 -term is only ordinary sheaf cohomology of the base with coefficients in the motivic cohomology groups of the fibre. It does not seem possible to get a Serre spectral sequence starting with motivic cohomology of the base with coefficients in the motivic cohomology of the fibres and converging to the motivic cohomology of the total space.

The reason is that two special cases of such a Serre spectral sequence would be an Atiyah-Hirzebruch spectral sequence for any “generalized motivic cohomology theory”, as well as a Künneth formula for Chow groups. These consequences however can only be obtained under cellularity assumptions:

One can only hope for an Atiyah-Hirzebruch spectral sequence for a generalized motivic cohomology theory if this cohomology theory is cellular over the motivic complexes $\mathbb{Z}(n)$. This is the case for algebraic K-theory and algebraic cobordism

[DI05, Theorem 6.2, Theorem 6.4]. Atiyah-Hirzebruch type spectral sequences have been obtained for K-theory by [FS02] and for algebraic cobordism in unpublished work of Hopkins and Morel.

On the other hand, a Künneth theorem for motivic cohomology can also only hold for schemes that are cellular in the sense of [DI05]. Under linearity assumptions, a Künneth spectral sequence has been obtained for Chow groups and algebraic K-theory in [Jos01], and under cellularity assumptions in [DI05, Theorem 8.6]. Maybe there exist spectral sequences for other cellular types of schemes, but these are then not spectral sequences in the category of modules over the motivic cohomology of the base scheme.

Since already the special cases of the Serre spectral sequence do only work in restricted settings, there is no hope for such a spectral sequence to work in general. The only form of the Serre spectral sequence that works in general settings is the local-global type sketched above.

Chapter 4

Fibrewise Localization

Whereas in previous chapters we have almost exclusively considered model categories of simplicial presheaves, we will now also consider localizations of such. Since the Bousfield localization is as complicated as the fibrant replacement, it is quite difficult to extract any geometric information from the local model structures. It is therefore necessary to develop tools to keep track of such information in the localization process.

One particular possibility to do so is the construction of fibrewise localizations for model categories of presheaves. From such fibrewise localizations it is possible to give precise criteria whether a given fibre sequence is local with respect to a given map $f : W \rightarrow *$. The main application we have in mind is the $\mathbb{A}^1$ -local model structure on $\Delta^{\text{op}}\mathcal{PShv}(\mathbf{Sm}_S)$ and fibre sequences in this model structure. Some results in this direction were already obtained by Morel [Mor06b] over base *fields*. We want to note however that our characterization results is sharp and applies to motivic homotopy theory over any base scheme.

Recall that our main techniques in Chapter 3 were based on the fact that fibre sequences and homotopy pullbacks are something local: It is possible to check if $F \rightarrow E \rightarrow B$ is a fibre sequence by looking at the points of the topos $\mathcal{Shv}(T)$. We will see in this chapter, that the situation for homotopy pullbacks in an f -localization of the model structure on $\Delta^{\text{op}}\mathcal{PShv}(T)$ is rather different. For a general localization functors, even localizations with respect to a general map $f : A \rightarrow B$, the problem of determining local fibre sequences seems rather intractable. Already in the easiest case of a localization with respect to a *null-homotopic* map $f : A \rightarrow B$, there exist *global* objects which provide obstructions to a fibre sequence being a local fibre sequence. This is the main result of this chapter, formulated in Theorem 4.3.10. However, the existence of fibrewise localization which is needed for this theorem depends on homotopy distributivity of the underlying model category.

We will start this chapter by recalling the notion of localization functors and general definitions concerning Bousfield localization in Section 4.1. Following this, we will describe the three possible ways of doing fibrewise localization in model categories of simplicial presheaves in Section 4.2. The existence of fibrewise localizations implies a characterization of local fibre sequences, which we will prove in Section 4.3. Some applications of this characterization result to $\mathbb{A}^1$ -homotopy theory are given in Section 4.4.

We note that all of the results proved in this chapter are known to hold for simplicial sets. We see again that most of the powerful localization theory can be generalized to the setting of simplicial presheaves, once we know the key fact homotopy distributivity.

4.1 Localization Functors for Simplicial Presheaves

This section collects elementary facts concerning localizations and nullifications of simplicial presheaves. These facts are well-known for simplicial sets resp. topological spaces and proofs mostly carry over directly. The only hard work is the theorem on existence, universality and continuity of localization functors which was established in [GJ98].

There are two approaches to localizations in the literature for simplicial sets: On the one hand, the whole book [Dro96] does not require a single word on model categories, and defines localizations as certain homotopy-universal coaugmented functors. On the other hand, starting from a proper, simplicial, cofibrantly generated model structure on the category $\mathcal{C}$, it is possible to define a new model structure by keeping the cofibrations, and defining a new class of *local weak equivalences*. The relation between the two approaches is easily described: The proof of existence of localization functors in [Dro96] is the same as the small object argument proving that Quillen's axioms hold for the local model structure. On the other hand, assuming known the existence of this model structure, the localization functor is basically the fibrant replacement in the local model structure.

First, we shortly repeat some of the basic notions for Bousfield localization, then we repeat the definitions for localization functors as well as some basic facts on localization functors.

Bousfield Localization: We repeat the standard definitions of local objects and local weak equivalences. These definitions can be found in [Dro96, Hir03] for the case of simplicial sets, and in [MV99] for the case of simplicial presheaves.

Let $\mathcal{C}$ be a model category, and let $f : A \rightarrow B$ be a morphism of cofibrant objects.

DEFINITION 4.1.1 (LOCAL OBJECTS, WEAK EQUIVALENCES) *An object $X \in \mathcal{C}$ is called f -local if X is fibrant and the following morphism is a weak equivalence:*

$$\mathbf{Hom}(B, X) \rightarrow \mathbf{Hom}(A, X).$$

A morphism $g : X \rightarrow Y \in \mathcal{C}$ is called an f -local weak equivalence if for any f -local object Z , the following morphism is a weak equivalence:

$$\mathbf{Hom}(Y, Z) \rightarrow \mathbf{Hom}(X, Z).$$

Of course, one can consider more general localizations, i.e. localizations with respect to a *set* of maps as in [MV99, Section 2.2], or homology localization as in [GJ98, Section 3]. If f is null-homotopic such a localization is also called *nullification*, and we also write L_W . The most important applications we have in mind are the $\mathbb{A}^1$ -nullification functors $L_{\mathbb{A}^1}$ on $\Delta^{\mathrm{op}}PShv(\mathbf{Sm}_S)$.

Localization Functors: This paragraph repeats the theorem on existence and universality of localization functors for simplicial presheaves. Most of the elementary facts in this section are easy consequences of this theorem, which is proved in [MV99, Theorem 2.2.5] and in similar form in [GJ98, Theorem 4.4].

We start recalling the definition of localization functor in a general model category.

DEFINITION 4.1.2 *A functor $F : \mathcal{C} \rightarrow \mathcal{C}$ is called coaugmented if there is a natural transformation $j : \text{id}_{\mathcal{C}} \rightarrow F$. A coaugmented functor F is called idempotent if the two natural maps $j_{FX}, Fj_X : FX \rightrightarrows FFX$ are weak equivalences and homotopic to each other. The coaugmentation map j_X is homotopy universal with respect to maps into local spaces if any map $X \rightarrow T$ into a local space T factors uniquely (up to homotopy) through $j_X : X \rightarrow FX$. The functor F is called simplicial if it is compatible with the simplicial structure, i.e. if there exist functorial morphisms $\sigma : (FX) \otimes K \rightarrow F(X \otimes K)$ for any object $X \in \mathcal{C}$ and any simplicial set K . These morphisms have to satisfy some rather obvious conditions described in [Dro96, Definition 1.C.8]. The functor F is called continuous if it induces a morphism on inner function spaces*

$$\mathbf{Hom}(X, Y) \rightarrow \mathbf{Hom}(FX, FY),$$

which is compatible with composition.

We recall the existence of localizations for simplicial presheaf categories from [GJ98, Theorem 4.4], which is the proper generalization of [Dro96, Theorem A.3]. The existence of the f -local model structure is proven in [GJ98, Theorem 4.8]. Note that the existence of localizations for simplicial presheaves is a global result, in the sense that it does not simply follow from the existence of localizations of simplicial sets by looking at the points of the topos.

THEOREM 4.1.3 *Let $f : A \rightarrow B$ be any cofibration in $\Delta^{\text{op}}\text{PShv}(T)$ and suppose α is an infinite cardinal which is an upper bound for the cardinalities of both B and the set of morphisms of T . Then there exists a functor L_f , called the f -localization functor, which is coaugmented and homotopically idempotent. Any two such functors are naturally weakly equivalent to each other. The map $X \rightarrow L_f X$ is a homotopically universal map to f -local spaces. Moreover, L_f can be chosen to be simplicial and continuous.*

There is a simplicial model structure on $\Delta^{\text{op}}\text{PShv}(T)$ where the cofibrations are monomorphisms, weak equivalences are f -local weak equivalences and fibrations are defined via the right lifting property.

REMARK 4.1.4 (PROPERNESS) *In [Hir03, Chapter 3], Bousfield localizations of general model categories are investigated. As shown in [Hir03, Proposition 3.4.4 and Theorem 4.1.1], left Bousfield localizations preserve left properness, i.e. the left Bousfield localization of a left proper model category is again left proper.*

The f -local model structure for a morphism $f : A \rightarrow B$ is not in general right proper. It is known [Jar00, Theorem A.5], that the f -local model structure is proper if f is of the form $* \rightarrow I$. A special case of this is the properness of the homotopy theory of a site with interval, which is proved in [MV99, Theorem 2.2.7 and Section

2.3]. Note however that in the formulation of the theorem left and right properness have to be interchanged. We will see at the end of Section 4.3, that properness only occurs for null-homotopic morphisms.

REMARK 4.1.5 (INTERNAL HOM) *As the internal mapping spaces play a fundamental role in the description of localizations, we recall some details and the difference between pointed and unpointed mapping spaces.*

The usual cartesian closed structure on $\mathbf{Shv}(T)$ consists of an adjunction between the product of simplicial presheaves, and the internal hom. That the product preserves cofibrations and trivial cofibrations is clear, since cofibrations are just monomorphisms. The weak equivalences can be checked at the points, and therefore the fact that the product of simplicial sets preserves trivial cofibrations implies the same fact for simplicial presheaves. The total derived functor theorem 2.1.13 therefore implies the existence of a Quillen adjunction, whose right derived part $\mathbf{RHom}$ is the internal mapping space of simplicial presheaves. For cofibrant X and fibrant Y , it can be defined as $\mathbf{Hom}(X \times \Delta^\bullet, Y)$, and its global sections yield the simplicial set mapping space belonging to the simplicial structure on $\Delta^{\text{op}}\mathbf{PShv}(T)$.

We want to note that the definition of (based) loop space in a pointed simplicial model category as given before Definition 2.1.8 implies the characterization as $\Omega X = \mathbf{RHom}_*(S^1, X)$.

There are versions of internal mapping spaces for both unpointed and pointed categories, and their relation is given by the following is a fibration sequence:

$$\mathbf{Hom}_*(X, Y) \rightarrow \mathbf{Hom}(X, Y) \rightarrow Y.$$

For simplicial presheaves X and Y on a site T with enough points, this can be easily proven using the above fibre sequence for simplicial sets [Dro96, Definition 1.A.1], the fact that the internal mapping spaces at the points are given by the simplicial set mapping spaces, and then putting everything together using Proposition 3.3.1. This implies that if we use the pointed mapping spaces in the definition of localization, only the connected component of the chosen base point will be localized, the others remain unchanged. This is one reason for connectivity restrictions on fibrewise localizations using pointed functors in Section 4.2.

Elementary facts concerning f -local spaces and L_f : There is a collection of elementary facts concerning localization functors in [Dro96]. These also hold for localization functors on categories of simplicial presheaves. We will neither state nor prove all of them, we only discuss those that are directly relevant to fibre sequences.

PROPOSITION 4.1.6 *Any homotopy limit of f -local simplicial presheaves is again f -local. In particular, we have the following conclusions:*

- (i) *The homotopy fibre of a morphism of f -local simplicial presheaves is f -local.*
- (ii) *The product of any family of f -local simplicial presheaves is f -local. The morphism*

$$L_f(X \times Y) \rightarrow L_f X \times L_f Y$$

is a homotopy equivalence.

(iii) For f -local Y and cofibrant X , the simplicial presheaves $\mathbf{Hom}(X, Y)$ and $\mathbf{Hom}_*(X, Y)$ are f -local. Therefore, the loop spaces $\Omega^n Y$ are f -local for f -local Y .

PROOF: Let T be a Grothendieck site, and let $\mathcal{I}$ be a small category. Furthermore, let $\mathcal{X} : \mathcal{I} \rightarrow \Delta^{\text{op}}\mathcal{PShv}(T)$ be a diagram of f -local simplicial presheaves. We want to show that

$$\mathbf{Hom}(f, \text{holim}_{\mathcal{I}} \mathcal{X}) : \mathbf{Hom}(B, \text{holim}_{\mathcal{I}} \mathcal{X}) \rightarrow \mathbf{Hom}(A, \text{holim}_{\mathcal{I}} \mathcal{X})$$

is a weak equivalence. By Proposition 3.1.5, this map is the same as the map $\text{holim}_{\mathcal{I}} \mathbf{Hom}(f, \mathcal{X})$. By assumption, the map

$$\mathbf{Hom}(f, \mathcal{X}(i)) : \mathbf{Hom}(B, \mathcal{X}(i)) \rightarrow \mathbf{Hom}(A, \mathcal{X}(i))$$

is a weak equivalence for any $i \in \mathcal{I}$. By Proposition 3.1.4, this implies that $\text{holim}_{\mathcal{I}} \mathbf{Hom}(f, \mathcal{X})$ is a weak equivalence as well.

(i) and (ii) are then clear by applying the above to the relevant diagrams, and the homotopy equivalence $L_f(X \times Y) \rightarrow L_f X \times L_f Y$ follows from universality and adjointness, cf. [Dro96, Section 1.G].

For (iii), we have by adjointness and locality of Y the weak equivalences:

$$\begin{aligned} \mathbf{Hom}(A, \mathbf{Hom}(X, Y)) &\simeq \mathbf{Hom}(X, \mathbf{Hom}(A, Y)) \\ &\simeq \mathbf{Hom}(X, \mathbf{Hom}(B, Y)) \\ &\simeq \mathbf{Hom}(B, \mathbf{Hom}(X, Y)). \end{aligned}$$

The statement on loop spaces follows from this using Remark 4.1.5. ■

Interaction of Localization and Homotopy Colimits:

PROPOSITION 4.1.7 ([Dro96], PROPOSITION 1.D.2) *Let T be a site, let $\mathcal{I}$ be a small category and let $f : A \rightarrow B$ be a morphism of cofibrant spaces. Furthermore, let $g : \mathcal{X} \rightarrow \mathcal{Y}$ be an $\mathcal{I}$ -diagram of cofibrant objects of simplicial presheaves in $\Delta^{\text{op}}\mathcal{PShv}(T)$. Assume that for any $i \in \mathcal{I}$ the morphism $g(i) : \mathcal{X}(i) \rightarrow \mathcal{Y}(i)$ is an L_f -equivalence. Then*

$$\text{hocolim}_{\mathcal{I}} g : \text{hocolim}_{\mathcal{I}} \mathcal{X} \rightarrow \text{hocolim}_{\mathcal{I}} \mathcal{Y}$$

is also an L_f -equivalence for both pointed and unpointed homotopy colimits.

PROOF: Using [MV99, Proposition 2.2.9], it suffices to show for any f -local object Z that

$$\mathbf{Hom}(\text{hocolim}_{\mathcal{I}} g, Z) : \mathbf{Hom}(\text{hocolim}_{\mathcal{I}} \mathcal{X}, Z) \rightarrow \mathbf{Hom}(\text{hocolim}_{\mathcal{I}} \mathcal{Y}, Z)$$

is a weak equivalence. Using Proposition 3.1.5, it suffices to show that the morphism $\text{holim}_{\mathcal{I}} \mathbf{Hom}(g, Z)$ is a weak equivalence. By assumptions the objects $\mathbf{Hom}(\mathcal{X}(i), Z)$ resp. $\mathbf{Hom}(\mathcal{Y}(i), Z)$ are fibrant and $\mathbf{Hom}(g(i), Z)$ is a weak equivalence for any $i \in \mathcal{I}$. Therefore, using Proposition 3.1.4, we conclude that the morphism $\text{holim}_{\mathcal{I}} \mathbf{Hom}(g, Z)$ is a weak equivalence. ■

Applying the above to the diagram of augmentation maps $g : \mathcal{X} \rightarrow L_f \mathcal{X}$, we obtain that

$$L_f(\operatorname{hocolim}_{\mathcal{I}} g) : L_f \operatorname{hocolim}_{\mathcal{I}} \mathcal{X} \rightarrow L_f \operatorname{hocolim}_{\mathcal{I}} L_f \mathcal{X}$$

is a weak equivalence, generalizing [Dro96, Theorem D.3]. In particular we get $L_f(X \vee Y) \simeq L_f(L_f X \vee L_f Y)$. Note that a homotopy colimit of local spaces need not be local again, which is one difficulty in establishing e.g. an $\mathbb{A}^1$ -local van-Kampen theorem. Note also that the interaction of homotopy limits and localizations is not as easy as the homotopy *colimit* case. Although we have seen in Proposition 4.1.6, that the homotopy fibre of a morphism of local spaces is local, it is not the case that the homotopy fibre of the localization of a morphism g is the localization of the homotopy fibre of g . The rest of this whole chapter is devoted to studying under which circumstances a homotopy pullback diagram in $\Delta^{\text{op}} PShv(T)$ is a homotopy pullback diagram in a localization of $\Delta^{\text{op}} PShv(T)$.

4.2 Fibrewise Localization for Simplicial Presheaves

In this section, we prove the existence of fibrewise versions of simplicial coaugmented functors in categories of simplicial presheaves. For a treatment of fibrewise localization for simplicial sets resp. topological spaces, see [Dro96, Section 1.F] and [Hir03, Chapter 6].

Fibrewise Localization: The main motivation for studying fibrewise localization is a gain of control on the behaviour of a fibration under a localization functor. For a locally trivial morphism $f : E \rightarrow B$ of topological spaces with fibre F , one can explain quite easily how to construct the fibrewise localization: Take a trivialization of f , i.e. a covering U_i of B over which $f|_{U_i} : E \times_B U_i \cong U_i \times F \rightarrow U_i$. Then apply the simplicial coaugmented functor: On the level of the trivialization one simply replaces the space F by the space LF . On the level of transition morphisms, one applies the functor L to the transition map. For this to work we need the functor L to be continuous. This produces an explicit recipe to construct an LF -bundle over B .

In this paragraph, we will formulate the above construction for general model categories of simplicial presheaves. The key to all the constructions described below will again be homotopy distributivity.

DEFINITION 4.2.1 *Let L be a simplicial coaugmented functor on $\Delta^{\text{op}} Shv(T)$. We say that L can be applied fibrewise resp. that there is a fibrewise version of L , if every fibre sequence of simplicial presheaves $F \rightarrow E \rightarrow B$ can be mapped via a homotopy commutative diagram*

$$\begin{array}{ccccc} F & \longrightarrow & E & \longrightarrow & B \\ \downarrow & & \downarrow & & \downarrow \\ LF & \longrightarrow & \overline{E} & \longrightarrow & B \end{array}$$

to a fibre sequence with fibre LF over B .

In the more specific case of localization functors, the morphism on total spaces should of course be a local weak equivalence.

DEFINITION 4.2.2 (FIBREWISE LOCALIZATION) *Let L_f be a localization functor. Then we say that L_f admits a fibrewise version, if there exists a homotopy commutative diagram as in Definition 4.2.1, where the morphism $E \rightarrow \overline{E}$ is an f -local weak equivalence.*

REMARK 4.2.3 *There is a difference between the above notion of fibrewise localization and the one used in [Dro96]. Whereas Dror Farjoun requires a concrete model for the fibrewise localization, the morphism $\overline{E} \rightarrow B$ being a fibration with fibre LF , we only require the existence of a fibre sequence. This is weaker, in that LF need not actually be the fibre of a fibrant replacement of $\overline{E} \rightarrow B$. For the homotopy groups this is not essential.*

We want to note that pointed and unpointed simplicial sets behave rather differently with respect to fibrewise localization. For unpointed simplicial sets, one can construct fibrewise localizations in various different ways, whereas for pointed simplicial sets, one always has to make special connectivity assumptions on the base resp. the fibre. This can be explained in two ways: First, there is no continuous choice of base point in a nontrivial fibre sequence $F \rightarrow E \xrightarrow{p} B$. Any such choice would yield a section $s : B \rightarrow E$ of p . By the Dold theorem [Dol63], this implies that the fibre sequence is trivial, cf. Proposition 4.3.13. The other way to see this difference is to note that the crucial step in the construction of fibrewise localization in [Dro96, Theorem 1.F.3] is a decomposition of the base B as an *unpointed* homotopy colimit of contractible spaces. This would not be possible with *pointed* homotopy colimits, because a pointed homotopy colimit of contractible spaces is again contractible.

This difference between the unpointed and the pointed setting is also illustrated by the following example [Hir03, Proposition 6.1.4]. See also the discussion in [Dro96, Remark 1.A.7].

EXAMPLE 4.2.4 Let $f : S^2 \rightarrow D^3$ be the inclusion in the category of pointed topological spaces. The fibration $\pi \circ p_2 : S^2 \times \mathbb{R} \rightarrow \mathbb{R} \rightarrow S^1$, where $\pi : \mathbb{R} \rightarrow S^1$ is the universal covering, does not have a fibrewise f -localization. The reason is that the fundamental group of S^1 would act transitively via unpointed weak equivalences on the fibres of such a localization. This is impossible since the fibrewise pointed localization would have $\mathbb{Z}$ -many components of which the base-point component is contractible and the others are still weakly equivalent to S^2 . $\square$

The problem in the above example is that the pointed localization functor does not localize all the components of the fibre $\coprod S^2$. Therefore when dealing with a pointed localization functor, we usually have to restrict to fibre sequences $F \rightarrow E \rightarrow B$ where the base B is simply connected, resp. ΩB is path-connected, cf. Definition 2.3.6. On the other hand, fibrewise localizations exist without any provisos for *unpointed* spaces. If an underlying unpointed category exists, we can apply the unpointed localization functor, which will yield the right homotopy types for connected spaces. However, this unpointed functor fails to satisfy the homotopy universality for the pointed category.

In [Dro96], several constructions of fibrewise localization are given. We will show that all of them also work in the category of simplicial presheaves on any

Grothendieck site T . The key input in all of these methods is again the homotopy distributivity from Section 3.1. Our results will be formulated in the pointed category, since fibre sequences are only defined for pointed model categories, cf. Definition 2.1.9. We will however use the *unpointed* localization functor, so that we can avoid connectivity assumptions on the base.

There are three methods for doing fibrewise localization described in [Dro96]. One proceeds via classifying spaces, one via the canonical homotopy colimit decomposition for a fibration discussed in Lemma 3.1.22, and the last one via a modification of the change-of-fibre morphism from Proposition 3.2.18. Only the last one has been generalized to model categories by [CS06], but we will see that the other methods work as well, at least for simplicial presheaves.

Fibrewise Nullification after Chataur and Scherer: One of the versions of fibrewise localization [Dro96, Proposition 1.F.7] has been extended to general model categories by Chataur and Scherer [CS06]. The advantage of this construction is that it works even for functors which are not simplicial resp. continuous. On the other hand, this approach works for *pointed spaces* only under the assumption that the fibre is connected.

There are some axioms that are discussed and used in [CS06]. We remark that the cube axiom used in that paper is the same thing as Mather's cube theorem 3.1.15, and the ladder axiom is a form of homotopy distributivity applied to sequential colimit diagrams saying that sequential homotopy colimits commute with homotopy pullbacks. From our discussion in Section 3.1 we know that both axioms are satisfied in any model category of simplicial presheaves.

In [CS06, Definition 3.6], Chataur and Scherer also introduce the following join axiom:

DEFINITION 4.2.5 (JOIN AXIOM) *A model category $\mathcal{M}$ satisfies the join axiom for the nullification functor L_W if the join of W with the cofibre of any generating fibration is weakly W -contractible.*

PROPOSITION 4.2.6 *Let T be a site. Then $\Delta^{\text{op}}\mathcal{PShv}(T)$ satisfies the join axiom for any nullification functor L_f .*

PROOF: We first note that the localization functors commute with finite products as remarked in the proof of [MV99, Lemma 2.2.32], cf. Proposition 4.1.6. Therefore $X \times W \rightarrow X$ is an f -local weak equivalence for any X . By left properness of the f -local model structure, cf. Theorem 4.1.3 the homotopy pushout of the f -local weak equivalence $X \times W \rightarrow W$ along $X \times W \rightarrow W$ will be an f -local weak equivalence again. Since W is weakly W -contractible, then so is $X * W$. ■

It is easy to see that the above proposition would hold in any left proper model category such that the localization preserves products. Moreover, we have proved in Proposition 4.1.6 that any localization functor commutes with products in a model category of simplicial presheaves, as a consequence of distributivity.

Now we can state the fibrewise localization result of Chataur and Scherer.

THEOREM 4.2.7 ([CS06], THEOREM 4.3) *Let $\mathcal{M}$ be a model category which is pointed, left proper, cellular and in which the cube, join and the ladder axiom holds.*

Let $L_f : \mathcal{M} \rightarrow \mathcal{M}$ be a localization functor, $p : F \rightarrow E \rightarrow B$ be a fibre sequence in $\mathcal{M}$, and assume that ΩB is path-connected in the sense of Definition 2.3.6.

Then there exists a fibrewise f -localization of p .

As discussed above, the conditions for the theorem hold for any nullification functor on $\Delta^{\text{op}}\text{PShv}(\text{Sm}_S)$, and in particular for $\mathbb{A}^1$ -nullification.

COROLLARY 4.2.8 *For any fibre sequence $p : F \rightarrow E \rightarrow B$ of pointed simplicial presheaves in $\Delta^{\text{op}}\text{PShv}(\text{Sm}_S)$ with ΩB path-connected, there exists a fibrewise $\mathbb{A}^1$ -localization of p .*

REMARK 4.2.9 *The cellularity assumption above is not necessary, it also works for the injective model structure, since we have good control on the codomains of the generating cofibrations and generating trivial cofibrations, cf. Remark 2.1.20.*

Note that the result of Chataur and Scherer needs the connectivity restriction on ΩB to work for general pointed model categories. We can get rid of this restriction if there is an unpointed model category underlying $\mathcal{M}$, as well as an unpointed version of the localization functor L_f . Note however, that applying the unpointed localization functor will localize *all the components* of the fibre of a given fibration. The next proposition shows how modify the proof from [CS06, Dro96] for unpointed model categories, without any assumption about ΩB . Although we are working in an unpointed setting, we will speak of fibre sequences. What we mean is that F is a homotopy fibre of $p : E \rightarrow B$ over a fixed base point $b \in B$, i.e. the obvious diagram is a homotopy pullback. This makes sense even in an unpointed setting.

PROPOSITION 4.2.10 *Let T be a Grothendieck site, and let $L_f : \Delta^{\text{op}}\text{PShv}(T) \rightarrow \Delta^{\text{op}}\text{PShv}(T)$ be a localization functor. For any fibration $p : E \rightarrow B$ and any base point $b \in B$ there exists a fibrewise f -localization of the homotopy pullback $p^{-1}(b) \rightarrow E \rightarrow B$.*

PROOF: Let $F \rightarrow E \xrightarrow{p} B$ be a fibre sequence, assuming that p is a fibration of fibrant simplicial presheaves on T , and that $F = p^{-1}(b)$ for a chosen base point $b \in B$.

(i) We construct a new fibre sequence. This argument is similar to what we have done in Proposition 3.2.18 and the same as [CS06, Proposition 4.1]. We assume without loss of generality that the augmentation map $F \rightarrow L_f F$ is a cofibration. Then we construct the following diagram:

$$\begin{array}{ccccc}
 F & \xrightarrow{\simeq_f} & L_f F & \longrightarrow & F_1 \\
 \downarrow & & \downarrow & & \downarrow \\
 E & \xrightarrow{\simeq_f} & E \cup_F L_f F & \xrightarrow{\simeq} & E_1 \\
 \downarrow p & & \downarrow q & & \downarrow p_1 \\
 B & \xrightarrow{=} & B & \xrightarrow{=} & B.
 \end{array}$$

The left column is the original fibre sequence. The space in the centre is constructed as the pushout of $F \rightarrow E$ along the augmentation $F \rightarrow L_f F$. We have assumed

that this is a cofibration, and therefore the centre space is also a homotopy pushout. The map q comes from the universal property of the pushout. Then we factor $q : E \cup_F L_f F \rightarrow B$ as a trivial cofibration $E \cup_F L_f F \rightarrow E_1$ and a fibration $p_1 : E_1 \rightarrow B$. Finally, F_1 is the fibre of p_1 over $b \in B$. An argument similar to the one applied in Proposition 3.2.18 implies that F_1 is the homotopy pushout of the homotopy fibres of $L_f F \leftarrow F \rightarrow E$. Therefore the composition $F \rightarrow L_f F \rightarrow F_1$ is an f -local weak equivalence.

(ii) The problem is that the rectification of $L_f F \rightarrow E \cup_F L_f F \rightarrow B$ to the fibre sequence $F_1 \rightarrow E_1 \rightarrow B$ destroys the property of the fibre being f -local. Therefore one has to use something similar to a small object argument, constructing ever better approximations to the total space of the fibrewise localization.

Transfinitely iterating this construction until a well-defined ordinal κ , see (iii), produces telescopes $E \rightarrow E_1 \rightarrow E_2 \rightarrow \dots$ resp. $F \rightarrow F_1 \rightarrow F_2 \rightarrow \dots$. Applying Puppe's theorem 3.1.16 to the diagram

$$\begin{array}{ccccccc} E & \longrightarrow & E_1 & \longrightarrow & E_2 & \longrightarrow & \dots \\ p \downarrow & & p_1 \downarrow & & p_2 \downarrow & & \\ B & \xrightarrow{=} & B & \xrightarrow{=} & B & \xrightarrow{=} & \dots \end{array}$$

we find that the homotopy fibre of $(\operatorname{hocolim}_{\kappa} E_i) \rightarrow B$ is $\operatorname{hocolim}_{\kappa} F_i$. We therefore have a morphism of fibre sequences

$$\begin{array}{ccccc} F & \longrightarrow & E & \longrightarrow & B \\ \downarrow & & \downarrow & & \downarrow = \\ \operatorname{hocolim}_{\kappa} F_i & \longrightarrow & \operatorname{hocolim}_{\kappa} E_i & \longrightarrow & B. \end{array}$$

Since $F \rightarrow F_i$ and $E \rightarrow E_i$ are local weak equivalences by (i), the vertical maps in the above are local weak equivalences. Moreover, any $F_i \rightarrow F_{i+1}$ factors through $L_f F$ and therefore we have a local weak equivalence $L_f F \simeq \operatorname{hocolim}_{\kappa} F_i$.

(iii) The size issues differ from the argument in [CS06]. There a countable number of steps is used, and the set of detectors is used to make sure countably many steps are enough. In an arbitrary topos, this does not work, since a set of detectors in the sense of [CS06, Definition 1.1] is the same as a set of compact generators in the sense of Definition 2.1.22. However, we can choose a cardinal in a similar way as [GJ98]: We let $f : A \rightarrow B$ be a cofibration of simplicial presheaves and assume L_f is the f -localization functor constructed in [GJ98, Theorem 4.4]. Then let α be a cardinal which is an upper bound for the cardinality of the sets of morphisms of the site T and all sets of sections of the simplicial presheaf B . The cardinal up to which the construction in (ii) has to be iterated is any cardinal κ with $\kappa > 2^\alpha$. ■

REMARK 4.2.11 *Note that the above construction is functorial since we have only used universal constructions such as the localization augmentation $F \rightarrow L_f F$ and pushouts along this morphism. This implies that there exists a natural transformation $\mathcal{H}(-, F) \rightarrow \mathcal{H}(-, L_f F)$ which by Theorem 3.2.17 is represented by a morphism of the corresponding classifying spaces $B^f F \rightarrow B^f L_f F$.*

Fibrewise Localization via Classifying Spaces: The construction of fibrewise localization using classifying spaces as in [Dro96, Theorem 1.F.1] is in some sense a converse to the above: While we have used a concrete construction of fibrewise localization to produce a morphism of classifying spaces via Brown representability, the classifying space approach assumes that we are given a morphism $B^f F \rightarrow B^f LF$ which classifies an LF -fibre sequence over $B^f F$ and therefore extends to a morphism of the corresponding universal fibre sequences.

From this morphism, one can produce the fibrewise localization as follows: A given fibre sequence $F \rightarrow E \rightarrow B$ with fibre F is classified by a morphism to the corresponding classifying space $B \rightarrow B^f F$. Composing with the morphism of classifying space we obtain a morphism $B \rightarrow B^f F \rightarrow B^f LF$. Pulling back the universal LF -fibre sequence along this morphism produces an LF -fibre sequence $LF \rightarrow \overline{E} \rightarrow B$ over B , which is the fibrewise localization of the fibre sequence we started with. This implies that in the above situation any F -fibre sequence of simplicial presheaves $F \rightarrow E \rightarrow B$ can be mapped via a homotopy commutative diagram

$$\begin{array}{ccccc} F & \longrightarrow & E & \longrightarrow & B \\ \downarrow & & \downarrow & & \downarrow \\ LF & \longrightarrow & \overline{E} & \longrightarrow & B \end{array}$$

to a fibre sequence over B , i.e. a fibrewise localization exists. Note that we can get such a morphism between the classifying spaces by applying Proposition 4.2.10 to the universal fibre sequence $F \rightarrow E^f F \rightarrow B^f F$. Then we obtain a new fibre sequence $LF \rightarrow \overline{E}^f F \rightarrow B^f F$ which then is classified by a morphism of the classifying spaces $B^f F \rightarrow B^f LF$.

Of course this construction is not as explicit as what we did in the previous paragraph, but it also proves that fibrewise localizations exist. It can in particular be used to understand fibrewise localizations in the case of torsors. As an example, let G be a solvable algebraic group over a perfect field k . We will see in the examples section, cf. Proposition 4.4.19, that its localization is a group H of multiplicative type. This morphism induces a morphism of classifying spaces $BG \rightarrow BH$, where BG and BH are the classifying spaces of torsors. This implies that the fibrewise localization can be constructed rather concretely: Take a G -torsor E over a simplicial presheaf X , together with a trivialization. Replace the fibres G by the group H , and replace the transition morphisms in $G(U)$ by their images in $H(U)$. This describes an H -torsor, which is the fibrewise localization of E .

Fibrewise Localization via Simplex Categories: We also provide the *rigid* fibrewise S -localization which for the category of simplicial sets is constructed in [Dro96, Theorem 1.F.3]. In the above, rigid means that the diagram of fibre sequences in the definition of fibrewise application of coaugmented functors does not only commute up to homotopy but *on the nose*.

This construction is easily seen to be functorial and to possess the following universal property:

Given a fibrewise space $E \rightarrow B$, the fibrewise S -localization $E \rightarrow \overline{E}$ as a B -morphism is the universal B -morphism from E into a B -space with homotopy fibre an S -local space.

However, the construction of [Dro96, Theorem 1.F.3] does not carry over directly to the case of simplicial presheaves. We shortly discuss the problems that appear when trying to carry over the proof of [Dro96, Theorem 1.F.3] to the setting of simplicial presheaves. The basic idea in the construction of fibrewise localization for fibrations of simplicial sets is to decompose the base into a diagram of contractible spaces. Then the total space of the fibration can be decomposed into the fibres over these contractible spaces. Using the fact that L takes weak equivalences to weak equivalences and therefore (weakly) contractible spaces to (weakly) contractible spaces, we know that applying the functor to the base diagram does not change the homotopy type of the homotopy colimit. This sort of argument does not work for categories of simplicial presheaves: Although we can canonically decompose any simplicial presheaf as homotopy colimit of $U \times \Delta^n$, the generators U are not contractible and can also not be written as a diagram of contractible spaces. But in order to get a morphism from the fibrewise localized diagram for $\overline{E}$ to the base we need to apply the functor L also to the diagram presenting the base. This discussion should explain the necessity of the extra hypothesis in the theorem below, requiring that the category of sheaves on T has a set of *local* generators. This allows to apply the functor L without changing the homotopy type of the base diagram.

THEOREM 4.2.12 *Let $L : \Delta^{\text{op}}\text{Shv}(T) \rightarrow \Delta^{\text{op}}\text{Shv}(T)$ be a functor of unpointed simplicial presheaves that sends contractible spaces to contractible spaces and preserves weak equivalences. Moreover, assume that for any representable $U \in T$ there exists a covering U_i of U such that $L(U_i) \simeq U_i$, resp. that there is a set of generators of $\text{Shv}(T)$ which are local with respect to L . Then for any fibre sequence $p : F \rightarrow E \rightarrow B$ there is another fibre sequence $LF \rightarrow \overline{E} \rightarrow B$.*

In case L is coaugmented we have a natural commutative (not just homotopy commutative) diagram over B :

$$\begin{array}{ccc} F & \longrightarrow & LF \\ \downarrow & & \downarrow \\ E & \longrightarrow & \overline{E} \\ \downarrow & & \downarrow \\ B & \xrightarrow{\text{id}} & B. \end{array}$$

PROOF: The structure of the proof is similar to that of [Dro96, Theorem 1.F.3], with some preliminary remarks on the extra hypothesis in (i). We then apply the canonical homotopy colimit decomposition from Proposition 3.1.20 in (ii) and use the homotopy distributivity from Proposition 3.1.14 to decompose the fibration into a similar diagram in (iii). We then apply the functor to the diagrams. This provides a homotopy commutative ladder diagram, which can be rectified using homotopy distributivity again.

(i) Recall from [MM92, p. 576] that a set of objects C_i , $i \in I$, generates the topos $\mathcal{E}$ if for any two parallel arrows $u, v : E \rightarrow E'$ in $\mathcal{E}$, the identity $u \circ w = v \circ w$ for all i and all morphisms $w : C_i \rightarrow E$ implies $u = v$. Since the elements of the site T generate $\mathcal{S}hv(T)$, the existence of L -local coverings for all objects $U \in T$ implies that L -local objects generate the topos $\mathcal{S}hv(T)$.

From a set of generators C_i , $i \in I$, of a topos $\mathcal{E}$, it is possible to construct a site, cf. [MM92, p. 580], by taking the full subcategory $\mathcal{C} \hookrightarrow \mathcal{E}$ and equipping it with a suitable Grothendieck topology. Giraud's theorem [MM92, Theorem 2.1 in Appendix] then implies an equivalence of categories $\mathcal{E} \cong \mathcal{S}hv(\mathcal{C})$.

Let now T be the site as in the assumptions above, let C_i , $i \in I$, be a set of local generators of $\mathcal{S}hv(T)$ and denote by $\mathcal{C}$ the full subcategory of $\mathcal{S}hv(T)$ generated by C_i . Then we have an equivalence of categories $\mathcal{S}hv(T) \cong \mathcal{S}hv(\mathcal{C})$. In particular, we consider from now on the model category $\Delta^{\text{op}}\mathcal{S}hv(\mathcal{C})$ instead of $\Delta^{\text{op}}\mathcal{S}hv(T)$, in which all representable objects are local with respect to L .

Then we proceed with the proof of [Dro96, Theorem 1.F.3] for the site $\mathcal{C}$.

(ii) Let $p : E \rightarrow B$ be a fibration of fibrant simplicial sheaves on $\mathcal{C}$. Then by Proposition 3.1.20 there exists a diagram $(T \times \Delta \downarrow B)$, such that the canonical morphism

$$\text{hocolim}_{\mathcal{I}}(\mathcal{C} \times \Delta \downarrow B) \rightarrow \text{colim}_{\mathcal{I}}(\mathcal{C} \times \Delta \downarrow B) \rightarrow B$$

is a weak equivalence. For convenience we call the diagram category $\mathcal{I}$, it is the category of all simplices $U \times \Delta^n \rightarrow B$ with $U \in \mathcal{C}$ and it maps $U \times \Delta^n$ to itself in $\Delta^{\text{op}}\mathcal{S}hv(T)$. The diagram will be denoted by $\Gamma(B)$.

(iii) Now we decompose the total space E as follows: Let $\Gamma(E)$ denote the following $\mathcal{I}$ -indexed diagram. We associate to each simplex $\sigma : U \times \Delta^n \rightarrow B$ the space $\sigma^*(E) = E \times_B (U \times \Delta^n)$, which is the pullback of the fibration p along σ . Since p is a fibration, the resulting square is a homotopy pullback. Moreover, by distributivity, we know that $\text{colim}_{\mathcal{I}} \Gamma(E) \rightarrow E$ is an isomorphism. Applying Homotopy Distributivity from Proposition 3.1.14, we find that $\Gamma(E)$ is a homotopy colimit diagram, i.e. the canonical morphism

$$\text{hocolim}_{\mathcal{I}} \Gamma(E) \rightarrow \text{colim}_{\mathcal{I}} \Gamma(E) \rightarrow E$$

is a weak equivalence and we have the following diagram:

$$\begin{array}{ccccc} \text{hocolim}_{\mathcal{I}} \Gamma(E) & \xrightarrow{\simeq} & \text{colim}_{\mathcal{I}} \Gamma(E) & \xrightarrow{=} & E \\ \text{hocolim } \sigma^*(p) \downarrow & & \downarrow & & \downarrow p \\ \text{hocolim}_{\mathcal{I}} \Gamma(B) & \xrightarrow{\simeq} & \text{colim}_{\mathcal{I}} \Gamma(B) & \xrightarrow{=} & B \end{array} \quad (4.1)$$

This process is also described in Lemma 3.1.22. We see, that the decomposition of the fibration works similar to the case of simplicial sets. The problem is however that over an elementary simplex $U \times \Delta^n$ the fibration need not be trivial, since U as an element of the site is already something quite global.

(iv) We now apply the functor objectwise to the diagrams $\Gamma(E)$ and $\Gamma(B)$. In case of a coaugmented functor we obtain morphisms $\Gamma(E) \rightarrow L\Gamma(E)$ and $b : \Gamma(B) \rightarrow L\Gamma(B)$. Therefore, there is a natural commutative diagram by taking homotopy colimits:

$$\begin{array}{ccc}
\mathrm{hocolim}_{\mathcal{I}} \Gamma(E) & \longrightarrow & \mathrm{hocolim}_{\mathcal{I}} L\Gamma(E) \\
\mathrm{hocolim} \sigma^*(p) \downarrow & & \downarrow p_L \\
\mathrm{hocolim}_{\mathcal{I}} \Gamma(B) & \xrightarrow{\mathrm{hocolim} b} & \mathrm{hocolim}_{\mathcal{I}} L\Gamma(B).
\end{array} \tag{4.2}$$

Since L takes weak equivalences to weak equivalences, we obtain $L(U \times \Delta^n) \simeq L(U) \simeq U \simeq U \times \Delta^n$, where the middle weak equivalence follows from the assumption that we used the site of L -local objects. This implies that in the morphism of diagrams $b : \Gamma(B) \rightarrow L\Gamma(B)$, we have $b(U \times \Delta^n \rightarrow B)$ is a weak equivalence for any simplex $U \times \Delta^n \rightarrow B$ of B . Therefore by the homotopy invariance of homotopy colimits, $\mathrm{hocolim}(b) : \mathrm{hocolim}_{\mathcal{I}} \Gamma(B) \rightarrow \mathrm{hocolim}_{\mathcal{I}} L\Gamma(B)$ is a weak equivalence.

(v) The last step in the construction of the commutative ladder is the rectification to obtain a diagram which commutes on the nose. This is done almost exactly as it is done in [Dro96]. Note that the diagram (4.1) is a homotopy pullback, and we therefore can replace $\mathrm{hocolim} \sigma^*(p)$ by a fibration without changing the homotopy type of the fibres. Now we denote by $\overline{E}_1$ the homotopy pullback of p_L along $\mathrm{hocolim} b$. We obtain a fibration $p_1 : \overline{E}_1 \rightarrow \mathrm{hocolim}_{\mathcal{I}} \Gamma(B)$ and composing this with the augmentation $\mathrm{hocolim}_{\mathcal{I}} \Gamma(B) \rightarrow B$, we obtain a morphism $\overline{E}_1 \rightarrow B$. Therefore we have a fibre sequence $LF \rightarrow \overline{E}_1 \rightarrow B$.

To get a morphism from E into the fibrewise localization, we consider the following diagram:

$$\begin{array}{ccccc}
E & \xleftarrow{\cong} & \mathrm{hocolim}_{\mathcal{I}} \Gamma(E) & \xrightarrow{l} & \overline{E}_1 \\
\downarrow & & \downarrow & & \downarrow \\
B & \xleftarrow{=} & B & \xrightarrow{=} & B
\end{array}$$

The morphism l is obtained from the usual construction of the homotopy pullback as point-set pullback of a factorization of p_L in diagram (4.2). We construct the fibrewise localization $\overline{E}$ as the homotopy pushout of the first line in the above diagram, again using Puppe's theorem 3.1.16. Since $\mathrm{hocolim}_{\mathcal{I}} \Gamma(E) \rightarrow E$ is a weak equivalence, so is $\overline{E}_1 \rightarrow \overline{E}$, whence the fibres of $\overline{E} \rightarrow B$ have the right homotopy type LF . By construction, we also have a B -morphism $E \rightarrow \overline{E}$. All the steps were natural, so the above construction is functorial in $p : E \rightarrow B$.

(vi) We started in the assumptions of the theorem with a fibre sequence $F \rightarrow E \rightarrow B$ in $\Delta^{\mathrm{op}}\mathcal{Shv}(T)$. Recall that in (i) we constructed an equivalence of categories $\rho : \mathcal{Shv}(T) \rightarrow \mathcal{Shv}(\mathcal{C})$. Then we produced a fibrewise localization $L\rho(F) \rightarrow \overline{E} \rightarrow \rho(B)$ of $\rho(F) \rightarrow \rho(E) \rightarrow \rho(B)$ in the category $\Delta^{\mathrm{op}}\mathcal{Shv}(\mathcal{C})$, together with the corresponding commutative ladder diagram. In the last step, we move this fibrewise localization back to $\Delta^{\mathrm{op}}\mathcal{Shv}(T)$: The fibrewise localization of $F \rightarrow E \rightarrow B$ is given by $\rho^{-1}(L\rho(F)) = LF \rightarrow \rho^{-1}(\overline{E}) \rightarrow \rho^{-1}\rho(B) = B$. The statements about commutativity still obtain, because we used an equivalence of categories.

(vii) The fact that $E \rightarrow \overline{E}$ is an f -local weak equivalence if $L = L_f$ is a localization functor follows from Proposition 4.1.7: The morphism $E \rightarrow \overline{E}$ is equivalent to the morphism $\mathrm{hocolim}_{\mathcal{I}} \Gamma(E) \rightarrow \mathrm{hocolim}_{\mathcal{I}} L\Gamma(E)$, which in turn is a homotopy colimit of localization maps. $\blacksquare$

REMARK 4.2.13 (i) *The condition of the above theorem is satisfied for the $\mathbb{A}^1$ -localization functors over a regular noetherian base scheme S . The following argument shows that any smooth scheme over S can be covered by $\mathbb{A}^1$ -rigid schemes.*

First, any connected smooth scheme is irreducible, and any irreducible scheme can be covered by quasi-projective schemes provided the base is noetherian, cf. Chow's lemma [Gro61, Theorem 5.6.1]. For a quasi-projective scheme $U \rightarrow S$, consider the embedding $U \hookrightarrow \mathbb{P}_S^n$. Now $\mathbb{P}_S^n$ can be covered by copies of $\mathbb{A}_S^n$, and $\mathbb{A}_S^n$ can be covered by 2^n copies of $\mathbb{G}_m^n$ as follows: We consider the hyperplanes $t_i = 0$ resp. $t_i = 1$ in $\mathbb{A}_S^n = \text{Spec } \mathbb{Z}[t_1, \dots, t_n]$. For a tuple $(a_1, \dots, a_n) \in \{0, 1\}^n$, we consider the complement of the union of the hyperplanes $t_i = a_i$. This is isomorphic to $\mathbb{G}_m^n$. It is also clear, that the union of these tori covers $\mathbb{A}_S^n$, and therefore $\mathbb{A}_S^n$ can be covered by tori for any S by pulling back this covering.

Finally, over a regular base scheme S , $\mathbb{G}_m$ and therefore $\mathbb{G}_m^n$ is $\mathbb{A}^1$ -rigid, hence $\mathbb{A}^1$ -local. Any open subscheme of an $\mathbb{A}^1$ -rigid scheme is again $\mathbb{A}^1$ -rigid, so the induced covering of U by $U \times_{\mathbb{P}^n} \mathbb{G}_m^n$ consists of $\mathbb{A}^1$ -rigid schemes.

We have managed to produce a set of $\mathbb{A}^1$ -local generators of $\text{Shv}(\text{Sm}_S)$ for any topology containing the Zariski topology, hence the theorem applies.

- (ii) *Note that although our main interest is in the fibrewise versions of localization resp. nullification functors, it is nice to have the possibility to apply other functors fibrewise. One example mentioned in [Dro96] is the case of the simplicial coaugmented functor*

$$L_W : X \mapsto \text{map}(W, X).$$

In the special case of $W = S^{p,q}$, we get fibrewise free loop spaces: For any fibre sequence $F \rightarrow E \rightarrow B$ we obtain another fibre sequence $\Omega^{p,q}F \rightarrow \overline{E} \rightarrow B$.

The same argument works for any “topological” functor one wants to apply fibrewise: Since all sheaves of constant simplicial sets U are fibrant and homotopy discrete, the free loop space will not change these sheaves, therefore we do not need the replacement site $\mathcal{C}$ constructed in step (i) of the proof of Theorem 4.2.12.

- (iii) *For simplicial sets, one can construct principalization functors using the above Theorem 4.2.12, by applying the functor $X \mapsto \text{Aut}_\bullet(X)$ which maps every simplicial X to the simplicial group of its automorphisms. This way one can associate to each fibre sequence a principal $\text{Aut}_\bullet(F)$ -bundle. The problem in a generalization of this to simplicial sheaves is to find a set of generators of the topos $\text{Shv}(T)$, such that the generators do not have any automorphisms. This seems to be more complicated than the construction employed in (i) for $\mathbb{A}^1$ -localization.*

Some Consequences: The existence of a fibrewise localization functor implies the following two results from [Dro96, Section 1.H]. It also yields the characterization of nullification functors given later on.

COROLLARY 4.2.14 *If $F \rightarrow E \rightarrow B$ is a fibre sequence with $L_f F \simeq *$ and B path-connected, then $L_f(p) : L_f E \rightarrow L_f B$ is a homotopy equivalence, and $* \rightarrow L_f E \rightarrow L_f B$ is a fibre sequence.*

Note that if we apply the pointed localization, we have to assume ΩB path-connected. For the unpointed version, the unconditional statement obtains, since the unpointed localization functor localizes the induced fibre sequences over each connected component of B . Of course this statement can also be proven without the use of fibrewise localization, cf. [MV99, Example 3.2.3]. Note however that in the above corollary there is no restriction like being locally trivial in the Nisnevich topology, the only restriction is that $F \rightarrow E \rightarrow B$ is a fibre sequence.

COROLLARY 4.2.15 *Let L_W be the W -nullification functor with respect to any space W . Let X be a pointed path-connected space. Then we have $L_W A_{L_W} X \simeq *$, where $A_{L_W} X$ is the homotopy fibre of the nullification $X \rightarrow L_W X$.*

The proofs carry over verbatim from [Dro96, Section 1.H], as we have established all the necessary facts about the nullification functors for simplicial presheaves.

4.3 Nullification and Local Fibre Sequences

The fibrewise localization we constructed for simplicial presheaves in Section 4.2 can be used to give criteria for locality of fibre sequences. We will see that a fibre sequence $F \rightarrow E \xrightarrow{p} B$ is f -local for a null-homotopic morphism $f : X \rightarrow Y$ of simplicial presheaves iff the pullback of the fibrewise localization of p to the non-local part of the base is a trivial fibre sequence.

The assumption that $f : X \rightarrow Y$ is null-homotopic, i.e. the localization functor L_f is equivalent to a nullification $L_{* \rightarrow W}$, is essential as we will see in Theorem 4.3.7.

Recall from Section 4.1 that there were two approaches to localization: One via an axiomatic definition of localization functors, and another one via the definition of an f -local model structure. These two approaches have a different terminology in dealing with local fibre sequences. Looking at the localization functors approach, the natural question to ask is if a fibre sequence is preserved by a localization functor L_f . On the other hand, the natural question for the model category approach is rather if a given sequence is a fibre sequence in the local model structure, using Definition 2.1.9. Although it seems almost clear that these two terminologies basically mean the same thing, we still give a proof of the fact. This fact also generalizes to homotopy pullbacks in case the local model structure is proper.

Localization of Fibre Sequences: In this paragraph, we show that to determine if $F \rightarrow E \rightarrow B$ is a fibre sequence in a localized model structure, it suffices to check if $LE \rightarrow LF \rightarrow LB$ is also a fibre sequence. A similar statement holds for homotopy pullbacks and their localizations. Note, that the fibre sequence statement does not depend on properness of the local model structure, since the localization of the fibre sequence automatically yields morphisms between fibrant objects, so no rectification as in Section 2.3 is necessary. However, the result for homotopy pullbacks is not unconditional, as we need properness of both $\mathcal{C}$ and its Bousfield

localization $L_f\mathcal{C}$ to even talk about homotopy pullbacks. Therefore the homotopy pullback part seems to depend on the fact that L_f is a nullification.

We begin with a definition of what it means for a fibre sequence resp. a homotopy pullback to be preserved by a localization functor.

DEFINITION 4.3.1 *Following [BF03] we say that a fibre sequence $F \rightarrow E \rightarrow B$ is preserved by a localization L_f if applying L_f to p yields a morphism of fibre sequences:*

$$\begin{array}{ccccc} F & \longrightarrow & E & \longrightarrow & B \\ \downarrow & & \downarrow & & \downarrow \\ LF & \longrightarrow & LE & \longrightarrow & LB \end{array}$$

Similarly, a homotopy pullback is preserved by the localization L_f if applying L_f to the homotopy pullback diagram yields a (homotopy) commutative diagram of homotopy pullbacks:

$$\begin{array}{ccccc} X & \xrightarrow{\quad} & Y & & \\ \downarrow & \searrow & \downarrow & \searrow & \\ & L_f X & \xrightarrow{\quad} & L_f Y & \\ \downarrow & \downarrow & \downarrow & \downarrow & \\ Z & \xrightarrow{\quad} & W & & \\ \downarrow & \searrow & \downarrow & \searrow & \\ & L_f Z & \xrightarrow{\quad} & L_f W & \end{array}$$

The next proposition shows that being an f -local fibre sequence is the same as being preserved by a localization.

PROPOSITION 4.3.2 *Let $p : F \rightarrow E \rightarrow B$ be a fibre sequence in a proper model category $\mathcal{C}$. Let $f : A \rightarrow B$ be a morphism with associated localization functor L_f . Then p is preserved by L_f iff p is a fibre sequence in the f -local model structure.*

PROOF: Assume that p is preserved by L_f . Then we have a commutative diagram of fibre sequences:

$$\begin{array}{ccccc} F & \longrightarrow & E & \xrightarrow{p} & B \\ \downarrow f & & \downarrow & & \downarrow \\ LF & \longrightarrow & LE & \longrightarrow & LB \end{array}$$

The morphism f is equivariant for the holonomy action of ΩB , because the functor L_f is continuous, and preserves products, cf. Theorem 4.1.3 and Proposition 4.1.6. The fibre sequence $LF \rightarrow LE \rightarrow LB$ is a fibre sequence in the f -local model structure since it is a non-local fibre sequence and both LE and LB are local, cf. [Hir03, Proposition 3.3.16]. Therefore $F \rightarrow E \rightarrow B$ is also a fibre sequence in the local model structure.

Conversely, let p be an f -local fibre sequence. Then there is an f -local fibre sequence $X \rightarrow Y \rightarrow Z$ where $q : Y \rightarrow Z$ is an f -local fibration of f -local fibrant

objects. Therefore q is homotopy equivalent to $L_f p$ and we have $\text{hofib } L_f p \simeq X \simeq L_f \text{hofib } p$. The latter weak equivalence follows since X is f -locally equivalent to F . So p is preserved by L_f . ■

The following proposition shows that a homotopy pullback is preserved by localization exactly when it is a homotopy pullback in the local model structure. This generalizes the previous statement on fibre sequences under the condition that both the model category $\mathcal{C}$ and its Bousfield localization $L_f \mathcal{C}$ are proper.

PROPOSITION 4.3.3 *Let the following square be a homotopy pullback in a proper model category $\mathcal{C}$:*

$$\begin{array}{ccc} A & \xrightarrow{\phi_0} & X \\ g \downarrow & & \downarrow h \\ B & \xrightarrow{\phi_1} & Y. \end{array}$$

Assume the Bousfield localization $L_f \mathcal{C}$ is also proper. Then the above diagram is a homotopy pullback in the f -local model structure iff its localization is a homotopy pullback in $\mathcal{C}$.

PROOF: We will occasionally refer to the pullback diagram as $\phi : g \rightarrow h$.

(i) Assume the square above is an f -local homotopy pullback. We can assume without loss of generality that h is an f -local fibration. To see this, consider the following diagram obtained by factoring h into the f -local trivial cofibration a and the f -local fibration c :

$$\begin{array}{ccccc} A & \xrightarrow{\quad} & X & & \\ & \searrow b & \downarrow h & \searrow a, \simeq_f & \\ & & B \times_Y Z & \xrightarrow{\quad} & Z \\ g \downarrow & \swarrow & \downarrow & \swarrow c & \\ B & \xrightarrow{\quad} & Y & & \end{array}$$

By assumption the morphism b is an f -local weak equivalence. As we also assumed functorial factorizations, this diagram continues to commute after f -localization, and the f -local weak equivalences descend to weak equivalences between the localized objects [Hir03, Theorem 3.2.18]. So the upper square is a homotopy pullback. It remains to show that the lower square is, then by the homotopy pullback lemma the localization $L_f g \rightarrow L_f h$ is a homotopy pullback. This argument uses properness of $L_f \mathcal{C}$ since the homotopy pullback lemma 3.1.6 only holds in a proper model category.

(ii) Let h be an f -local fibration, and consider the following diagram:

$$\begin{array}{ccccc} A & \longrightarrow & X & \longrightarrow & L_f X \\ \downarrow & & \downarrow & & \downarrow \\ B & \longrightarrow & Y & \longrightarrow & L_f Y. \end{array}$$

The outer rectangle is an f -local homotopy pullback by the homotopy pullback lemma 3.1.6, since the left-hand square is an f -local homotopy pullback by assumption, and the right-hand square is a homotopy pullback by [Hir03, Proposition 3.4.8(1)]. By the characterization of f -local homotopy pullbacks, the morphism $A \rightarrow L_f X \times_{L_f Y} L_f B$ is an f -local weak equivalence.

Since we assumed h to be an f -local fibration which therefore is preserved by L_f , cf. Proposition 4.3.2, we can assume that $L_f h$ is a fibration, and therefore an f -local fibration [Hir03, Proposition 3.3.16(1)]. By properness the following pullback is also an f -local homotopy pullback:

$$\begin{array}{ccc} L_f B \times_{L_f Y} L_f X & \longrightarrow & L_f X \\ \downarrow & & \downarrow L_f h \\ L_f B & \xrightarrow{L_f \phi_1} & L_f Y \end{array}$$

By 2-out-of-3, all morphisms in the composition $A \rightarrow L_f A \rightarrow L_f B \times_{L_f Y} L_f X$ are f -local weak equivalences, where the latter map is the universal one from the definition of pullbacks. By assumption, $L_f h$ is an f -local fibration, so is $L_f B \times_{L_f Y} L_f X \rightarrow L_f B$, and in particular $L_f B \times_{L_f Y} L_f X$ is f -local. Using again [Hir03, Theorem 3.2.18], $L_f A \rightarrow L_f B \times_{L_f Y} L_f X$ is a weak equivalence, which readily implies (using properness) that $L_f g \rightarrow L_f h$ is a homotopy pullback.

(iii) Assume now $L_f g \rightarrow L_f h$ is a homotopy pullback. Factor h into an f -local trivial cofibration a and an f -local fibration c . We need to prove that the pullback map $b : A \rightarrow B \times_Y X$ is an f -local weak equivalence. Consider the following diagram:

$$\begin{array}{ccccc} L_f A & \xrightarrow{\quad} & L_f X & & \\ & \searrow L_f b & & \searrow L_f a & \\ & L_f(B \times_Y Z) & \xrightarrow{\quad} & L_f Z & \\ & \downarrow f & & \downarrow d & \\ & L_f B \times_{L_f Y} \tilde{X} & \xrightarrow{\quad} & \tilde{X} & \\ & \swarrow & & \swarrow e & \\ L_f B & \xrightarrow{L_f \phi_1} & L_f Y & & \end{array}$$

Since we can not guarantee that $L_f c$ is again an f -local fibration, we have to factor it as f -local trivial cofibration d followed by the f -local fibration e . We have by construction that the square $\phi_1^* c \rightarrow c$ is an f -local homotopy pullback, and using (ii), the square $L_f(\phi_1^* c) \rightarrow L_f c$ is a homotopy pullback. Therefore $f : L_f(B \times_Y Z) \rightarrow L_f B \times_{L_f Y} \tilde{X}$ is a weak equivalence. By assumption, $L_f g \rightarrow L_f h$ is a homotopy pullback, so the weak equivalence $d \circ L_f a$ induces a weak equivalence $f \circ L_f b$, which by 2-out-of-3 implies that $L_f b$ is a weak equivalence. Both $L_f A$ and $L_f(B \times_Y Z)$ are f -local, so employing [Hir03, Theorem 3.2.18] again, we find that $b : A \rightarrow B \times_Y Z$ is an f -local weak equivalence, which proves the claim. ■

REMARK 4.3.4 *Note that the properness hypothesis in the above result is really necessary: If any homotopy pullback in the proper model category $\mathcal{C}$ with one morphism an f -local fibration is preserved by f -localization, then the Bousfield localization $L_f\mathcal{C}$ is already proper. The following is a short argument for this.*

Let $\mathcal{C}$ be a proper model category, and let the following homotopy pullback in $\mathcal{C}$ be given:

$$\begin{array}{ccc} A & \xrightarrow{\phi_0} & X \\ g \downarrow & & \downarrow h \\ B & \xrightarrow{\phi_1} & Y. \end{array}$$

Let ϕ_1 be an f -local weak equivalence, let h be an f -local fibration. We have assumed that the homotopy pullback is preserved by f -localization. The morphism $L_f\phi_1$ is a weak equivalence, and by properness of $\mathcal{C}$, the morphism $L_f\phi_0$ is also a weak equivalence. Therefore $\phi_0 : A \rightarrow X$ is an f -local weak equivalence, and we have proved properness of the f -local model structure.

The argument that a homotopy pullback with one weak equivalence has another parallel weak equivalence is as follows: Consider the following diagram:

$$\begin{array}{ccc} A & \xrightarrow{j} & X \\ d \downarrow & & \downarrow c \\ B \times_Y \tilde{X} & \xrightarrow{k} & \tilde{X} \\ \downarrow & & \downarrow p \\ B & \xrightarrow{i} & Y \end{array}$$

We assume the outer rectangle is a homotopy pullback. Then we consider a factorization of $X \rightarrow Y$ as a trivial cofibration c and a fibration p . We assume i is a weak equivalence and we want to show j is a weak equivalence. Then k is a weak equivalence by properness. Since the rectangle is a homotopy pullback, the morphism $d : A \rightarrow B \times_Y \tilde{X}$ is a weak equivalence. By 2-out-of-3 and since c is a weak equivalence we obtain that j is a weak equivalence.

Characterizing Nullifications: This paragraph is concerned with special properties that distinguish nullification functors, i.e. localizations with respect to *null-homotopic* morphisms $W \rightarrow *$, from localizations with respect to general morphisms $f : A \rightarrow B$. This is a generalization of results that appeared in [BF03] to the case of simplicial presheaves. The first is a simple lemma appearing as [BF03, Lemma 1.7].

LEMMA 4.3.5 *Let T be a Grothendieck site, and let $F \rightarrow E \xrightarrow{p} B$ be a fibre sequence in $\Delta^{\text{op}}\text{PShv}(T)$. Let L_f be an arbitrary localization functor on $\Delta^{\text{op}}\text{PShv}(T)$. Consider a fibrewise localization $LF \rightarrow \overline{E} \xrightarrow{\overline{p}} B$. Then L_f preserves p iff L_f preserves $\overline{p}$.*

PROOF: The definition of fibrewise localization, cf. Definition 4.2.2, requires that the morphism $a : E \rightarrow \overline{E}$ is an f -local weak equivalence, hence $L_f a : L_f E \rightarrow L_f \overline{E}$ is a weak equivalence and therefore $\text{hofib } p \simeq \text{hofib } \overline{p}$. ■

We need yet another lemma on ladder diagrams of simplicial presheaves, which is derived from the corresponding fact on simplicial sets:

LEMMA 4.3.6 *Let T be a site, and let a commutative ladder diagram of fibre sequences be given:*

$$\begin{array}{ccccc} F_1 & \longrightarrow & E_1 & \longrightarrow & B_1 \\ f \downarrow & & e \downarrow & & \downarrow b \\ F_2 & \longrightarrow & E_2 & \longrightarrow & B_2 \end{array}$$

Assume that f and b are weak equivalences. Then e is also a weak equivalence.

PROOF: Fibre sequences are preserved by passing to points, cf. Proposition 3.1.11. Also weak equivalences are determined on points by definition. It therefore suffices to prove the above assertion for simplicial sets.

If we consider the above diagram in the category of simplicial sets, then we can assume without loss of generality that $B_1 \rightarrow B_2$ is a weak equivalence of *connected* simplicial sets. Then we have a diagram of the corresponding long exact sequences

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \pi_n(F_1) & \longrightarrow & \pi_n(E_1) & \longrightarrow & \pi_n(B_1) \longrightarrow \cdots \\ & & \pi_n(f) \downarrow & & \pi_n(e) \downarrow & & \downarrow \pi_n(b) \\ \cdots & \longrightarrow & \pi_n(F_2) & \longrightarrow & \pi_n(E_2) & \longrightarrow & \pi_n(B_2) \longrightarrow \cdots \end{array}$$

Using the five lemma together with the fact that $\pi_n(f)$ and $\pi_n(b)$ are isomorphisms for all $n \geq 0$, we obtain that $\pi_n(e)$ is an isomorphism for all $n \geq 0$. Hence e is a weak equivalence. ■

The following characterization of nullification functors was proved for the case of simplicial sets resp. topological spaces in [BF03, Thm. 2.1]. Using the elementary facts and the existence of fibrewise localization, we find that it holds for any model category of simplicial presheaves.

THEOREM 4.3.7 *Let T be a Grothendieck site, and let $f : X \rightarrow Y$ be a morphism of simplicial presheaves in $\Delta^{\text{op}}\text{PShv}(T)$. Denoting by L_f the localization functor, consider the following statements:*

- (i) L_f is equivalent to a nullification, i.e. there exists a simplicial presheaf W such that L_f and $L_{*\rightarrow W}$ have the same local spaces resp. induce Quillen-equivalent Bousfield localizations.
- (ii) If in any fibre sequence $F \rightarrow E \xrightarrow{p} B$ both F and B are local, then so is E .
- (iii) Every fibre sequence with B local is preserved by L_f .
- (iv) For every space Z the space $L_f A_{L_f} Z$ is contractible.

Then (ii), (iii) and (iv) are equivalent, and (i) implies (ii). If we assume that X and Y are π_0 -connected, as defined in Definition 2.3.5, then (iv) implies (i).

PROOF: Note that L_f and L_g for $g : * \rightarrow W$ have the same local spaces iff the corresponding local model structures are Quillen equivalent. This can be seen from the following argument: The local model categories have the same underlying categories, namely the category $\Delta^{\text{op}}\text{PShv}(T)$. The cofibrations in both model structures are the same, i.e. the monomorphisms of simplicial presheaves. Then we have that $h : X \rightarrow Y$ is a f -local weak equivalence iff $L_f h : L_f X \rightarrow L_f Y$ is a weak equivalence iff $L_g h : L_g X \rightarrow L_g Y$ is a weak equivalence iff $h : X \rightarrow Y$ is a g -local weak equivalence. The outer equivalences follow from [Hir03, Theorem 3.2.18], the inner equivalence is in turn equivalent to the assertion that L_f and L_g have the same local spaces. Therefore both local model structures have the same cofibrations and weak equivalences, therefore also the same fibrations. Hence the identity on $\Delta^{\text{op}}\text{PShv}(T)$ is both a left and right Quillen functor.

Now we prove the following implications: (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (ii).

(i) implies (ii) is elementary fact (e.6) in [Dro96, Section 1.A.8]. To get rid of the connectedness assumption, we have to use a slightly extended argument. This is proved as follows: Let $F \rightarrow E \rightarrow B$ be any fibre sequence. Then the following is also a fibre sequence $\mathbf{Hom}(W, F) \rightarrow \mathbf{Hom}(W, E) \rightarrow \mathbf{Hom}(W, B)$ for any cofibrant W , cf. Remark 4.1.5 resp. Proposition 3.1.5. Moreover, this holds for any choice of base points, so it does not need connectedness. For a pointed and *connected* space X , we have that X is W -local iff $\mathbf{Hom}_*(W, X) \simeq *$. If this holds for B and F , then by the long exact homotopy sequence associated to the $\mathbf{Hom}$ -sequence, this also holds for E . For the general case, we note that the internal hom-functor is coaugmented $X \rightarrow \mathbf{Hom}(W, X)$ by the constant maps. Therefore we obtain a morphism of fibre sequences

$$\begin{array}{ccccc} F & \longrightarrow & E & \longrightarrow & B \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{Hom}(W, F) & \longrightarrow & \mathbf{Hom}(W, E) & \longrightarrow & \mathbf{Hom}(W, B). \end{array}$$

Now X is W -local iff $X \rightarrow \mathbf{Hom}(W, X)$ is a weak equivalence. Assuming F and B are W -local in the above diagram, the left and right vertical morphisms are weak equivalences. This implies that also the morphism $E \rightarrow \mathbf{Hom}(W, E)$ is a weak equivalence by Lemma 4.3.6. Therefore E is W -local.

For (ii) implies (iii) it suffices to show by Lemma 4.3.5 that the fibrewise localization $L_f F \rightarrow \overline{E} \xrightarrow{\overline{p}} B$ is preserved. But this follows from (ii), since both B and $L_f F$, hence by assumption also $\overline{E}$ are local. Note that this is the place where we use the existence of fibrewise localization for simplicial presheaves!

(iii) implies (iv): By assumption the fibre sequence $A_{L_f} Z \rightarrow Z \rightarrow L_f Z$ is preserved by L_f , so $L_f A_{L_f} Z$ is the fibre of the homotopy equivalence $L_f Z \rightarrow L_f L_f Z$, hence contractible.

(iv) implies (ii): Let $F \rightarrow E \rightarrow B$ be a fibre sequence with F and B local. Then $E \rightarrow B$ factors as $l_E : E \rightarrow L_f E$ and $q : L_f E \rightarrow B$, by (homotopy) universality of L_f . Consider the following diagram:

$$\begin{array}{ccccc} F & \xrightarrow{m} & \text{hofib}(q) & \longrightarrow & * \\ \downarrow & & \downarrow & & \downarrow \\ E & \xrightarrow{l_E} & L_f E & \xrightarrow{q} & B \end{array}$$

From [Hir03, Proposition 3.3.16], a fibration between local objects is an f -local fibration. Therefore $\mathrm{hofib}(q)$ is local. This also implies that $\mathrm{hofib}(m)$ is local, since F is local by assumption. Now we apply the (non-local!) homotopy pullback lemma: The outer rectangle is a homotopy pullback, since $F \rightarrow E \rightarrow B$ is a fibre sequence. The same holds for the right square. So the left square is a homotopy pullback and therefore $A_{L_f}E = \mathrm{hofib}(l_E) \simeq \mathrm{hofib}(m)$, which is local. Hence $A_{L_f}E \simeq L_f A_{L_f}E$ is contractible by assumption. Then l_E is a weak equivalence, so E is already local. The above is independent of the choice of base points, since the homotopy pullback argument and the comparison of the spaces can be done at the points of the topos, and we apply an *unpointed* localization functor, which localizes *all* components of a space.

(iv) implies (i): We start with the remark, that if $f : X \rightarrow Y$ is a morphism of π_0 -connected simplicial presheaves, then $L_f Z$ is π_0 -connected for any π_0 -connected simplicial presheaf Z . This can be seen by the direct construction of L_f e.g. in [GJ98, Section 4]. The functor L_f is constructed as transfinite composition of pushouts

$$\begin{array}{ccc} X \times \Delta^n \cup_{X \times \partial \Delta^n} Y \times \partial \Delta^n & \longrightarrow & L_f^\alpha Z \\ \downarrow & & \downarrow \\ Y \times \Delta^n & \longrightarrow & Z', \end{array}$$

where $L_f^{\alpha+1}Z = \mathrm{Ex}_T^\infty Z'$ and Ex_T^∞ is a functorial fibrant replacement for the simplicial model structure on $\Delta^{\mathrm{op}}\mathcal{PShv}(T)$. These pushouts can be computed at the points, and in particular $x^*(L_f^\alpha Z)$ and $x^*(Y \times \Delta^n)$ are connected simplicial sets for any point x of the topos. The pushout Z' of these simplicial sets is again connected, and a fibrant replacement does not change the homotopy types at the points. Therefore $L_f Z$ is π_0 -connected for any π_0 -connected simplicial presheaf Z , in particular for X and Y .

Now we can proceed with the argument from [BF03, Theorem 2.1]: From Proposition 4.1.7 we know that $L_f(X \vee Y) \simeq L_f(L_f X \vee L_f Y)$. Define $W = A_{L_f}X \vee A_{L_f}Y$. By assumption and the formula above, $L_f W \simeq *$, so any L_f -local space is also W -local.

Letting $g : W \rightarrow *$, we have $\mathrm{pt} \simeq L_g W \simeq L_g(L_g A_{L_f}X \vee L_g A_{L_f}Y)$, the latter weak equivalence again from Proposition 4.1.7. From the definition of locality, $L_g Z \simeq *$ iff $Z \rightarrow *$ is an g -local weak equivalence iff $\mathbf{Hom}_*(Z, P) \simeq *$ for any g -local space P . Note that this uses the π_0 -connectedness of Z . It follows that

$$\mathbf{Hom}_*(L_g A_{L_f}X \vee L_g A_{L_f}Y, L_g A_{L_f}X) \simeq *.$$

Because the morphisms $*, \mathrm{id} \vee * : L_g A_{L_f}X \vee L_g A_{L_f}Y \rightarrow L_g A_{L_f}X$ are then homotopic, this implies $L_g A_{L_f}X \simeq *$. The same holds for $A_{L_f}Y$. Consider the commutative diagram

$$\begin{array}{ccc} L_g X & \xrightarrow{L_g f} & L_g Y \\ \downarrow & & \downarrow \\ L_g L_f X & \xrightarrow{L_g L_f f} & L_g L_f Y. \end{array}$$

Both vertical arrows are homotopy equivalences, since their homotopy fibres are contractible, cf. Corollary 4.2.14. $L_f f$ is a homotopy equivalence, because it is an f -local equivalence of f -local spaces, cf. [Hir03, Theorem 3.2.18]. Therefore, $L_g L_f f$ is a weak equivalence and by 2-out-of-3 also $L_g f$. Thus every W -local space is L_f -local and the two localization functors agree up to homotopy. ■

REMARK 4.3.8 *It seems plausible that the implication from (iv) to (i) also holds under weaker connectedness assumptions than used above. In view of [MV99, Corollary 2.3.22], we know that for an interval I on site, the localization $L_I X$ of a π_0 -connected X is again π_0 -connected. The interval $\mathbb{A}^1$ in $\mathbf{Sm}_S$ is not π_0 -connected, so the π_0 -connectedness of the spaces X and Y in Theorem 4.3.7 is certainly not necessary.*

The Characterization Result: We now finally give the criterion for a fibre sequence to be preserved by a nullification. There is one preparatory lemma we need, which is a presheaf version of [BF03, Lemma 3.2]. All we need for the simplicial set proof of this lemma to work also for simplicial presheaves is Theorem 4.3.7.

LEMMA 4.3.9 *Let T be a Grothendieck site, and let L_f be a nullification functor on $\Delta^{\text{op}}\mathbf{PShv}(T)$. Consider the following diagram of simplicial presheaves, in which the square is a homotopy pullback and the lower sequence is a fibre sequence with local base C :*

$$\begin{array}{ccccc} D & \longrightarrow & E & & \\ q \downarrow & & \downarrow p & & \\ A & \longrightarrow & B & \xrightarrow{r} & C \end{array}$$

Then there is a homotopy equivalence of homotopy fibres $\text{hofib}(Lq) \simeq \text{hofib}(Lp)$, and p is preserved by L iff q is.

PROOF: Consider the following diagram:

$$\begin{array}{ccc} D & \longrightarrow & E \\ q \downarrow & & \downarrow p \\ A & \longrightarrow & B \\ \downarrow & & \downarrow r \\ * & \longrightarrow & C \end{array}$$

As $A \rightarrow B \rightarrow C$ is a fibre sequence, the lower square is a homotopy pullback. By the homotopy pullback lemma 3.1.6, the sequence $D \rightarrow E \xrightarrow{r \circ p} C$ is also a fibre sequence. Since L_f is a nullification and C is f -local, the two fibre sequences r and $r \circ p$ are preserved by L , cf. Theorem 4.3.7. Then we consider the localized diagram

$$\begin{array}{ccc}
L_f D & \longrightarrow & L_f E \\
L_f q \downarrow & & \downarrow L_f p \\
L_f A & \longrightarrow & L_f B \\
\downarrow & & \downarrow L_f r \\
* & \longrightarrow & L_f C
\end{array}$$

By the pullback lemma and properness of the (non-local!) model structure, the upper square is a homotopy pullback. This proves the assertion $\mathrm{hofib}(L_f q) \simeq \mathrm{hofib}(L_f p)$. Also, $\mathrm{hofib}(p) \simeq \mathrm{hofib}(q)$, because the non-localized upper square was assumed to be a homotopy pullback, and this implies that p is preserved by L precisely when q is. $\blacksquare$

The following theorem provides a criterion for locality of fibre sequences which appeared in [BF03, Theorem 4.1] for the case of simplicial sets resp. topological spaces. Although the statement below is more general and applies to general model categories of simplicial presheaves, the proof is still almost the original one. Note that we need the hypothesis that $f : X \rightarrow Y$ is null-homotopic in order to apply the previous lemmas.

THEOREM 4.3.10 *Let T be a site, and let $f : X \rightarrow Y$ be a null-homotopic morphism of simplicial sheaves in $\Delta^{\mathrm{op}}\mathrm{Shv}(T)$ such that L_f admits a fibrewise version. Furthermore, let $F \rightarrow E \xrightarrow{p} B$ be a fibre sequence.*

Let $L_f F \rightarrow \overline{E} \rightarrow B$ be a fibrewise localization of p , and let $L_f F \rightarrow E' \rightarrow A_{L_f} B$ be the pullback of this fibrewise localization of p to $A_{L_f} B = \mathrm{hofib}(B \rightarrow L_f B)$. Then the following are equivalent:

- (i) *The fibre sequence p is an f -local fibre sequence,*
- (ii) *the fibre sequence p is preserved by L_f ,*
- (iii) *$E' \simeq L_f F \times_{A_{L_f} B} B$ and therefore there is an f -local weak equivalence $E' \simeq L_f F$, and*
- (iv) *the following composition of morphisms is null-homotopic:*

$$A_{L_f} B \rightarrow B \xrightarrow{p} B^f F \rightarrow B^f L_f F.$$

The first morphism is the natural inclusion of the homotopy fibre $A_{L_f} B = \mathrm{hofib}(B \rightarrow L_f B)$ into B , then follows the morphism $B \rightarrow B^f F$ classifying the fibre sequence p , and finally the morphism $B^f F \rightarrow B^f L_f F$ induced by fibrewise localization.

PROOF: (i) and (ii) are equivalent by Proposition 4.3.2. (iii) and (iv) are equivalent, as (iv) describes via morphisms on classifying spaces what the construction in (iii) does on the level of fibre sequences.

We write $L = L_f$ omitting f from the notation. We know from Lemma 4.3.5 that p is preserved by L iff $\overline{p}$ is. Applying Lemma 4.3.9 to the diagram

$$\begin{array}{ccccc}
E_1 & \longrightarrow & \overline{E} & & \\
q \downarrow & & \downarrow \overline{p} & & \\
A_L B & \xrightarrow{d_B} & B & \xrightarrow{l_B} & LB
\end{array}$$

we see that $\overline{p}$ is preserved iff q is. Since $LA_L B$ is contractible, q is preserved iff there is a map of fibre sequences:

$$\begin{array}{ccccc}
LF & \longrightarrow & E_1 & \longrightarrow & A_L B \\
= \downarrow & & \downarrow & & \downarrow \\
LF & \xrightarrow{\simeq} & LE_1 & \longrightarrow & *
\end{array}$$

Such a morphism of fibre sequences can only exist if the upper fibre sequence is trivial, i.e. there is a weak equivalence $E_1 \simeq LF \times A_L B$. By the Dold theorem 4.3.13 discussed in the next paragraph, the fibre sequence q is equivalent to a trivial one iff the inclusion of the fibre $LF \hookrightarrow E_1$ has a left homotopy inverse. Therefore q is preserved by L iff q is equivalent to a trivial fibre sequence. $\blacksquare$

We can even prove a more general result describing sharp conditions under which a homotopy pullback of simplicial presheaves descends to a nullification model structure.

COROLLARY 4.3.11 *Let T be a site and let a homotopy pullback of simplicial presheaves in $\Delta^{\text{op}}\text{Shv}(T)$ be given:*

$$\begin{array}{ccc}
X & \xrightarrow{i} & Y \\
j \downarrow & & \downarrow q \\
Z & \xrightarrow{p} & W.
\end{array}$$

Furthermore, let $f : A \rightarrow B$ be a null-homotopic morphism of simplicial sheaves in $\Delta^{\text{op}}\text{Shv}(T)$ such that L_f admits a fibrewise version, and the f -local model structure is proper. Then the above homotopy pullback is an f -local homotopy pullback iff the fibrewise localization of the fibre sequence $\text{hofib } q \rightarrow Y \rightarrow W$ splits over $A_{L_f} Z$ iff the fibrewise localization of the fibre sequence $\text{hofib } p \rightarrow Z \rightarrow W$ splits over $A_{L_f} Y$.

PROOF: The proof is similar to the one of Theorem 4.3.10. Assume the homotopy pullback is f -local. By properness, it suffices to check one factorization, i.e. one of the conclusions, cf. [GJ99, Lemma II.8.18]. Since the diagram is a homotopy pullback, we have a fibre sequence $\text{hofib } q \rightarrow Y \rightarrow W$. We also consider the local fibre sequence $\text{hofib } Lq \rightarrow LY \rightarrow LW$. It is not necessary that both agree, i.e. that L preserves the fibre sequence q . We pull these two fibre sequences back along $p : Z \rightarrow W$. Since we assumed the diagram is a local homotopy pullback, the pullback of $Lq : LY \rightarrow LW$ is locally weakly equivalent to $j : X \rightarrow Z$, which is the pullback of the fibre sequence q . Therefore the pullback of the fibre sequence q along $Z \rightarrow W$ is preserved by L , and the equivalence of (ii) and (iii) in Theorem 4.3.10 implies that the pullback of q along $Z \rightarrow W$ splits over the non-local part of Z . This is what we wanted to prove.

Assume that the fibrewise localization of $\text{hofib } q \rightarrow Y \rightarrow W$ splits over $A_{L_f} Z$. Then the pullback of $\text{hofib } q \rightarrow Y \rightarrow W$ to Z is an f -local fibre sequence by

Theorem 4.3.10. This implies that after replacing q by a local fibration $\tilde{Y} \rightarrow W$, there is a local weak equivalence $\tilde{Y} \times_W Z \simeq X$. Therefore the homotopy pullback above is also a local homotopy pullback. ■

REMARK 4.3.12 (i) Note that since $X \rightarrow \mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(X)$ is an $\mathbb{A}^1$ -local weak equivalence, the statement that $F \rightarrow E \rightarrow B$ is an $\mathbb{A}^1$ -local fibre sequence implies also that $\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(F) \rightarrow \mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(E) \rightarrow \mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(B)$ is an $\mathbb{A}^1$ -local fibre sequence and vice versa. We will see in Chapter 5 that there is considerably more geometry in proving that $\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(F) \rightarrow \mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(E) \rightarrow \mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(B)$ is a fibre sequence in the simplicial model structure. On the other hand, the localization theorem above is easier to apply to singular resolutions of schemes.

(ii) In Section 4.4, we will show how to use the above theorem to construct fibre sequences in the $\mathbb{A}^1$ -local model structure. Since there are some difficulties in connecting the geometry of schemes with the algebraic topology of the corresponding simplicial presheaves, there are yet no fully worked out geometric examples of fibre sequences that are not $\mathbb{A}^1$ -local, see the paragraph on non-local fibre sequence on page 130. It is nevertheless nice to have a sharp criterion for $\mathbb{A}^1$ -locality of fibre sequences.

The Dold Theorem: For the proof of Theorem 4.3.10, we needed a result characterizing trivial fibre sequences. This was originally proved by Dold [Dol63], and we show that it also holds in categories of simplicial presheaves. This is again proved by looking at the points of the topos.

PROPOSITION 4.3.13 Any fibre sequence $F \rightarrow E \rightarrow B$ in a category $\Delta^{\mathrm{op}}\mathcal{Shv}(T)$ of simplicial presheaves is trivial in the sense that there is a weak equivalence $E \simeq F \times B$, iff the inclusion of the fibre admits a left homotopy inverse.

PROOF: One direction is easy: Let $F \rightarrow E \rightarrow B$ be a fibre sequence which is equivalent to a trivial one. We consider a nice model, i.e. a fibration $p : E \rightarrow B$ of fibrant objects and a weak equivalence $F \rightarrow p^{-1}(*)$, and we assume F fibrant. By the assumption that p is trivial, there is a homotopy equivalence $E \rightarrow F \times B$, which composed with the projection $F \times B \rightarrow F$ provides the left homotopy inverse of $i : F \rightarrow E$.

For the converse, let a fibration of fibrant objects $p : E \rightarrow B$ be given, together with a morphism $l : E \rightarrow F = p^{-1}(*)$. This yields a morphism $l \times p : E \rightarrow F \times B$. We assume that l is left homotopy inverse to $i : p^{-1}(*) \rightarrow E$ and we want to show that $l \times p$ is a weak equivalence. Assume for now that T has enough points. By definition of the injective model structure, it suffices to check this on points. Let x be any point of the topos, and consider $x^*(F) \rightarrow x^*(E) \rightarrow x^*(B)$. This is a fibre sequence by [MV99, Lemma 2.1.20]. Moreover, the morphism $x^*(l) : x^*(E) \rightarrow x^*(F)$ is a left homotopy inverse of $x^*(p)$. By the usual Dold theorem [Dol63, Theorem 6.1] the morphism $x^*(l \times p) = x^*(l) \times x^*(p)$ is a weak equivalence of simplicial sets. Therefore, $l \times p$ induces a weak equivalence for any point x of T and thus is a local weak equivalence.

More generally, since we allowed ourselves to choose a nice model, a weak equivalence can be checked on sections: If $(l \times p)(U) : E(U) \rightarrow (F \times B)(U)$ is a

weak equivalence for any $U \in T$, and all of E , B and F are fibrant, then $l \times p$ is a weak equivalence [MV99, Lemma 2.1.10]. This again follows from the usual Dold theorem for simplicial sets. The latter argument of course works for any topos, even the ones without points. ■

Pullback Stability: In the following $L = L_W$ will be a nullification functor. The following is an immediate corollary to Theorem 4.3.10. Again the proof follows the proof of [BF03, Theorem 0.2].

PROPOSITION 4.3.14 *Let T be a Grothendieck site, and let $L = L_W$ be a nullification functor on $\Delta^{\text{op}}\mathcal{PShv}(T)$. If a pointed fibre sequence $F \rightarrow E \xrightarrow{p} B$ of simplicial presheaves is preserved by L , then any pointed fibre sequence p_1 induced from p by pullback along $f : X \rightarrow B$ is also preserved by L .*

PROOF: Let $F \rightarrow E \xrightarrow{p} B$ be a fibre sequence, and let $f : X \rightarrow B$ be a morphism of simplicial presheaves. Then the fibre sequence p is classified by a morphism $p : B \rightarrow B^f F$, and the pullback of p along $f : X \rightarrow B$ is classified by the composition $X \rightarrow B \rightarrow B^f F$. By Theorem 4.3.10, it suffices to show that if the composition

$$A_L B \rightarrow B \rightarrow B^f F \rightarrow B^f L F$$

is null-homotopic, then so is the composition

$$A_L X \rightarrow X \rightarrow B \rightarrow B^f F \rightarrow B^f L F.$$

This follows since we have a commuting square

$$\begin{array}{ccc} A_L X & \longrightarrow & A_L B \\ \downarrow & & \downarrow \\ X & \longrightarrow & B, \end{array}$$

which is obtained by taking the homotopy fibres of the coaugmentation morphisms $X \rightarrow LX$ resp. $B \rightarrow LB$. ■

It is also possible under some connectedness assumptions to prove this result in a way similar to [BF03, Proposition 3.1].

Nullifications and Properness: Denis-Charles Cisinski has kindly pointed out to me that (iii) in Theorem 4.3.7 holds if we know that both the original and the local model category are proper. This basically raises the question of the relation between nullification functors L_f and properness of the f -local model category, which we partially answer below.

Cisinski's argument that Theorem 4.3.7 (iii) follows from properness runs something like this: Consider a local and fibrant space B and a simplicial fibre sequence $F \rightarrow E \xrightarrow{p} B$ over it. We want to show that F is the f -local homotopy fibre of p . Using properness of the f -local model structure, it suffices to take a factorization of $* \rightarrow B$ as an f -local trivial cofibration $c : * \rightarrow \tilde{B}$ and an f -local fibration $q : \tilde{B} \rightarrow B$, and to prove that the induced morphism $d : F \rightarrow \tilde{B} \times_B E$ is an f -local weak equivalence, cf. [GJ99, Section II.8]. The situation is depicted in the following diagram:

$$\begin{array}{ccccc}
F & \xrightarrow{d} & \tilde{B} \times_B E & \longrightarrow & E \\
\downarrow & & \downarrow & & \downarrow p \\
* & \xrightarrow{c} & \tilde{B} & \xrightarrow{q} & B
\end{array}$$

Since $\tilde{B}$ is f -locally contractible but also local, it is contractible, i.e. c is a (non-local) weak equivalence. Since the outer square is a (non-local) homotopy pullback by definition, the right one as well, so the left square is a (non-local) homotopy pullback. Then the (non-local) weak equivalence $* \rightarrow \tilde{B}$ pulls back to an (non-local) and therefore also f -local weak equivalence $d : F \rightarrow \tilde{B} \times_B E$. This follows from properness of the non-local model structure. The f -local homotopy fibre of p is $\tilde{B} \times_B E$ and we have shown it f -locally is the same as F . Therefore $F \rightarrow E \rightarrow B$ is an f -local fibre sequence.

Under suitable assumptions on connectedness of f , we now obtain the following corollary. Probably this can also be obtained under weaker assumptions, one of the most interesting cases being a morphism of representable objects $f : U_1 \rightarrow U_2$ in T .

COROLLARY 4.3.15 *Let T be a Grothendieck site, and let $f : X \rightarrow Y$ be a morphism of π_0 -connected simplicial presheaves in $\Delta^{\text{op}}\text{PShv}(T)$. If the f -local model structure on $\Delta^{\text{op}}\text{PShv}(T)$ is right proper, then there exists a simplicial presheaf W such that the localization functors L_f and L_W are equivalent.*

PROOF: Cisinski's argument above shows that if the f -local model structure is proper, then (iii) in Theorem 4.3.7 holds. By Theorem 4.3.7, the localization has to be equivalent to L_W for some simplicial presheaf W . $\blacksquare$

Using the characterization of properness due to Cisinski in [Cis02], we can also prove the other direction:

COROLLARY 4.3.16 *Let T be a Grothendieck site, and let $f : X \rightarrow Y$ be a morphism of simplicial presheaves on T . If f is null-homotopic, then the local model structure $L_f\Delta^{\text{op}}\text{PShv}(T)$ is proper.*

PROOF: Using [Cis02, Theorem 4.8], it suffices to show that for a fibration $p : E \rightarrow B$ with fibrant base B , for any f -local weak equivalence $i : X \rightarrow Y$ and every morphism $u : Y \rightarrow B$, the morphism $j : E \times_B X \rightarrow E \times_B Y$ is also an f -local weak equivalence.

By Theorem 4.3.7, the fibre sequence $\text{hofib}(p) \rightarrow E \rightarrow B$ is preserved by f -localization. By Proposition 4.3.14, the same also holds for the pulled-back fibre sequences $\text{hofib}(p) \rightarrow E \times_B Y \rightarrow Y$ and $\text{hofib}(p) \rightarrow E \times_B X \rightarrow X$. We therefore obtain a commutative ladder of fibre sequences:

$$\begin{array}{ccccc}
L_f \text{hofib}(p) & \longrightarrow & L_f(E \times_B X) & \longrightarrow & L_f X \\
\downarrow & & \downarrow & & \downarrow \\
L_f \text{hofib}(p) & \longrightarrow & L_f(E \times_B Y) & \longrightarrow & L_f Y.
\end{array}$$

It is clear that the morphism on the fibres is a weak equivalence, and since i is an f -local weak equivalence, $L_f i : L_f X \rightarrow L_f Y$ is also a weak equivalence. From

Lemma 4.3.6, we conclude that $L_f(E \times_B X) \rightarrow L_f(E \times_B Y)$ is a weak equivalence. Therefore j is an f -local weak equivalence.

Note that all the results cited from Section 4.3 do not use properness of the local model structure, only properness of $\Delta^{\text{op}}PShv(T)$. ■

This provides another proof of the properness results from [MV99, Theorem 2.3.2] resp. [Jar00, Theorem A.5].

REMARK 4.3.17 *Note also the following interesting relation between properness and sharp maps. After Definition 3.3.2 we have seen that in a proper model category all fibrations are sharp. Cisinski's result in [Cis02, Theorem 4.8] is some kind of converse of this: The f -local model category is right proper if all non-local fibrations are f -local sharp maps.*

4.4 Applications and Examples

In this section, we will mainly focus on how to apply Theorem 4.3.10 to construct fibre sequences in $\mathbb{A}^1$ -homotopy. There are several situations in which one can directly apply this theorem, namely if the base is local or the fibre has a local classifying space. Everything beyond that involves more geometry, as we will discuss in Chapter 5. There are however interesting fibre sequences which are easy corollaries of Theorem 4.3.10 and we will discuss these in the following. These fibre sequences will allow to reduce the computation of homotopy groups of algebraic groups over a field to the computation of homotopy groups of Chevalley groups. We also deduce a description of the fundamental group of the projective line $\mathbb{P}^1$ from an $\mathbb{A}^1$ -local version of Ganea's theorem.

Examples of Geometric Fibre Sequences: We start with the case of a local base. The following corollary is already a consequence of Theorem 4.3.7:

PROPOSITION 4.4.1 *Let $F \rightarrow E \rightarrow B$ be a fibre sequence of simplicial presheaves in $\Delta^{\text{op}}PShv(\text{Sm}_S)$. If B is $\mathbb{A}^1$ -local, then this fibre sequence is an $\mathbb{A}^1$ -local fibre sequence.*

Recall from [MV99, Example 3.2.4] that a scheme X over S is called $\mathbb{A}^1$ -rigid if for any smooth scheme U over S , the morphism

$$\text{Hom}_S(U, X) \rightarrow \text{Hom}_S(U \times \mathbb{A}^1, X)$$

is a bijection. In the following, we will list some examples of $\mathbb{A}^1$ -rigid schemes.

EXAMPLE 4.4.2 Let S be any scheme, and let $X \rightarrow S$ be a smooth morphism whose fibres are smooth curves of genus $g \geq 1$ or abelian varieties. Then X is an $\mathbb{A}^1$ -rigid S -scheme, cf. [MV99, Example 3.2.4]. □

EXAMPLE 4.4.3 Let S be a regular scheme. Then $\mathbb{G}_m$ is $\mathbb{A}^1$ -rigid. More generally, from the elementary facts in Proposition 4.1.6 it follows that $(\mathbb{G}_m)^n$ is $\mathbb{A}^1$ -rigid for any $n \geq 1$. This can even be extended to more general tori: Let T be an isotrivial torus over S , i.e. there exists a finite étale morphism $S' \rightarrow S$ such that

$S' \times_S T \cong (\mathbb{G}_{m,S'})^n$. The theory of étale descent then shows that S -morphisms $U \rightarrow T$ are in bijective correspondence with S' -morphisms $U \times_S S' \rightarrow (\mathbb{G}_{m,S'})^n$ equipped with descent data. By rigidity, we have a bijection

$$\mathrm{Hom}_{S'}(U \times_S S', (\mathbb{G}_{m,S'})^n) \rightarrow \mathrm{Hom}_{S'}((U \times \mathbb{A}^1) \times_S S', (\mathbb{G}_{m,S'})^n).$$

Since this morphism is induced by the projection $U \times_S S' \times \mathbb{A}_{S'}^1 \rightarrow U \times_S S'$, any morphism in the right-hand side which has descent data comes from a morphism on the left-hand side which also has descent data. This implies rigidity of non-split tori over a regular base.

This is not true for non-regular schemes. Over $\mathrm{Spec} k[t]/(t^2)$, the sheaf $\mathbb{G}_m$ does not have homotopy invariance, since a polynomial ring can have non-constant invertible polynomials whose non-constant coefficients are nilpotent, cf. [MV99, Example 3.2.10]. $\square$

EXAMPLE 4.4.4 Let S be a regular scheme, and let X be a semi-abelian variety, i.e. there is an exact sequence of groups

$$(\mathbb{G}_m)^n \longrightarrow X \longrightarrow A,$$

where A is an abelian variety. It follows from Example 4.4.2, Example 3.3.10 and Proposition 4.4.1 that this is an $\mathbb{A}^1$ -local fibre sequence. From Example 4.4.3, the fibre is $\mathbb{A}^1$ -local, therefore the total space X is $\mathbb{A}^1$ -local as well. $\square$

EXAMPLE 4.4.5 Let S be any base scheme, and let G be a finite étale group schemes over S . Then G is $\mathbb{A}^1$ -local, cf. [MV99, Proposition 4.3.5]. $\square$

As remarked earlier, the $\mathbb{A}^1$ -rigid schemes are almost the analogues of hyperbolic manifolds. In complex analytic geometry, there are no holomorphic maps $\mathbb{C} \rightarrow X$ for X a hyperbolic manifold, since this is uniformized by some hyperbolic space $\mathbb{H}^n$. Therefore, anabelian varieties should be $\mathbb{A}^1$ -rigid. This is however not a precise result since there is no precise definition of anabelian variety. Note however, that also tori are $\mathbb{A}^1$ -rigid, whereas a complex torus is not hyperbolic. The reason is that the morphism $\exp : \mathbb{C} \rightarrow \mathbb{C}^*$ is holomorphic, but can not be made algebraic.

It follows from the above discussion, that any geometric fibration $p : E \rightarrow B$ of smooth schemes with an $\mathbb{A}^1$ -local base B , induces a fibre sequence in the $\mathbb{A}^1$ -local model structure. One particular case of this is a family of schemes over a curve of genus ≥ 1 . Geometric fibre sequence with an $\mathbb{A}^1$ -rigid scheme as base are however not very helpful, since the base is homotopy discrete, and therefore contributes only to the π_0 -part of the long exact sequence of homotopy groups. The local version of the Ganea theorem we will discuss later also applies Proposition 4.4.1, but to simplicial presheaves which are not discrete. These results are much more interesting, cf. Proposition 4.4.25.

Next, we want to study fibre sequences such that the classifying space of the (localization of the) fibre is $\mathbb{A}^1$ -local. This implies that any $L_f F$ -fibre sequence over a weakly f -contractible simplicial presheaf will then be trivial.

PROPOSITION 4.4.6 *Let $F \rightarrow E \xrightarrow{p} B$ be a fibre sequence of simplicial presheaves in $\Delta^{\text{op}}\text{PShv}(\text{Sm}_S)$. If $B^f F$ is $\mathbb{A}^1$ -local, then this fibre sequence is an $\mathbb{A}^1$ -local fibre sequence. To obtain this conclusion, it suffices that the morphism $B \rightarrow B^f F$ classifying p factors through a local simplicial presheaf.*

The following important classifying spaces are known to be $\mathbb{A}^1$ -local: The classifying space of the multiplicative group $\mathbb{G}_m$ is $\mathbb{A}^1$ -local over a regular base scheme S , cf. [MV99, Proposition 4.3.8] and the classifying space $B_{\text{ét}}G$ of a finite étale group scheme G over a base S is $\mathbb{A}^1$ -local if the order of G is prime to the characteristic of S , cf. [MV99, Proposition 4.3.1]. From this we get the following examples of $\mathbb{A}^1$ -local fibre sequences:

EXAMPLE 4.4.7 Any finite étale covering $g : E \rightarrow B$ induces an $\mathbb{A}^1$ -local fibre sequence $\text{Gal}(E/B) \rightarrow E \rightarrow B$ if either the degree of g is prime to the characteristic of S or the base B is $\mathbb{A}^1$ -local.

The first one is Proposition 4.4.6, the second one is Proposition 4.4.1 applied to the simplicial fibre sequence $\text{Gal}(E/B) \rightarrow E \rightarrow B$ from Example 3.3.9. This has also been proven in [Mor06b, Lemma 4.5]. $\square$

EXAMPLE 4.4.8 Let F be a simplicial presheaf which is $\mathbb{A}^1$ -weakly equivalent to a hypersurface of general type. Let $f : E \rightarrow B$ be a (étale) locally trivial morphism with fibre F . Then the transition maps in a trivialization are automorphisms of F , and it is known that the automorphism group of a hypersurface of general type is a finite group scheme. We assume it is étale of order prime to the characteristic of the base scheme S , which is clear if S is a field of characteristic zero.

Then $f \in B^f L_{\mathbb{A}^1} F$ is in the image of the change-of-fibre morphism $B_{\text{ét}}G \rightarrow B^f L_{\mathbb{A}^1} F$, cf. Proposition 3.2.20, since its transition maps are point-set automorphisms. Therefore the second condition in Proposition 4.4.6 is satisfied: f is classified by a morphism which factors through the $\mathbb{A}^1$ -local space $B_{\text{ét}}G$.

Therefore we obtain an $\mathbb{A}^1$ -local fibre sequence $F \rightarrow E \rightarrow B$. $\square$

EXAMPLE 4.4.9 Let S be any regular scheme. Any $\mathbb{G}_m$ -torsor over a smooth S -scheme X induces a fibre sequence $\mathbb{G}_m \rightarrow E \rightarrow X$. This follows since $B\mathbb{G}_m$ is $\mathbb{A}^1$ -local. This was also noted in [Mor06b, Example 4.6(1)].

One of the more interesting examples of such a situation are the Hopf fibrations

$$\mathbb{G}_m \rightarrow \mathbb{A}^n \setminus \{0\} \rightarrow \mathbb{P}^{n-1}.$$

These are $\mathbb{A}^1$ -local fibre sequences for any regular base scheme S and therefore the homotopy type of $\mathbb{P}^n$ is almost the same as the homotopy type of the sphere: The projection $\mathbb{A}^n \setminus \{0\} \rightarrow \mathbb{P}^{n-1}$ induces isomorphisms on sheaves of homotopy groups

$$\pi_n^{\mathbb{A}^1}(\mathbb{A}^n \setminus \{0\}) \rightarrow \pi_n^{\mathbb{A}^1}(\mathbb{P}^{n-1})$$

for any $n \geq 2$.

Similarly, the complement of a hyperbolic quadric can be described using $\mathbb{G}_m$ -torsors, cf. [DI05, Proposition 5.6]: Let k be a field and let $q(x_1, \dots, x_n)$ be a quadratic form. The equation $q(x_1, \dots, x_n) = 0$ defines an affine quadric $\text{AQ}(q) \hookrightarrow \mathbb{A}^n$ and a projective quadric $\text{PQ}(q) \hookrightarrow \mathbb{P}^{n-1}$. As in [DI05, Proposition 5.6], we

assume that q is hyperbolic and non-degenerate. The proof of [DI05, Proposition 5.6] shows that there are fibre sequences

$$\mathbb{G}_m \times \mathbb{A}^{k-1} \rightarrow \mathbb{A}^{2k} \setminus \text{AQ}(q) \rightarrow \mathbb{A}^k \setminus \{0\},$$

where $n = 2k$, and the total space is the complement of the affine quadric $\text{AQ}(q)$ in $\mathbb{A}^n$. By Proposition 4.4.6, we know that these are $\mathbb{A}^1$ -local fibre sequences and therefore

$$\pi_n^{\mathbb{A}^1}(\mathbb{A}^{2k} \setminus \text{AQ}(q)) \rightarrow \pi_n^{\mathbb{A}^1}(\mathbb{A}^k \setminus \{0\})$$

is an isomorphism of sheaves of groups for any $n \geq 2$. Similarly, the torsor

$$\mathbb{G}_m \rightarrow \mathbb{A}^{2k} \setminus \text{AQ}(q) \rightarrow \mathbb{P}^{2k-1} \setminus \text{PQ}(q)$$

is an $\mathbb{A}^1$ -local fibre sequence and therefore the homotopy group sheaves π_n for $n \geq 2$ are those of the projective space $\mathbb{P}^{2k-1}$. $\square$

EXAMPLE 4.4.10 Finally, we can consider an $\mathbb{A}^1$ -contractible fibre F . It follows that $B^f L_{\mathbb{A}^1} F \simeq B^f(*) \simeq *$ and therefore any smooth morphism $f : X \rightarrow Y$ of smooth schemes, which is locally trivial in the Nisnevich topology and has an $\mathbb{A}^1$ -contractible fibre F , is a trivial fibration in the $\mathbb{A}^1$ -local model structure. This was already stated in [MV99, Example 3.2.3].

Note, that in the paper [AD07a] many examples of $\mathbb{A}^1$ -contractible smooth quasi-affine schemes are produced. From the above, we find that smooth families of such schemes would also induce fibre sequences.

We further want to note that forms of $\mathbb{A}^n$ which may exist over a non-perfect field are also contractible. A form of $\mathbb{A}^n$ is a scheme X over a field k such that there exists a finite extension L/k over which $\mathbb{A}^n \times_k L \cong X \times_k L$. Contractibility of such a scheme follows since Proposition 4.1.6 implies that

$$\pi_n^{\mathbb{A}^1}\left(\prod X_\alpha\right) \cong \prod \pi_n^{\mathbb{A}^1}(X_\alpha).$$

Since the projection $\mathbb{A}^n \times_k L \rightarrow \text{Spec } L$ is an $\mathbb{A}^1$ -local weak equivalence, we have $\pi_n^{\mathbb{A}^1}(X \times_k L) \cong \pi_n^{\mathbb{A}^1}(\text{Spec } L)$, but also $\pi_n^{\mathbb{A}^1}(X \times_k L) = \pi_n^{\mathbb{A}^1}(X) \times \pi_n^{\mathbb{A}^1}(\text{Spec } L)$. Such a form is therefore $\mathbb{A}^1$ -weakly contractible. $\square$

We note that we do not know $\mathbb{A}^1$ -locality of classifying spaces for more general groups. However, as Fabien Morel has shown, the classifying space of simplicial group $\text{Sing}_\bullet^{\mathbb{A}^1}(GL_n)$ is local. This implies that torsors under GL_n induce $\mathbb{A}^1$ -local fibre sequences, cf. Chapter 5. It does not seem plausible that $H \rightarrow G \rightarrow G/H$ is an $\mathbb{A}^1$ -local fibre sequence for any homogeneous space under a linear algebraic group. Some assumptions on the groups involved are probably necessary.

Homotopy Types of Algebraic Groups: In this paragraph we will describe fibre sequences similar to those developed by Jardine in [Jar81, Jar83, Jar84]. This allows to reduce the computations of the sheaves of homotopy groups of an algebraic group to the case of semisimple, simply-connected algebraic groups. The homotopy groups of the latter are rather complicated, related to unstable algebraic K-theory. We refer to Chapter 5 for a detailed discussion of such issues.

We henceforth assume that the base scheme $S = \operatorname{Spec} k$ is the spectrum of a field. Some of the results are only valid over a perfect field, due to problems with unipotent groups. In this situation, we reduce the description of the homotopy type of a general algebraic group to a description of semisimple, simply connected groups. We will apply several reduction steps: In the first step, we use Chevalley's theorem to show that it suffices to describe the case of a linear algebraic group. The next steps reduce to the case of a connected linear algebraic group resp. semisimple groups. Passing to the universal cover, the problem of computing homotopy groups of algebraic groups boils down to the homotopy groups of split, semisimple, simply-connected groups. These groups are Chevalley groups, whose homotopy groups can be identified with Karoubi-Villamayor K-theory, at least for affine schemes over a discrete valuation ring, cf. Chapter 5. The case of non-split algebraic groups is not yet clear at the moment, all we can say is that these groups are at least not $\mathbb{A}^1$ -rigid.

Recall that an algebraic group is a group object G in the category of schemes over a field k . The general references for linear algebraic groups we will use are [Bor91, Wat79]. We will assume that our groups are smooth.

We begin recalling the theorem of Chevalley which states that each algebraic group is locally the product of a linear algebraic group and an abelian variety. This allows to reduce to the case of linear algebraic groups. For a statement of the theorem, and a readable proof we refer to [Con02].

PROPOSITION 4.4.11 (CHEVALLEY'S THEOREM) *Let k be a perfect field, and let G be a smooth and connected algebraic group over k . Then there exists a unique normal linear algebraic closed subgroup H in G for which G/H is an abelian variety. Therefore, there is an exact sequence of algebraic groups*

$$H \longrightarrow G \longrightarrow A$$

with H a linear algebraic group and A an abelian variety.

COROLLARY 4.4.12 *Let k be a perfect field, and let G be an algebraic group over k . Then there is an $\mathbb{A}^1$ -local fibre sequence*

$$H \xrightarrow{i} G \xrightarrow{p} A,$$

with H a linear algebraic group and A an abelian variety.

The induced morphism on π_0 is a short exact sequence of sheaves of groups

$$0 \longrightarrow \pi_0^{\mathbb{A}^1}(H) \xrightarrow{i_*} \pi_0^{\mathbb{A}^1}(G) \xrightarrow{p_*} A = \pi_0^{\mathbb{A}^1}(A) \longrightarrow 0,$$

and the inclusion of the fibre induces isomorphisms $i_* : \pi_n^{\mathbb{A}^1}(H, x) \rightarrow \pi_n^{\mathbb{A}^1}(G, i(x))$ for any $n \geq 1$ and any choice of base point $x \in H$.

PROOF: From Example 3.3.10 we know that the sequence of morphisms of sheaves of constant simplicial sets $H \rightarrow G \rightarrow A$ is a fibre sequence in the simplicial model structure. The base is an abelian variety, so $\mathbb{A}^1$ -rigid and therefore local. From Theorem 4.3.10, this fibre sequence is preserved by $\mathbb{A}^1$ -localization.

Since the abelian variety A is $\mathbb{A}^1$ -local, we have $\pi_n^{\mathbb{A}^1}(A, *) = 0$ for any $n \geq 1$ and any choice of base point. From the long exact sequence for the fibre sequence, cf. Proposition 2.1.11, we get the isomorphisms claimed. ■

A similar proof yields the following statement which reduces the homotopy type of a linear algebraic group to that of a *connected* group.

PROPOSITION 4.4.13 *Let G be a linear algebraic group over k . Then there exists an $\mathbb{A}^1$ -local fibre sequence*

$$G^0 \xrightarrow{i} G \xrightarrow{p} H.$$

where H is a finite étale group over k . Therefore there exists a short exact sequence

$$0 \longrightarrow \pi_0^{\mathbb{A}^1}(G^0) \xrightarrow{i_*} \pi_0^{\mathbb{A}^1}(G) \xrightarrow{p_*} H = \pi_0^{\mathbb{A}^1}(H) \longrightarrow 0,$$

and the inclusion of the fibre induces isomorphisms $i_* : \pi_n^{\mathbb{A}^1}(G^0, x) \rightarrow \pi_n^{\mathbb{A}^1}(G, i(x))$ for any $n \geq 1$ and any choice of base point $x \in G^0$.

PROOF: That the group of connected components of a linear algebraic group is a finite étale group scheme is stated in [Wat79, Theorem 6.7]. The $\mathbb{A}^1$ -rigidity for a finite étale group scheme follows from [MV99, Proposition 4.3.5]. ■

Next we will study the homotopy types of solvable groups, which will allow to show that the quotient $G \rightarrow G/RG$ induces an $\mathbb{A}^1$ -local fibre sequence with fibre the solvable radical RG of G . We begin with unipotent groups. Recall [DG70, Exposé XVII, Proposition 4.1.1] that a smooth connected linear algebraic group is unipotent if it has a filtration $\{1\} = U_n \subseteq U_{n-1} \subseteq \cdots \subseteq U_1 \subseteq U_0 = U$ such that U_{i+1} is normal in U_i and the quotient U_i/U_{i+1} is isomorphic to (a form of) the additive group $\mathbb{G}_a$. Note that such forms only exist over non-perfect fields, since for a perfect field k we have $H^1(k, \mathbb{G}_a) = 0$.

PROPOSITION 4.4.14 *Let G be a smooth connected linear algebraic group scheme over k , and let $R_u G$ denote the unipotent radical. Then there is an $\mathbb{A}^1$ -local weak equivalence $\phi : G \rightarrow G/R_u G$. Therefore any connected linear algebraic group has the homotopy type of a reductive group.*

PROOF: We use induction on the filtration with subquotients (forms of) $\mathbb{G}_a$. These are contractible, cf. Example 4.4.10. For the inductive step, one uses the $\mathbb{A}^1$ -local fibre sequence

$$U_{i+1} \longrightarrow U_i \rightarrow \mathbb{G}_a.$$

By assumption U_{i+1} is contractible, and Example 4.4.10 implies that this is an $\mathbb{A}^1$ -local trivial fibration with contractible base. Therefore U_i is contractible. We conclude that any smooth connected unipotent group over k is contractible.

By Example 4.4.10 again, we obtain that $\phi : G \rightarrow G/R_u G$ is an $\mathbb{A}^1$ -local trivial fibration. ■

REMARK 4.4.15 *There are of course also non-smooth unipotent groups, the key examples are given by*

$$\alpha_{p^n} = \ker \left(\mathbb{G}_a \xrightarrow{(-)^{p^n}} \mathbb{G}_a \right).$$

These are also contractible, since any morphism from a smooth local henselian k -scheme $\mathrm{Spec} R \rightarrow \alpha_{p^n}$ factors through $\mathrm{Spec} k$. It follows that any connected unipotent linear algebraic group is $\mathbb{A}^1$ -contractible.

We are going to reproduce Jardine's result [Jar84, Theorem 1] in the setting of $\mathbb{A}^1$ -homotopy theory. For this we need to determine when the classifying space of a group T of multiplicative type is $\mathbb{A}^1$ -local. We introduce the following notation: For the torus T there is the following exact sequence of groups of multiplicative type

$$1 \rightarrow T^0 \rightarrow T \rightarrow \mu \rightarrow 1,$$

where μ is finite étale and T^0 is connected.

PROPOSITION 4.4.16 *Let k be a field. Then any surjective homomorphism $\phi : G \rightarrow H$ of algebraic groups with kernel a group of multiplicative type T induces an $\mathbb{A}^1$ -local fibration sequence*

$$T \xrightarrow{i} G \xrightarrow{p} H,$$

if one of the following conditions are satisfied:

- (i) *The characteristic of k is zero, or*
- (ii) *the connected part of T is split.*

In this case, the inclusion of the fibre induces isomorphisms $i_* : \pi_n^{\mathbb{A}^1}(G, x) \rightarrow \pi_n^{\mathbb{A}^1}(H, p(x))$ for any $n \geq 2$ and any choice of base point $x \in G$.

PROOF: We have seen in Example 4.4.3 that a group T of multiplicative type over a field k is $\mathbb{A}^1$ -rigid. It follows from [CTO92, Theorem 4.2] that the sheaf of groups $U \mapsto H^1(U, T)$ satisfies $\mathbb{A}^1$ -invariance if k has characteristic zero or if the order of μ is prime to the characteristic of k and the connected part is split.

Therefore the classifying space BT of a group T of multiplicative type satisfying the above restrictions is $\mathbb{A}^1$ -local.

The group G is a T -torsor over H , therefore it induces a simplicial fibre sequence and is thus classified by a morphism into BT , the classifying space of T -torsors. This is local by the argument above, so Proposition 4.4.6 implies that $T \rightarrow G \rightarrow H$ is an $\mathbb{A}^1$ -local fibre sequence. ■

REMARK 4.4.17 *It seems reasonable to expect that the methods in the proof of [CTO92, Theorem 4.2] generalize to show that the classifying space of a torus T over a field is $\mathbb{A}^1$ -local if the following two conditions are satisfied:*

- (i) *The order of μ is prime to the characteristic of k , and*
- (ii) *the connected part T^0 splits over an extension L/k of degree $[L : k]$ prime to the characteristic of k .*

Using the above results on unipotent groups and groups of multiplicative type, we show how to describe the homotopy type of solvable algebraic groups. The next intermediate step is the commutative linear algebraic groups, which can be decomposed into a unipotent and a multiplicative part after [Wat79, Theorem 9.5]. This implies that commutative linear algebraic groups are homotopy discrete.

LEMMA 4.4.18 *Let k be a perfect field, and let G be a commutative linear algebraic group over k . Then there exists a unipotent group G_u , a group of multiplicative type G_s and an isomorphism $G \cong G_u \times G_s$.*

There are two ways to describe the homotopy type of a solvable linear algebraic group. One way is to use the derived series $G \supseteq DG \supseteq D^2G \supseteq \cdots \supseteq D^nG = 1$ in an induction argument together with the fact that commutative groups can be decomposed as above, cf. [Bor91, Definition I.2.4].

Another way is to use the fact that for any smooth connected solvable linear algebraic group G over a perfect field k there exists a unique connected normal algebraic subgroup G_u of G such that G_u is unipotent and G/G_u is of multiplicative type, cf. [Wat79, Theorem 10.3]. This implies the existence of an $\mathbb{A}^1$ -local fibre sequence $G_u \rightarrow G \rightarrow G/G_u$, from which we get an $\mathbb{A}^1$ -local weak equivalence $G \rightarrow G/G_u$.

PROPOSITION 4.4.19 *For any solvable group G over a perfect field k there exists a unipotent subgroup $G_u \hookrightarrow G$ such that G/G_u is of multiplicative type. The projection $\phi : G \rightarrow G/G_u$ is an $\mathbb{A}^1$ -local weak equivalence. The classifying space $BL_{\mathbb{A}^1}G$ of Nisnevich torsors is therefore local if G/G_u satisfies the conditions of Proposition 4.4.16.*

Under these conditions, any principal fibration with structure group G induces an $\mathbb{A}^1$ -local fibre sequence.

COROLLARY 4.4.20 *For any smooth connected linear algebraic group G there is an fibre sequence $RG \rightarrow G \rightarrow G/RG$, where RG denotes the solvable radical of G . This fibre sequence is $\mathbb{A}^1$ -local if the multiplicative quotient of RG satisfies the conditions of Proposition 4.4.16.*

A linear algebraic group is semisimple, if its radical is trivial. In the above, the group G/RG is semisimple. Such groups can still be further decomposed. On the one hand, there are simply-connected semisimple groups, i.e. those which do not have any nontrivial étale covering. On the other hand, one can decompose a group into its simple factors.

PROPOSITION 4.4.21 *Let G be a connected semisimple linear algebraic group over a field k . Then there exists a fibre sequence $P \rightarrow \tilde{G} \rightarrow G$, where $\tilde{G}$ is a semisimple, simply connected algebraic group, and P is a finite central subgroup of $\tilde{G}$. This fibre sequence is $\mathbb{A}^1$ -local if the order of P is prime to the characteristic of k .*

The group $\tilde{G}$ is usually called the *universal covering* of the semisimple group G , cf. [Sko01, Section 3.2]. Note that the results of this section imply that this is not the universal covering in the sense of $\mathbb{A}^1$ -homotopy theory.

Another way to make a semisimple group “simpler” is to decompose it into an almost direct product of simple groups. We therefore get an $\mathbb{A}^1$ -homotopy version of [Jar81, Theorem 3.2.5 and Corollary 3.2.6].

PROPOSITION 4.4.22 *Let G be a connected smooth semisimple linear algebraic group over k , and let $S_1, \dots, S_n$ be the finite set of minimal connected normal*

algebraic subgroups of G of nonzero dimension. Then the multiplication homomorphism

$$m : S_1 \times \cdots \times S_n \rightarrow G : (s_1, \dots, s_n) \mapsto s_1 \cdots s_n$$

is a surjective homomorphism with finite, étale, central kernel P . Therefore, if the order of P is prime to the characteristic of k , the following is an $\mathbb{A}^1$ -local fibre sequence:

$$P \hookrightarrow S_1 \times \cdots \times S_n \xrightarrow{m} G.$$

In the latter situation, we have a short exact sequence of $\mathbb{A}^1$ -homotopy group sheaves:

$$0 \rightarrow \bigoplus_{i=1}^n \pi_1^{\mathbb{A}^1}(S_i) \rightarrow \pi_1^{\mathbb{A}^1}(G) \rightarrow P \rightarrow 0,$$

and for all $l \geq 2$, the multiplication map induces isomorphisms

$$\bigoplus_{i=1}^n \pi_l^{\mathbb{A}^1}(S_i) \xrightarrow{\cong} \pi_l^{\mathbb{A}^1}(G)$$

PROOF: The algebraic argument works as in the proof of [Jar81, Theorem 3.2.5]. That the corresponding morphism induces a fibre sequence if the order of P is prime to the characteristic of k follows from Proposition 4.4.6.

The right exactness of the short exact sequence of π_1 -sheaves follows since the functor $K_1(G, R)$ is trivial for any simple, simply connected group G and semi-local ring R . The group of connected components can be identified with K_1 . ■

We have seen how to reduce the computation of homotopy groups of algebraic groups to those of simple algebraic groups, using fibre sequences in $\mathbb{A}^1$ -homotopy theory. Together with the results of Chapter 5, this implies that the higher homotopy group sheaves for an algebraic group G are direct sums of unstable K-groups if the maximal torus of G is split. The fundamental groupoid then depends on the various pieces encountered in the reduction steps, like the connected components, the solvable radical etc. We will conclude the discussion of homotopy types of algebraic groups with two remarks on the homotopy types of anisotropic groups and on the situation over more general base schemes.

REMARK 4.4.23 *It is not yet clear how to describe the homotopy groups of non-split simple algebraic groups, i.e. groups, whose maximal torus does not split over the base field. First we note, that although the unit of a non-split semisimple group is the only k -rational point, such a group is not $\mathbb{A}^1$ -rigid: Let L/k be a splitting field for the anisotropic group G , which exists since any connected, reductive group splits over a finite separable extension of the base field [Bor91, Corollary V.18.8]. Then there are nontrivial morphisms $\mathbb{A}^n \times_k L \rightarrow G \times_k L$, and therefore there can not be a bijective correspondence*

$$\mathrm{Hom}_L(L, G \times L) = \mathrm{Hom}_k(L, G) \rightarrow \mathrm{Hom}_k(\mathbb{A}^1 \times L, G) = \mathrm{Hom}_L(\mathbb{A}^1 \times L, G \times L).$$

The idea is to consider $G \times_k L$ as a split semisimple group over L , where we can determine its $\mathbb{A}^1$ -homotopy type as an L -scheme. Then we push forward this homotopy type along the finite morphism $f : \mathrm{Spec} L \rightarrow \mathrm{Spec} k$. Using [MV99,

Proposition 3.1.27 and Proposition 3.2.12], we find that the following composition is a weak equivalence for any simplicial presheaf X :

$$f_*(X) \rightarrow \mathbf{R}f_*(X) \rightarrow \mathbf{R}^{\mathbb{A}^1}f_*(X).$$

In this, $\mathbf{R}^{\mathbb{A}^1}f_*(G \times_k L)$ is the right homotopy type for $G \times_k L$ as a k -scheme, and its homotopy groups at the henselian local k -algebra R over k are given by computing the homotopy groups of $G \times_k L$ at the henselian local L -algebra $R \otimes_k L$. Having the homotopy type of $G \times_k L$ as a k -scheme, we could use the product formula

$$\pi_n^{\mathbb{A}^1}(G \times_k L) \cong \pi_n^{\mathbb{A}^1}(G) \times \pi_n^{\mathbb{A}^1}(L/k).$$

Therein, G is an anisotropic semisimple group and L is a splitting field. The latter description does not however seem to account for the geometry of forms of algebraic groups.

REMARK 4.4.24 *Over an arbitrary base scheme S , the whole theory does not work properly, due to several reasons: First, if the base is not regular, the multiplicative group $\mathbb{G}_m$ is not $\mathbb{A}^1$ -local, and it seems to be difficult to determine its $\mathbb{A}^1$ -homotopy type and the homotopy type of its classifying space, cf. [MV99, Example 3.2.10]. Therefore the immediate restriction is to regular base schemes. Second, it is not yet clear if the restrictions of Proposition 4.4.16 can be lifted, so we are not sure if a group of multiplicative type has an $\mathbb{A}^1$ -local classifying space over a regular base scheme. Finally, all geometric decomposition results in which unipotent groups appear are only handled over (perfect) fields. These results include the Chevalley theorem, the structure theorem for commutative linear algebraic groups, and the definition of reductive groups via the unipotent radical. Summing up, we do not know enough about the geometry of algebraic groups over general bases to completely describe their $\mathbb{A}^1$ -homotopy types.*

A Local Version of Ganea's Theorem: In this paragraph we apply the localization result from Theorem 4.3.10 to the fibre sequences coming out of Ganea's theorem which we discussed in Section 3.3.

The fibre of the fold map of a local simplicial presheaf can be determined by the following proposition. Note however, that although $\Omega L_f X$ is local by Proposition 4.1.6, it is not necessarily the localization of ΩX .

PROPOSITION 4.4.25 *Let T be a Grothendieck site, and let L_f be a nullification functor on $\Delta^{\text{op}}\text{PShv}(T)$. Then for any simplicial presheaf X , the following is an f -local fibre sequence:*

$$\Sigma \Omega L_f X \rightarrow L_f X \vee L_f X \xrightarrow{\vee} L_f X.$$

This is a direct corollary from Proposition 3.3.11 and Theorem 4.3.10, since $L_f X$ is of course local. The generalized version for a finite set of simplicial presheaves follows similarly.

PROPOSITION 4.4.26 *For any finite set of simplicial presheaves $X_0, \dots, X_n$, the following fibre sequence is f -local:*

$$\Omega L_f X_0 * \dots * \Omega L_f X_n \rightarrow W(L_f X_0, \dots, L_f X_n) \rightarrow L_f X_0 \times \dots \times L_f X_n.$$

Finally, also Proposition 3.3.18 has a local version, which should have interesting applications to $\mathbb{A}^1$ -homotopy related to the Hilton-Milnor theorem, the EHP-sequence [Gan65] or construction of homotopy decompositions [Sel96].

Note that the $\Omega L_f X$ in the fibre is there because a priori the spaces in the fibre sequence need not be local. This loop space is the only homotopy limit, and we make sure it has the right homotopy type. The other constructions are homotopy colimits, and these “commute” with localization, cf. Proposition 4.1.7.

PROPOSITION 4.4.27 *Let again T be a Grothendieck site, and L_f be a localization functor on $\Delta^{\text{op}} PShv(T)$. For any f -local fibre sequence $F \rightarrow E \rightarrow B$ of simplicial presheaves there is another f -local fibre sequence*

$$F * \Omega L_f B \rightarrow E/F \rightarrow B.$$

PROOF: For a given f -local fibre sequence $F \rightarrow E \xrightarrow{p} B$ we can rectify it to an f -local fibration p of f -local and fibrant objects E and B , cf. Section 2.3. This uses properness of the local model structure, cf. Theorem 4.3.7. Applying Proposition 3.3.18 to this fibre sequence of simplicial presheaves, we get a fibre sequence of the desired form. This fibre sequence is an f -local fibre sequence by Theorem 4.3.10 since B was chosen to be f -local. ■

The Fundamental Group of $\mathbb{P}^1$: As already mentioned there are particularly beautiful instantiations of Ganea’s theorem. Applying Proposition 4.4.25 to Example 3.3.12 yields an interesting fibre sequence which can be used to give a description of $\pi_1^{\mathbb{A}^1}(\mathbb{P}^1)$ different from the one in [Mor06b].

Let S be a regular base scheme. Then the simplicial presheaf $B\mathbb{G}_m$ defined on the site Sm_S with the Nisnevich topology is $\mathbb{A}^1$ -local: Its homotopy groups are given by the group of units and the Picard group, and these are known to satisfy $\mathbb{A}^1$ -invariance for regular schemes, cf. [MV99, Proposition 4.3.8]. Since we have the $\mathbb{A}^1$ -local weak equivalence $\Sigma_S^1 \mathbb{G}_m \simeq \mathbb{P}^1$, we obtain the following beautiful $\mathbb{A}^1$ -local fibre sequence:

$$L_{\mathbb{A}^1} \Sigma_S^1 \mathbb{G}_m \simeq \mathbb{P}^1 \rightarrow \mathbb{P}^\infty \vee \mathbb{P}^\infty \rightarrow \mathbb{P}^\infty.$$

The first general conclusion about the $\mathbb{A}^1$ -local homotopy type of $\mathbb{P}^1$ we can draw is that for $n \geq 2$, there are isomorphisms of homotopy group sheaves

$$\pi_n^{\mathbb{A}^1}(\mathbb{P}^1) \cong \pi_n^{\mathbb{A}^1}(\mathbb{P}^\infty \vee \mathbb{P}^\infty),$$

which follows since $\pi_n^{\mathbb{A}^1}(\mathbb{P}^\infty) \cong \pi_n^{\mathbb{A}^1}(B\mathbb{G}_m) \cong 0$ for $n \geq 2$.

To say something about the fundamental group, we need a version of the van-Kampen theorem, enabling us to compute the homotopy of the wedge sum. We would of course expect that

$$\pi_1^{\mathbb{A}^1}(\mathbb{P}^\infty \vee \mathbb{P}^\infty) \cong \pi_1^{\mathbb{A}^1}(\mathbb{P}^\infty) * \pi_1^{\mathbb{A}^1}(\mathbb{P}^\infty),$$

where the $*$ denotes the free product of (strongly $\mathbb{A}^1$ -invariant) sheaves of groups. The van-Kampen theorem holds of course for simplicial sheaves over an arbitrary base, but it does not necessarily hold for the $\mathbb{A}^1$ -local model structure, due to complications with the localization functors. It is not clear if we can expect a

van-Kampen theorem in situations where $\mathbb{A}^1$ -connectivity fails [Ayo06]. It seems plausible to expect something like this over a base $S = \operatorname{Spec} R$ for R a Dedekind ring, but at the moment we only know it over a field, cf. [Mor06b].

PROPOSITION 4.4.28 *Let k be any field. Then $\pi_1^{\mathbb{A}^1}(\mathbb{P}^1)$ as a sheaf of groups is isomorphic to the kernel of the fold map:*

$$\vee : \mathbb{G}_m * \mathbb{G}_m \rightarrow \mathbb{G}_m : (u_1, \dots, u_n) \mapsto u_1 \cdots u_n.$$

The notation $(u_1, \dots, u_n)$ describes a general reduced word element of the free product, written as a sequence of units, which under the fold map is mapped to the product of these units.

Note that $\mathbb{G}_m * \mathbb{G}_m : U \mapsto \mathcal{O}_k^*(U) * \mathcal{O}_k^*(U)$ is already a sheaf. We hence can describe the fundamental group sheaf of $\mathbb{P}^1$ as follows:

$$\pi_1(\mathbb{P}_k^1)(U) \cong \ker(\mathcal{O}_k^*(U) * \mathcal{O}_k^*(U) \rightarrow \mathcal{O}_k^*(U)).$$

This provides a slightly different perspective on Morel's results from [Mor06b]: On the one hand, we obtain a computation of $\pi_1^{\mathbb{A}^1}(\mathbb{P}^1)$ which is independent of the definition of Milnor-Witt K-theory. On the other hand, using Morel's identification of this fundamental group as Milnor-Witt K-theory, we obtain a description of the K_2^{MW} directly in terms of units. Note also that we get a fibre sequence which is valid for a general regular base scheme S and therefore we can compute the non-localized homotopy groups of $\Sigma\mathbb{G}_m$, cf. Example 3.3.12. Any information on the $\mathbb{A}^1$ -local fundamental group of $\mathbb{P}^\infty \vee \mathbb{P}^\infty$ will yield a presentation of $\pi_1^{\mathbb{A}^1}(\mathbb{P}^1)$ as above over more general bases.

In a similar vein, we can get a general fibre sequence describing the fundamental group of $\mathbb{A}^2 \setminus \{0\}$, yielding an $\mathbb{A}^1$ -local counterpart to Example 3.3.14. Using Proposition 4.4.26 with $X_0 = X_1 = B\mathbb{G}_m$, we obtain for any regular base scheme S an $\mathbb{A}^1$ -local fibre sequence

$$\mathbb{G}_m * \mathbb{G}_m \simeq \mathbb{A}^2 \setminus \{0\} \rightarrow \mathbb{P}^\infty \vee \mathbb{P}^\infty \rightarrow \mathbb{P}^\infty \times \mathbb{P}^\infty.$$

The $\mathbb{A}^1$ -local weak equivalence $\mathbb{G}_m * \mathbb{G}_m \simeq \mathbb{A}^2 \setminus \{0\}$ follows, since the join $\mathbb{G}_m * \mathbb{G}_m$ is defined as the homotopy pushout of the following diagram:

$$\mathbb{G}_m \xleftarrow{p_1} \mathbb{G}_m \times \mathbb{G}_m \xrightarrow{p_2} \mathbb{G}_m.$$

A particular $\mathbb{A}^1$ -local cofibrant resolution of this diagram is the covering of $\mathbb{A}^2 \setminus \{0\}$ by $\mathbb{G}_m \times \mathbb{A}^1$ and $\mathbb{A}^1 \times \mathbb{G}_m$, glued together along $\mathbb{G}_m \times \mathbb{G}_m$.

In the case of a base field k , we can again identify the fundamental group of $\mathbb{P}^\infty \vee \mathbb{P}^\infty$, and we get the following presentation of the fundamental group of $\mathbb{A}^2 \setminus \{0\}$:

$$\pi_1(\mathbb{A}_k^2 \setminus \{0\})(U) \cong \ker(\mathcal{O}_k^*(U) * \mathcal{O}_k^*(U) \rightarrow \mathcal{O}_k^*(U) \times \mathcal{O}_k^*(U)).$$

The morphism above is induced from the “abelianization” $\mathbb{P}^\infty \vee \mathbb{P}^\infty \rightarrow \mathbb{P}^\infty \times \mathbb{P}^\infty$, and therefore takes a reduced word $(a_1, b_1, \dots, a_n, b_n)$ in $\mathcal{O}_k^*(U) * \mathcal{O}_k^*(U)$ with a_i from the first factor and b_i from the second factor to the pair of products $(a_1 \cdots a_n, b_1 \cdots b_n) \in \mathcal{O}_k^*(U) \times \mathcal{O}_k^*(U)$.

We want to note that the above presentations also allow for a simple explanation of the morphisms in the exact sequence

$$0 \rightarrow \pi_1^{\mathbb{A}^1}(\mathbb{A}^2 \setminus \{0\}) \xrightarrow{p_*} \pi_1^{\mathbb{A}^1}(\mathbb{P}^1) \xrightarrow{\partial} \mathbb{G}_m \rightarrow 0$$

induced from the $\mathbb{A}^1$ -local fibre sequence $\mathbb{G}_m \xrightarrow{i} \mathbb{A}^2 \setminus \{0\} \xrightarrow{p} \mathbb{P}^1$ via the long exact sequence of Proposition 2.1.11. The morphism p_* is basically the identity, taking a reduced word $(a_1, b_1, \dots, a_n, b_n) \in \mathcal{O}_k^*(U) * \mathcal{O}_k^*(U)$ with $a_1 \cdots a_n = b_1 \cdots b_n = 1$ to the same word in $\pi_1^{\mathbb{A}^1}(\mathbb{P}^1)$ since $a_1 \cdots a_n b_1 \cdots b_n = 1$. On the other hand, the boundary map ∂ can be described as follows: For any reduced word $(a_1, b_1, \dots, a_n, b_n)$ representing a class in $\pi_1^{\mathbb{A}^1}(\mathbb{P}^1)$ we have $a_1 \cdots a_n = (b_1 \cdots b_n)^{-1}$ but not necessarily $a_1 \cdots a_n = 1$. This provides the element $\partial(a_1, b_1, \dots, a_n, b_n) = a_1 \cdots a_n$ in $\mathcal{O}_k^*(U)$.

Note that the non-commutativity of the fundamental groups of $\mathbb{P}^1$ and $\mathbb{A}^2 \setminus \{0\}$ is rather clearly visible from the above $\mathbb{A}^1$ -local fibre sequences.

To round off the discussion, let us note that using Proposition 4.4.27 we can also describe the fibre of the adjunction morphism $\Sigma\Omega X \rightarrow X$, also called the homology suspension in [Gan65]. For any $\mathbb{A}^1$ -local simplicial presheaf X , we have an $\mathbb{A}^1$ -local fibre sequence

$$\Omega X * \Omega X \rightarrow \Sigma\Omega X \rightarrow X.$$

This follows by applying Proposition 4.4.27 to the path-space fibre sequence $\Omega X \rightarrow * \rightarrow X$. In the special case of $X = B\mathbb{G}_m$ we obtain over any regular base scheme S the fibre sequence associated to the $\mathbb{G}_m$ -covering of $\mathbb{P}^1$:

$$\mathbb{A}^2 \setminus \{0\} \rightarrow \mathbb{P}^1 \rightarrow \mathbb{P}^\infty.$$

It seems plausible that Proposition 4.4.27 can also be used to provide generalizations of [Gan65, Proposition 2.1 and Theorem 2.2], assuming we have the van-Kampen theorem and the Hurewicz homomorphism available. Such results would yield other methods of computing with $\mathbb{A}^1$ -homology sheaves at least over a field, where the above theorems are known. We will however not discuss these issues in this thesis.

$\mathbb{A}^1$ -Local Classifying Spaces: In this paragraph, we will discuss the existence of classifying spaces for fibre sequences in the $\mathbb{A}^1$ -local model structure. Such statements can not be proven using the machinery of Chapter 3, since we do not know homotopy distributivity for these model structures.

By applying the localization criterion from Theorem 4.3.10 to the universal fibre sequences produced by Theorem 3.2.17, we obtain sufficient criteria for the representability of the fibre sequence functor in the $\mathbb{A}^1$ -local model structures:

PROPOSITION 4.4.29 *Let T be a Grothendieck site, let $g : X \rightarrow Y$ be a morphism of simplicial presheaves in $\Delta^{\text{op}}PShv(T)$ such that L_g admits a fibrewise version, and let F be a simplicial presheaf. Then $L_g B^f F$ represents the functor*

$$\mathcal{H}^{\mathbb{A}^1}(-, F) : B \mapsto \{p : F \rightarrow E \rightarrow B \mid p \text{ is an } \mathbb{A}^1\text{-local fibre sequence}\}$$

if the fibre sequence

$$L_g F \rightarrow E' \rightarrow A_{L_g} B^f F$$

is trivial. As in Theorem 4.3.10, this fibre sequence is the pullback of the fibrewise localization of the universal fibre sequence $F \rightarrow E^f F \rightarrow B^f F$ along

$$A_{L_g} B^f F = \text{hofib}(B^f F \rightarrow L_g B^f F) \rightarrow B^f F.$$

One example of a situation where this is satisfied is the $\mathbb{A}^1$ -localization of $\Delta^{\text{op}} PShv(\text{Sm}_k)$ where k is a field of characteristic zero, applied to $\mathbb{A}^1$ -local classifying spaces such as the classifying space BT of Nisnevich torsors under a group T of multiplicative type or the classifying space $B_{\text{ét}} G$ of étale torsors under a finite group G . Thereby we recover the known results that the above spaces are classifying spaces also in the $\mathbb{A}^1$ -local model structures. Such results could be extended with the usual provisos, like the order of G should be prime to the characteristic of the field k .

Another way to express the condition of Proposition 4.4.29 is the following: The fibrewise localization construction of Section 4.2 implies the existence of a morphism $B^f F \rightarrow B^f L_g F$ classifying this fibrewise localization. The universal fibre sequence is a g -local fibre sequence if the composition $A_{L_g} B^f F \rightarrow B^f F \rightarrow B^f L_g F$ is null-homotopic.

We see that the locality of $B^f F$ is not necessary for the fibre sequence functor to be representable in the local model structure. If the conditions of the above proposition are satisfied, then

$$L_g F \rightarrow L_g E^f L_g F \rightarrow L_g B^f L_g F$$

is the g -local universal fibre sequence with fibre F .

Since the classical methods of constructing classifying spaces resp. universal fibre sequences are not known to work in the localized model structure, it seems difficult to construct local universal fibre sequences for spaces F for which the sequence $F \rightarrow E^f F \rightarrow B^f F$ is not local. Moreover, since we have seen throughout this chapter, that being $\mathbb{A}^1$ -local is a rather “global” property for a fibre sequence, it seems rather unlikely that the local fibre sequence functor

$$\mathcal{H}^{\mathbb{A}^1}(-, F) : B \mapsto \{p : F \rightarrow E \rightarrow B \mid p \text{ is an } \mathbb{A}^1\text{-local fibre sequence}\}$$

should satisfy homotopy-continuity in general. This would require control on how the homotopy fibre of the localization behaves under homotopy colimits, which is impossible.

We can even say more: If the universal fibre sequence $F \rightarrow E^f F \rightarrow B^f F$ for a simplicial presheaf F is not local, but the fibre sequence is locally trivial in the topology of the site T , then the fibre sequence functor

$$\mathcal{H}^{\mathbb{A}^1}(-, F) : B \mapsto \{p : F \rightarrow E \rightarrow B \mid p \text{ is an } \mathbb{A}^1\text{-local fibre sequence}\}$$

does not have the Mayer-Vietoris property. Hence it simply can not be representable. The same conclusion can be obtained if there exists some F -fibre sequence which is locally trivial and not local. The argument is as follows: Since the fibre sequence $F \rightarrow E \rightarrow B$ is locally trivial, we can write it in $\Delta^{\text{op}} PShv(T)$ as

$$F \rightarrow \text{hocolim}_{T \times \Delta \downarrow B} \sigma^*(E) \rightarrow \text{hocolim}_{T \times \Delta \downarrow B} \sigma,$$

where σ runs over all simplices $\sigma : U \times \Delta^n \rightarrow B$. Moreover, we can replace the site T with a site T' such that $\mathcal{S}hv(T) \cong \mathcal{S}hv(T')$ and all the above fibre sequences are actually trivial, cf. the proof of Theorem 4.2.12. Since the resulting fibre sequences are *trivial*, they are in particular g -local. Moreover, the homotopy colimits commute with the localization, in the sense that

$$L_g \operatorname{hocolim}_{T \times \Delta \downarrow B} \sigma^*(E) \simeq L_g \operatorname{hocolim}_{T \times \Delta \downarrow B} L_g \sigma^*(E),$$

cf. Proposition 4.1.7. Therefore the g -local homotopy colimit of the above local fibre sequences is the sequence $L_g F \rightarrow L_g E \rightarrow L_g B$. But this is by assumption not a fibre sequence, $L_g F$ is not the local homotopy fibre of $L_g E \rightarrow L_g B$. Therefore the Mayer-Vietoris property fails for the local fibre sequence functor, whence it can not be representable.

We therefore see, that it is basically no restriction to first construct classifying spaces for simplicial presheaves and then check if the corresponding universal fibre sequences localize. The above argument also shows that if there are fibre sequences which are not g -local but T -locally trivial, then homotopy distributivity fails in the g -local model category. This implies the failure of fibrewise localization, and therefore the behaviour of fibre sequences under further localizations is not clear: The methods of this chapter can not be applied in the $\mathbb{A}^1$ -local category to study its localizations. The localization criterion Theorem 4.3.10 fails, and new methods have to be developed to understand how $\mathbb{A}^1$ -local fibre sequences behave under further localizations.

As already remarked in Section 3.2, the most interesting universal fibre sequence to investigate for locality is the one classifying spherical fibrations

$$L_{\mathbb{A}^1} S^{2n,n} \rightarrow E^f(L_{\mathbb{A}^1} S^{2n,n}) \rightarrow B^f L_{\mathbb{A}^1} S^{2n,n},$$

for its relevance to a motivic surgery program. Showing that the above fibre sequence is local however needs much more geometry than we can afford at the moment.

Non-Local Fibre Sequences: From the discussion in the previous paragraph, we can now extract the difficulties in constructing non-local fibre sequences. To show that a fibre sequence $F \rightarrow E \rightarrow B$ is not $\mathbb{A}^1$ -local, we would have to first compute the fibrewise nullification $L_{\mathbb{A}^1} F \rightarrow \overline{E} \rightarrow B$, then compute the $\mathbb{A}^1$ -nonlocal part $A_{L_{\mathbb{A}^1}} B$, and then prove that the pulled back fibre sequence $L_{\mathbb{A}^1} F \rightarrow E' \rightarrow A_{L_{\mathbb{A}^1}} B$ does not split up to homotopy.

Considering the difficulties in each task, it seems better to look for cases in which one of the steps above is easy. The approach which is most promising due to the amount of literature is to come up with a fibre sequence over an $\mathbb{A}^1$ -contractible variety. We list some examples, without giving definite results.

EXAMPLE 4.4.30 There is a very simple example of non-local fibre sequence: Let k be a field of characteristic $p > 0$, and recall that the work of Harbater and Raynaud on the Abhyankar conjecture implies the existence of a whole lot of étale coverings of $\mathbb{A}_k^1$. Note that by [MV99, Proposition 4.3.3], the classifying space $B_{\text{ét}} G$ for G an étale p -group scheme is contractible in the $\mathbb{A}^1$ -local model structure

over k . Therefore we consider a quasi p -group G of order divisible by p but not a power of p . As an example, we note that the alternating group A_n of degree $n > 2$ is a quasi 3-group whose order is $(n!/2)$ and therefore not a power of 3. By the work of Raynaud [Ray94], every quasi- p group appears as étale covering of the affine line in characteristic p , and we obtain a simplicial fibre sequence

$$G \rightarrow X \rightarrow \mathbb{A}^1$$

as in Example 3.3.9. The two steps in the proof of non-locality become rather easy: Since G is just a finite group, it is $\mathbb{A}^1$ -local and we do not have to do a fibrewise localization. Since the base is $\mathbb{A}^1$, we do not have to compute the non-local part. All we have to do to apply Theorem 4.3.10 is check if this fibre sequence is not trivial, which follows since G is $\mathbb{A}^1$ -local and obviously not contractible. This implies that BG is not $\mathbb{A}^1$ -local, since there exists a non-contractible map $\mathbb{A}^1 \rightarrow BG$ classifying the above nontrivial and non-local étale cover with covering group G . This moreover implies that the universal G -covering $G \rightarrow EG \rightarrow BG$ is not $\mathbb{A}^1$ -local, since the fibre sequence produced above is a pullback of the universal fibre sequence.

Note that the BG we are using above can be either $B_{\text{ét}}G$ in the terminology of [MV99] or the classifying space for fibre sequences with fibre G . This is wrong for the classifying space of G -torsors BG , by [MV99, Proposition 4.3.5] which says that the classifying space of G -torsors is $\mathbb{A}^1$ -local over a general base. This is due to the fact that the p -coverings we were considering above are not locally trivial in the Nisnevich topology. $\square$

EXAMPLE 4.4.31 Another interesting example to look at are the smooth quasi-affine $\mathbb{A}^1$ -contractible varieties produced in [AD07a]. There do exist lots of non-trivial vector bundles on these varieties, which are described in [AD07b]. Of course, by homotopy invariance of K-theory, all of these vector bundles have to be stably trivial. Considered as fibre sequences with fibre $\mathbb{A}^n$, they are also $\mathbb{A}^1$ -local fibre sequences which do not yield any information at all, since all spaces involved are contractible. However, the corresponding GL_n -torsors are non-trivial, and the question is, if the corresponding simplicial fibre sequence $GL_n \rightarrow E \rightarrow B$ with B smooth quasi-affine contractible is an $\mathbb{A}^1$ -local fibre sequence. Note that the nontriviality of the above sequence is not enough, we have to compute the fibrewise localization of the GL_n -torsor to decide this question.

It would suffice to show that the morphism $H^1(X, GL_n) \rightarrow H^1(X, K_{1,n})$ induced by $GL_n \rightarrow K_{1,n} = GL_n/E_n$ applied to the above GL_n -torsors yields trivial K_1 -torsors: The change-of-fibre for $GL_n \rightarrow \text{Sing}_{\bullet}^{\mathbb{A}^1}(GL_n)$ would result in constant transition maps, which are all homotopic to the identity if the corresponding $K_1 = \pi_0(\text{Sing}_{\bullet}^{\mathbb{A}^1}(GL_n))$ -torsor is trivial. It should be possible to do this, since the quotients on which these nontrivial GL_n -torsors live are all quotients of affine space $\mathbb{A}^n$ modulo the action of a unipotent group.

One way to see it is the following: From [AD07b, Corollary 3.1], we get non-trivial vector bundles of arbitrarily large rank. We can therefore choose the corresponding GL_n -torsors such that n is in the stable range. Then we have $H^1(X, K_{1,n}) \cong H^1(X, K_1)$. The latter is trivial for contractible schemes, by homotopy invariance and descent for K-theory.

However, it should also be possible to show by an explicit description of the cocycles involved that the corresponding GL_n -torsors have a reduction to E_n , and therefore applying the change-of-fibre construction for $GL_n \rightarrow \mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(GL_n)$ should result in *trivial* $\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(GL_n)$ -torsors. We conjecture that this is always the case: For any non-trivial vector bundle $E \rightarrow B$ on a smooth $\mathbb{A}^1$ -contractible variety B , the corresponding GL_n -torsor $GL_n \rightarrow E' \rightarrow B$ becomes trivial after fibrewise replacing GL_n by $\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(GL_n)$. Therefore the fibre sequence $GL_n \rightarrow E' \rightarrow B$ is an $\mathbb{A}^1$ -local fibre sequence and induces a $\mathbb{A}^1$ -local weak equivalence $GL_n \rightarrow E'$.

This implies that the isomorphism

$$H_{\mathrm{Zar}}^1(X, GL_n) \xrightarrow{\cong} H_{\mathrm{Zar}}^1(X, \mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(GL_n))$$

from [Mor07, Theorem 2.2.1] can not be extended to arbitrary smooth schemes X over a field k . What we have seen in this paragraph are non-trivial elements in the kernel of the above map for a smooth quasi-affine $\mathbb{A}^1$ -contractible scheme. On the other hand, it follows from [Mor07, Theorem 12 and Remark 2.1.1] that the classifying space of $\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(GL_n)$ is $\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(Gr_n)$, which is $\mathbb{A}^1$ -local. Therefore $H_{\mathrm{Zar}}^1(X, \mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(GL_n))$ satisfies homotopy invariance, and describes the pointed set of connected components of the Grassmannians Gr_n .

Summing up, the exotic GL_n -torsors do not actually provide examples of non-local fibre sequences. The localization theorem can be applied, because the fibrewise localizations over $\mathbb{A}^1$ -contractible schemes yield trivial $\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(GL_n)$ -torsors, and therefore any exotic GL_n -torsor $GL_n \rightarrow E \rightarrow B$ over an $\mathbb{A}^1$ -contractible smooth scheme B yields an $\mathbb{A}^1$ -local fibre sequence

$$\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(GL_n) \rightarrow \mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(E) \rightarrow \mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(B).$$

In particular, the total spaces of these torsors provide schemes which are $\mathbb{A}^1$ -weakly equivalent to GL_n . $\square$

EXAMPLE 4.4.32 The quadratic situation, i.e. the theory of inner product spaces provides similar examples to look at: Stably, the functors associating to a scheme the category of inner product spaces satisfy homotopy invariance and patching, and therefore the Witt groups and hermitian K-theory are representable in the $\mathbb{A}^1$ -local model structures [Hor05]. On the other hand, the unstable situation differs much from the stable situation, and the quadratic analogue of Serre's conjecture fails, cf. [Lam06, Chapter VII]. This implies that one does not have to pass to exotic $\mathbb{A}^1$ -contractible schemes to get the failure of homotopy invariance as in the case of projective modules, there are already quadratic modules over ordinary affine space $\mathbb{A}^n$ which are not extended.

Again the mere existence of non-extended inner product structures on modules over a polynomial ring does not imply that the corresponding torsor is not an $\mathbb{A}^1$ -local fibre sequence. It is still necessary to apply a fibrewise localization. In the paper [KS82], there are some criteria for quadratic bundles over $\mathbb{A}^n$ to be extended. I have not yet come by a description of the fibrewise localization of non-extended bundles. All I can say at this moment is the following corollary of the localization result Theorem 4.3.10: If such non-extended bundles do not become trivial after a fibrewise localization, then the universal fibre sequence

$$\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(O_n) \rightarrow E^f(\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(O_n)) \rightarrow B^f \mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(O_n)$$

is not $\mathbb{A}^1$ -local: Any morphism $\mathbb{A}^n \rightarrow B^f \text{Sing}_{\bullet}^{\mathbb{A}^1}(O_n)$ would factor through the non-local part $A_{L_{\mathbb{A}^1}} B^f \text{Sing}_{\bullet}^{\mathbb{A}^1}(O_n)$ of the classifying space, and therefore the pullback of the universal fibre sequence to the non-local part can therefore not be trivial. $\square$

Applications Beyond $\mathbb{A}^1$ -Homotopy: By now, we have only considered applications of the localization theory developed to the nullifications of $\Delta^{\text{op}} \mathcal{PShv}(\text{Sm}_S)$ at the affine line $\mathbb{A}^1$. Of course there are other spaces at which we can localize and hopefully get useful information.

On the one hand, unstable localization theory for topological spaces has been used in [Bou94] to provide unstable analogues of the chromatic homotopy theory, which aims at an understanding of the homotopy groups of spheres. A similar goal to which to apply unstable localization theory to would be an understanding of the global structure of the homotopy category of simplicial presheaves on a site T , at least in algebraic geometric contexts like the Nisnevich or étale sites. We note however that these methods do not extend to the localized model structures of $\mathbb{A}^1$ -homotopy theory, since fibrewise localization only works for simplicial presheaves. Thus we can only track the behaviour of spaces under a localization functor if we start from a model category of simplicial presheaves; if we start from an already localized model structure, we do not know homotopy distributivity, which blocks the straightforward way of getting any information about what a localization functor does to a space.

Another interesting localization functor to study would be $\mathbb{P}^1$ -nullification. The result of a suitable construction should be something like a “birational homotopy type”, similar to the birational motives that are obtained by localizing the category of motives at the motive of $\mathbb{P}^1$. In dealing with $\mathbb{A}^1$ -localization and in particular connecting the homotopical behaviour with geometric phenomena, we always found it tedious that the morphisms $\mathbb{A}^1 \rightarrow X$ can not be assembled into something which is meaningful from an algebraic geometry point of view. The space $\mathbb{P}^1$ is much better behaved in this respect, there are well-studied moduli spaces of rational curves describing the morphisms $\mathbb{P}^1 \rightarrow X$.

Finally, we want to note that it might be profitable not only to study localizations, but also cellularizations of simplicial presheaves. In particular the $\mathbb{G}_m$ -cellular spaces of [DI05] seem to be a very interesting subject to study, there is a rich supply of complicated problems about higher homotopy group sheaves generalizing K-theory yet to be discovered.

Chapter 5

Geometry and Fibre Sequences

As we have seen in the previous chapter, the locality criterion Theorem 4.3.10 can only be applied in situations where we have quite good control on the classifying space of the fibre resp. on the base of the fibration. Therefore, the situation in which we can apply the above theorem is that of a fibre sequence of schemes $F \rightarrow E \rightarrow B$, where e.g. the base is local. Such fibre sequences tend to yield only rather simple information: An exact sequence for π_0 and π_1 and isomorphisms for the higher homotopy group sheaves, cf. Corollary 4.4.12 or Proposition 4.4.16. Most of the more interesting fibre sequences involve spaces with nontrivial, i.e. non-discrete, homotopy types, where it is more difficult to apply Theorem 4.3.10.

In this chapter, we will discuss geometric situations in which we can construct fibre sequences between non-discrete spaces. What we are interested in are conditions under which the spaces

$$\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(X)(U) := \mathrm{Hom}_{\mathrm{Sm}_S}(U \times \Delta_{\mathbb{A}^1}^{\bullet}, X).$$

are as close as possible to being fibrant. Since the sites we consider are given by cd-structures, we can use Theorem 2.2.9 to reduce to the question when a simplicial presheaf $\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(X)$ has the Brown-Gersten property. This is however only rarely the case, as an example of Morel shows: GL_n -torsors over $\mathbb{P}^1$ are not the same as GL_n -torsors over $\mathbb{P}^1 \times \mathbb{A}^1$ and therefore $\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(Gr_n)$ can not have the Brown-Gersten property on the category of all smooth schemes. It seems likely that only discrete simplicial sheaves have the Brown-Gersten property over *all* smooth schemes over S . The next best approximation to fibrancy is the *affine Brown-Gersten property*, where one requires only the Brown-Gersten property for the class of affine schemes over S .

This affine Brown-Gersten property has been investigated by Morel in [Mor07], and is similar to the h-principle in complex analysis. In particular, the affine Brown-Gersten property already follows, if we can patch together morphisms defined on a covering of an affine scheme, which are only *homotopic* over the intersections. Our main interest will be in proving this sort of patching theorems.

In [Mor07] it is shown that the simplicial presheaf $\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(GL_n)$ has the affine Brown-Gersten property over a field k , and the main result of this chapter is a generalization of this assertion to general Chevalley groups over an excellent Dedekind ring. Having this affine Brown-Gersten property for Chevalley groups, we sketch how to extend it to suitable homogeneous spaces.

We will also be interested in the question when a morphism $f : X \rightarrow Y$ of schemes with fibre F yields a fibre sequence of simplicial presheaves

$$\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(F) \rightarrow \mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(X) \rightarrow \mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(Y).$$

In the special case when the morphism is a fibre bundle, i.e. comes from a G -torsor over Y via an action $G \times F \rightarrow F$, this question can be reduced to the Serre problem for G -torsors. This implies in particular that every projective space bundle over a field of characteristic zero induces a fibre sequence of simplicial presheaves

$$\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(\mathbb{P}^n) \rightarrow \mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(E) \rightarrow \mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(B).$$

In this chapter, we will exclusively work in the model category $\Delta^{\mathrm{op}}\mathrm{PShv}(\mathrm{Sm}_S)$, where we usually assume that the base S is the spectrum of an excellent Dedekind ring. We use the Nisnevich topology, whence the points of the topos can be identified with henselian local rings smooth and of finite type over S . The methods used in this chapter are very much inspired by Morel's preprint [Mor07].

The chapter is structured as follows: In Section 5.1 we show that patching of morphisms suffices to get the affine Brown-Gersten property for a simplicial presheaf $\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(X)$. This is an argument concerning only simplicial sets. In Section 5.2, we explain the main steps in the proof of patching theorems for morphisms. Then Section 5.3 proves the patching theorem for morphisms into a Chevalley group over an excellent Dedekind ring. Finally, in Section 5.4 we apply the results to give criteria which imply that principal fibre bundles under Chevalley groups induce $\mathbb{A}^1$ -local fibre sequences.

In the following paragraph, we shortly describe the relation between the affine Brown-Gersten property and the complex analytic h-principle.

Homotopy Principles in Complex Analysis: Gromov's homotopy-principle [Gro89] is a powerful way to relate continuous morphisms between complex analytic manifolds to holomorphic maps: Maps between complex analytic manifolds satisfy the h-principle if every continuous map $f : X \rightarrow Y$ is homotopic to a holomorphic one. A stronger statement is that there is a weak equivalence between the function spaces $\mathrm{map}_{\mathrm{hol}}(X, Y)$ of holomorphic resp. $\mathrm{map}_{\mathrm{cont}}(X, Y)$ of continuous maps. In his paper [Gro89], Gromov gave conditions for complex manifolds to satisfy the h-principle. Roughly, for maps $f : X \rightarrow Y$ to satisfy the h-principle, it is necessary that there are sufficiently many holomorphic maps $X \rightarrow \mathbb{C}$ and sufficiently many holomorphic maps $\mathbb{C} \rightarrow Y$. For the former, it suffices to assume that X is a Stein manifold. For the latter, Gromov defined the notion of a spray, which is a geometric object providing enough holomorphic maps $\mathbb{C} \rightarrow Y$. In such a situation, it is then possible to approximate any continuous map $f : X \rightarrow Y$ locally by holomorphic maps $g_i : X_i \rightarrow Y$, such that $f|_{X_i}$ is homotopic to g_i . The spray then allows to modify these g_i within their homotopy classes such that they can be patched to a global holomorphic map $g : X \rightarrow Y$.

In a series of papers [Lár03, Lár04], Lárusson has investigated formulations of Gromov's Oka principle in a model category framework. He defines model structures on the category of simplicial sheaves on the site of Stein manifolds with holomorphic maps, such that a complex manifold X viewed as a representable simplicial sheaf satisfies the homotopy principle exactly if it is quasi-fibrant.

In this chapter, we are going to do something similar in the situation of $\mathbb{A}^1$ -homotopy, at least for group schemes over Dedekind rings. The affine Brown-Gersten property, which will play a prominent role in Section 5.1, is almost exactly the condition for quasi-fibrancy, cf. Theorem 2.2.9. However, the direct translation of the notion of a spray does not work for $\mathbb{A}^1$ -homotopy: The original spray condition from [Gro89] is a condition about infinitesimal liftings, and in the analytic topology such liftings suffice to produce homotopies. For $\mathbb{A}^1$ -homotopies, this is not enough, as a morphism $\mathbb{A}^n \rightarrow X$ can not be constructed simply by infinitesimal liftings. As an example, the fact that the elementary subgroups form a spray for the Chevalley groups is related to the K_1 -analogue of the Serre conjecture on projective modules. By contrast, any complex Lie group as a complex manifold has a spray given by the exponential map. Therefore, sprays in $\mathbb{A}^1$ -homotopy are more complicated and more global objects than in complex analysis, the reason being that the homotopies given by morphisms of polynomial rings are more complicated and more global than holomorphic homotopies defined on a small analytic disc.

Finally, we want to note that homotopy principles should work for a suitable homotopy theory of rigid-analytic varieties. Using as interval the Tate algebra $T\langle\zeta\rangle$ instead of a polynomial ring as the interval allows to construct homotopies by infinitesimal extensions, so a direct translation of Gromov's work should yield a rigid-analytic homotopy principle. There are some results in the literature on rigid-analytic and formal patching that are reminiscent of results related to the Oka principle.

5.1 Patching and the Affine Brown-Gersten Property

As mentioned above, the Brown-Gersten property for a bounded, complete and regular cd-structure is a quite powerful tool to compute homotopy groups: If a simplicial sheaf has the Brown-Gersten property, then the homotopy groups of the simplicial sets of sections do not change in the course of a fibrant replacement. However, it is usually not the case that $\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(X)$ has the Brown-Gersten property for a scheme X . A good replacement seems to be the affine Brown-Gersten property, a weaker property which was introduced in [Mor07]. The restriction to affine smooth schemes is similar to the restriction to Stein manifolds in the complex-analytic h-principle.

DEFINITION 5.1.1 (AFFINE BROWN-GERSTEN PROPERTY) *Let $\mathcal{F}$ be a simplicial presheaf on Sm_S with $S = \mathrm{Spec} R$. Then $\mathcal{F}$ is said to have the affine Brown-Gersten property for the Nisnevich topology if for any regular R -algebra A , any étale A -algebra B and each $f \in A$ such that $A/f \rightarrow B/f$ is an isomorphism, the following diagram of simplicial sets is a homotopy pullback:*

$$\begin{array}{ccc} \mathcal{F}(\mathrm{Spec} A) & \longrightarrow & \mathcal{F}(\mathrm{Spec} A_f) \\ \downarrow & & \downarrow \\ \mathcal{F}(\mathrm{Spec} B) & \longrightarrow & \mathcal{F}(\mathrm{Spec} B_f). \end{array}$$

Note that the above conditions on A and B are just the conditions for an elementary Nisnevich covering as in Example 2.2.5, i.e. $\mathrm{Spec} B \rightarrow \mathrm{Spec} A$ is an

étale morphism which is an isomorphism on the complement of the open subset $\mathrm{Spec} A_f \hookrightarrow \mathrm{Spec} A$.

There are two things that need to be investigated. First, we need conditions when a simplicial sheaf of the form $\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(X)$ has the affine Brown-Gersten property. The second thing we need to know is if the affine Brown-Gersten property really implies that the homotopy groups over affine schemes do not change after a fibrant replacement.

The first question has partially been answered by Morel in [Mor07, Lemma 1.3.5] for the case of sheaves of simplicial groups over a field. In view of the analogy with the complex-analytic h-principle, it seems reasonable to try to generalize this result to simplicial sets. Moreover, one would like to have an easy-to-check geometric criterion for the diagrams in the affine Brown-Gersten property to be homotopy pullbacks. We provide one possible such result. Assuming that a scheme X satisfies a property we call *analytic patching*, we show that $\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(X)$ has the affine Brown-Gersten property. The property of analytic patching allows to glue together morphisms on an elementary Nisnevich covering which are only homotopic over the intersection. The name analytic patching is due to the fact that the corresponding patching result for K-theory is called “patching along analytic isomorphisms”.

DEFINITION 5.1.2 *An affine analytic patching situation over R consists of a regular R -algebra A , an étale A -algebra B and an $f \in A$ such that $A/f \rightarrow B/f$ is an isomorphism.*

Let X be a sheaf. We say that X satisfies the analytic patching property if for any patching situation as above, and sections $\alpha \in X(\mathrm{Spec} A_f[t_1, \dots, t_n])$ and $\beta \in X(\mathrm{Spec} B[t_1, \dots, t_n])$ such that there is $H \in X(\mathrm{Spec} B_f[t_1, \dots, t_n][t])$ with $H(t=0) = \alpha$ and $H(t=1) = \beta$ in $X(\mathrm{Spec} B_f[t_1, \dots, t_n])$, there exists a section

$$\gamma \in X(\mathrm{Spec} A[t_1, \dots, t_n])$$

and homotopies $\gamma \sim \alpha$ resp. $\gamma \sim \beta$ in the sense that there exists

$$H_\alpha \in X(\mathrm{Spec} A_f[t_1, \dots, t_n][t])$$

with $H_\alpha(t=0) = \gamma$ and $H_\alpha(t=1) = \alpha$ and similarly for β . We also require that $H = H_\beta \circ H_\alpha$, i.e. the composition of the two homotopies is the original homotopy.

This patching along elementary homotopies $(\mathrm{Spec} B_f) \times \mathbb{A}^1 \rightarrow X$ is already sufficient to get the affine Brown-Gersten property for $\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(X)$, as we will see in the next result. The above definition is however only suitable for the case that $\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(X)(A)$ is fibrant for any affine scheme. However, since our main interest is in group schemes, this suffices, since in this case $\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(G)(U)$ is fibrant by a result of Moore, cf. [GJ99, Lemma I.3.4].

PROPOSITION 5.1.3 *Let X be a presheaf. Assume that X has the analytic patching property and that $\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(X)(U)$ is a fibrant simplicial set for any smooth affine scheme U . Then $\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(X)$ has the affine Brown-Gersten property.*

PROOF: Let the following analytic patching diagram be given:

$$\begin{array}{ccc}
A & \xrightarrow{i} & B \\
h \downarrow & & \downarrow j \\
A_f & \xrightarrow{k} & B_f.
\end{array}$$

We introduce shortcut notation $X(A)$ for $\text{Sing}_{\bullet}^{\mathbb{A}^1}(X)(\text{Spec } A)$. Using this notation, we want to show that the following square is homotopy cartesian:

$$\begin{array}{ccc}
X(A) & \longrightarrow & X(B) \\
\downarrow & & \downarrow p_1 \\
X(A_f) & \xrightarrow{p_2} & X(B_f).
\end{array}$$

For this, it suffices to show that for one factorization of $X(B) \rightarrow X(B_f)$ as a trivial cofibration $i : X(B) \rightarrow Y$ and a fibration $p : Y \rightarrow X(B_f)$, the induced morphism $X(A) \rightarrow X(A_f) \times_{X(B_f)} Y$ is a weak equivalence. We factor this morphism $X(A) \rightarrow X(A_f) \times_{X(B_f)} Y$ into a trivial cofibration $j : X(A) \rightarrow Z$ and a fibration $q : Z \rightarrow X(A_f) \times_{X(B_f)} Y$. It now suffices to show that q is actually a trivial fibration, i.e. it satisfies the right lifting property w.r.t. all morphisms $\partial\Delta^n \rightarrow \Delta^n$.

(i) Consider the following diagram:

$$\begin{array}{ccc}
\partial\Delta^n & \xrightarrow{\rho} & Z \\
\downarrow & & \downarrow q \\
\Delta^n & \xrightarrow{\sigma} & X(A_f) \times_{X(B_f)} Y
\end{array} \tag{5.1}$$

Given this lifting problem, it suffices to show that a solution exists after possible modifying the simplex σ to a homotopic n -simplex σ' without changing the boundary: The homotopy $\tau : \sigma \sim \sigma'$ is an $(n+1)$ -simplex in $X(A_f) \times_{X(B_f)} Y$, and if σ' can be lifted to a simplex θ in Z , then we have a horn Λ_n^{n+1} whose only non-degenerate simplex is the $(n+1)$ -st face given by θ . Since q is a fibration, there also exists a lift of σ , which is the missing face of the lift of τ .

We will show in (iii) that this allows us to assume that the simplex σ in the base is a patching situation as in Definition 5.1.2. This means that the n -simplex is given by an n -simplex $\sigma_A \in X(A_f)$, an n -simplex σ_B in $X(B)$ and an $(n+1)$ -simplex σ_H providing a homotopy $\sigma_A \sim \sigma_B$ in $X(B_f)$.

Then we use that σ comes from a patching situation. We conclude that there exists a simplex $\tau : \Delta^n \rightarrow Z$ such that τ is homotopic to σ_A in $X(A_f)$, and to σ_B in $X(B_f)$. We call the respective homotopies H_A resp. H_B . Note that the requirement that these two homotopies should compose to σ_H in $X(B_f)$ implies that $H_A|_{\partial\sigma_A} = H_B|_{\partial\sigma_B}$ in $X(B_f)$. Therefore we can also glue these homotopies to something global, i.e. we have a homotopy from $\partial\tau$ to ρ . Since q is a fibration, the same argument as above implies that τ can be moved to a lift of σ whose boundary agrees with ρ .

(ii) We still need to show that any n -simplex in the base $X(A_f) \times_{X(B_f)} Y$ is homotopic to a patching situation. For this, we do an induction on the construction of Y .

We first recall the small objects argument for simplicial sets to obtain a description of the simplices in $X(A_f) \times_{X(B_f)} Y$. Consider a morphism $f : K \rightarrow L$ of simplicial sets. We want to factor it into a trivial cofibration and a fibration. Recall that Λ_k^n is the k -th horn of the n -simplex, i.e. the subcomplex of Δ^n obtained by taking out the k -th face. The inclusion $\Lambda_k^n \rightarrow \Delta^n$ is a monomorphism and hence a cofibration, which in addition is a weak equivalence, since both spaces are contractible. The factorization $K \rightarrow \tilde{K} \rightarrow L$ is obtained as

$$\tilde{K} = \varinjlim K_i,$$

where K_{i+1} is obtained from K_i as the pushout

$$\coprod_D \Delta^{n_D} \longleftarrow \coprod_D \Lambda_{k_D}^{n_D} \longrightarrow K_i.$$

The indexing above is over all commutative diagrams D of the form

$$\begin{array}{ccc} \Lambda_{k_D}^{n_D} & \longrightarrow & K_i \\ \downarrow & & \downarrow \\ \Delta^{n_D} & \longrightarrow & L \end{array}$$

The small object argument then states that any lifting problem in $\tilde{K}$ is given by a lifting problem in K_i for some i , and by construction can be solved in K_{i+1} . Hence $\tilde{K} \rightarrow Y$ is a fibration.

Now the spaces we deal with are constructed exactly this way, i.e. Y is given by applying the above construction to the morphism $X(B) \rightarrow X(B_f)$. We denote (somewhat abusively) by $\text{Ex}^n X(B)$ the successive stages in the small object argument.

(iii) From the above, we find that a general n -simplex in

$$X(A_f) \times_{X(B_f)} \text{Ex}^{m+1} X(B)$$

is given by an n -simplex $\sigma_A \in X(A_f)$, a Λ_k^{n+1} -horn in $\text{Ex}^m X(B)$ and an $(n+1)$ -simplex τ in $X(B_f)$ whose boundary consists of the images of the n -simplices from $X(A_f)$ and $\text{Ex}^m X(B)$. Then we want to show that this simplex is homotopic to one coming from a patching situation. This is where we need that $X(B)$ is already fibrant. Indeed, for the Λ_k^{n+1} -horn in $\text{Ex}^m X(B)$, there exists an $(n+1)$ -simplex σ_B in $\text{Ex}^m X(B)$. This $(n+1)$ -simplex will not necessarily map to τ , but it will have the same boundary horn. Therefore, since $X(B_f)$ is also fibrant, τ and σ_B will be homotopic in $X(B_f)$. The k -th face of σ_B is an n -simplex in $\text{Ex}^m(B)$, which is homotopic in $X(B_f)$ to σ_A . Therefore, any simplex in $X(A_f) \times_{X(B_f)} \text{Ex}^{m+1} X(B)$ is homotopic to a simplex given by a patching situation. $\blacksquare$

The second question, i.e. if the homotopy groups over an affine scheme do not change after a fibrant replacement, is a little harder to answer. In some special cases, [Mor07, Theorem A.4.2] can be used to decide this question. The problem is that the affine schemes do not form a BG-class, cf. [MV99, Definition 3.1.12]. Moreover, restricting the usual cd-structures to the category of smooth affine schemes over a base scheme seems to destroy boundedness of the cd-structures. Therefore, there is no abstract way of dealing with the second problem.

In [Mor07], the affine replacement of a simplicial sheaf $\mathcal{F}$ is defined by

$$\mathcal{F}^{af}(X) := \operatorname{holim}_{Y \rightarrow X} \operatorname{Ex}^\infty \mathcal{F}(Y),$$

with the homotopy limit running over all smooth affine (over the base) schemes Y with a morphism to X . For the Zariski topology, the answer is rather complete, and will be given in [Mor07, Theorem A.2.2].

PROPOSITION 5.1.4 *Let X be a scheme over a Dedekind ring S which has the analytic patching property. Then the affine replacement of $\operatorname{Sing}_\bullet^{\mathbb{A}^1}(X)$ has the Brown-Gersten property for the Zariski topology.*

The proof basically uses the Jouanolou-trick to extend results from affine schemes to quasi-projective schemes. The quasi-projective schemes then form a BG-class, and applying [MV99, Proposition 3.1.16] one obtains the required weak equivalences $\operatorname{Sing}_\bullet^{\mathbb{A}^1}(X)(U) \rightarrow \operatorname{Ex}^\infty \operatorname{Sing}_\bullet^{\mathbb{A}^1}(X)(U)$ for any affine scheme. Here, the term Ex^∞ denotes a fibrant replacement functor in the category of simplicial sheaves on the Zariski site. This argument extends to the category of smooth schemes over a Dedekind ring.

For the Nisnevich topology, we need more control on the geometry of covering situations, and it is not clear to me how to extend the proof of [Mor07, Theorem A.4.2] to the case of Dedekind rings. However, it seems plausible to expect a Dedekind ring version of this theorem to hold.

5.2 Patching and Local-Global Principles

In this section, we prepare the main ingredients in the proof of the patching result for Chevalley groups over Dedekind rings. We review the necessary literature on local-global principles.

Axiom Q: The key step in the proof of the local-global principle seems to be what Bass, Connell and Wright described as “Axiom Q” in [BCW76, Paragraph 1.1].

We first need to introduce the relevant notation. Let R be a commutative ring, and let G be a functor from R -algebras to groups. For R -algebras A and B , and an R -algebra homomorphism $f : A[t] \rightarrow B : t \mapsto s$, we denote $g \in G(A[t])$ by $g(t)$ and its image under $G(f)$ in $G(B)$ by $g(s)$. Applied to $B = A$, $s \in A$ and $A[t] \rightarrow A[t] : t \mapsto st$, this defines $g(st)$. We denote

$$G(A[t], (t)) = \{g(t) \in G(A[t]) \mid u(0) = 1\},$$

which is the kernel of the evaluation at zero morphism $G(A[t]) \rightarrow G(A)$.

DEFINITION 5.2.1 *Let R be any commutative ring, and let G be a functor from R -algebras to groups. We say that G satisfies Axiom Q if for any given R -algebra A , an element $s \in A$ and an element $u(t)$ in $G(A_s[t], (t))$ there exists an integer $n \geq 0$ and an element $v(t) \in G(A[t], (t))$ such that $u(s^n t) = v(t)_s$.*

There are many functors which satisfy Axiom Q. We list the ones most important to us.

EXAMPLE 5.2.2 Elementary subgroups $E(\Phi) \hookrightarrow G(\Phi)$ satisfy Axiom Q. In other words, let Φ be an irreducible and reduced root system with $\text{rk } \Phi \geq 2$. For any commutative ring R , any $a \in R$, and $\sigma \in E(\Phi, R_a[t], (t))$, there exists $\tau \in E(\Phi, R[t], (t))$ such that $\tau_a = \sigma(a^p t)$ for some integer p .

This is proven in [Abe83, Lemma 1.11]. It is a straightforward generalization of the case GL_n originally proven in [Sus77] and described in detail in [Lam06, Chapter VI.1]. $\square$

EXAMPLE 5.2.3 It is also stated in [BCW76, Remark 1.8 resp. Proposition 1.12] that automorphism group functors of algebras satisfy Axiom Q. This implies Quillen's local-global principle for projective as well as quadratic modules. $\square$

REMARK 5.2.4 *The fact emphasized in [BRK05, Theorem 3.1] that the local-global principle is related to normality of subgroup functors follows already from [BCW76, Remark 1.4]. Let $G(\Phi)$ be a Chevalley group, and let $E(\Phi)$ be the elementary subgroup. If E is normal in G , then we get Axiom Q for the quotient $K_1(\Phi, -) = G(\Phi, -)/E(\Phi, -)$. An element in K_1 which is trivial at all the maximal ideals has to be trivial, since Axiom Q for K_1 implies the local-global principle.*

Local-Global Principle: The patching technique for projective modules resp. elementary subgroups is a method to deduce structure theorems like the above factorization of matrices, cf. Proposition 5.3.2, for a ring R from the corresponding structure result for the localizations $R_{\mathfrak{p}}$ for prime ideals $\mathfrak{p}$ resp. only the maximal ideals $\mathfrak{m}$. Such a local-global principle was first applied in the solution of the Serre problem by Quillen [Qui76] resp. Suslin [Sus77]. There the goal was to deduce that a projective module P over $R[t_1, \dots, t_n]$ is extended from R iff P is free over $R_{\mathfrak{m}}[t_1, \dots, t_n]$ for every maximal ideal $\mathfrak{m}$.

Later, various versions of the local-global principle have been applied to problems such as describing the structure of locally polynomial algebras [BCW76] or rational triviality of torsors [CTO92]. For an overview of applications of local-global principles see [BRK05].

The following form of the local-global principle is given in [Abe83, Theorem 1.15], and generalizes the corresponding result of Suslin for the case GL_n . See also the exposition in [Lam06, Chapter VI.1].

THEOREM 5.2.5 (LOCAL-GLOBAL PRINCIPLE) *Let R be a commutative ring. For any element $\sigma \in G(\Phi, R[t], (t))$, the following assertions hold:*

(A_n) *The Quillen set $Q(\sigma) = \{s \in R \mid \sigma_s \in E(\Phi, R_s[t])\}$ is an ideal in R .*

(B_n) *If $\sigma_{\mathfrak{m}} \in E(\Phi, R_{\mathfrak{m}}[t])$ for every maximal ideal $\mathfrak{m}$ in R , then $\sigma \in E(\Phi, R[t])$.*

COROLLARY 5.2.6 *Let $\sigma \in G(\Phi, R[t])$ with $\text{rk } \Phi \geq 2$, and let $\sigma_{\mathfrak{m}} \in G(\Phi, R_{\mathfrak{m}}) \cdot E(\Phi, R_{\mathfrak{m}}[t])$ for every maximal ideal $\mathfrak{m}$ of R , then $\sigma \in G(\Phi, R) \cdot E(\Phi, R[t])$.*

The key step in the proof is exactly Axiom Q described above, cf. [BRK05]. The relation between subgroup normality and the local-global principle sketched there suggests that the above form of the local-global principle does not generalize to SL_2 . It also seems that the methods used in the proofs of the normality of the

elementary subgroups for general Chevalley groups over commutative rings (with rank of the root system ≥ 2) given in [HV03] have some relation to Axiom Q and the local-global principle.

Nisnevich Neighbourhoods and Lindel's Theorem: There are some structure theorems for local rings that are needed in order to prove affine Brown-Gersten type properties for $\text{Sing}_{\bullet}^{\mathbb{A}^1}(X)$. This paragraph describes these geometric structure results.

By Lindel's theorem we refer to a characterization of smooth local algebras as étale neighbourhoods of very special smooth local algebras, namely localizations of $R[t_1, \dots, t_n]$ at a maximal ideal. This theorem makes it possible to extend statements about the K_0 - resp. the K_1 -form of the Serre problem from localizations of polynomial algebras to general smooth local rings. In its original form the theorem appeared in [Lin81]:

THEOREM 5.2.7 *Let L be a field, let C be a finitely generated L -algebra with $d = \dim C \geq 1$, and let $\mathfrak{m}$ be a maximal ideal of C . Assume that $C_{\mathfrak{m}}$ is regular and that the residue field $K = C/\mathfrak{m}$ is a simple separable extension of L . Then there exist elements $x_1, \dots, x_t \in C$ with the following properties:*

- (i) $C = L[x_1, \dots, x_t]$ and $\mathfrak{m} = (f(x_1), x_2, \dots, x_t)$ with $f(x_1) \notin \mathfrak{m}^2$, where f is the minimal polynomial of the image of x_1 in K .
- (ii) $C_{\mathfrak{m}}$ is an étale neighbourhood of $D_{\mathfrak{n}}$ where $D = L[x_1, \dots, x_d]$ and $\mathfrak{n} = \mathfrak{m} \cap D$.

This result has been extended to the case of excellent discrete valuation rings by Dutta in [Dut99, Theorem 1.3]. A similar form of the extension to discrete valuation rings can be found already in [Pop89, Proposition 2.1].

THEOREM 5.2.8 *Let $(A, \mathfrak{m}, K)$ be a regular local ring of dimension $d+1$, essentially of finite type over an excellent discrete valuation ring (V, π) such that $A/\mathfrak{m}$ is separably generated over $V/\pi V$. Let $a \in \mathfrak{m}^2$ be such that $a \notin \pi A$. Then there exists a regular local ring $(B, \mathfrak{n}, K) \hookrightarrow (A, \mathfrak{m}, K)$ with the following properties:*

- (i) B is the localization of a polynomial ring $W[x_1, \dots, x_d]$ at a maximal ideal of the form $(\pi, f(x_1), x_2, \dots, x_d)$ where f is a monic irreducible polynomial in $W[x_1]$ and $(W, \pi) \subseteq A$ is an excellent discrete valuation ring. Moreover A is an étale neighbourhood of B .
- (ii) There exists an element $h \in B \cap aA$ such that $B/hB \rightarrow A/aA$ is an isomorphism, and furthermore $hA = aA$.

REMARK 5.2.9 *It does not seem reasonable to expect anything like this over a general base. This might be one of the geometric motivations behind the failure of the $\mathbb{A}^1$ -connectivity conjecture [Ayo06]. The above results nevertheless show that the $\mathbb{A}^1$ -localization behaviour over excellent discrete valuation rings is the same as over a field.*

Note that the conditions in the above theorem hold for local rings of schemes which are smooth and essentially of finite type over the base discrete valuation ring V .

The next thing we need is control on the behaviour of the structures we are interested in with respect to Nisnevich neighbourhoods. The prototypical results from the literature are [Lin81, Proposition 1] for the case of projective modules, and [Vor81, Lemma 2.4].

The following patching theorem for the elementary subgroups in the *Nisnevich topology* is the generalization of [Vor81, Lemma 2.4] to arbitrary Chevalley groups, and can be found in [Abe83, Lemma 3.7].

PROPOSITION 5.2.10 *Let $A \hookrightarrow B$ be a subring, $f \in B$ not a nilpotent element.*

- (i) *If $Bf + A = B$ then there exists for every $\alpha \in E(\Phi, B_f)$ some $\beta \in E(\Phi, A_f)$ and $\gamma \in E(\Phi, B)$ such that $\alpha = \gamma_f \beta$.*
- (ii) *If moreover $Bf \cap A = Af$ and f is not a zero-divisor in B then there exists for every $\alpha \in G(\Phi, B)$ with $\alpha_f \in E(\Phi, B_f)$ some $\beta \in G(\Phi, A)$ and $\gamma \in E(\Phi, B)$ such that $\alpha = \gamma \beta$.*

It seems reasonable to expect that there is also a version of Axiom Q which also takes care of the fact that $A \hookrightarrow B$ is not necessarily a localization, but more generally a localization of an étale A -algebra. Such an axiom would imply the above proposition exactly like Definition 5.2.1 implies the patching for the Zariski topology.

Note also that the above proposition already implies that the elementary subgroups have patching for the Nisnevich topology, and therefore $\text{Sing}_{\bullet}^{\mathbb{A}^1}(E(\Phi))$ has the affine Brown-Gersten property. To get the affine Brown-Gersten property for Chevalley groups, we still need to control the difference between the Chevalley groups and their elementary subgroups, i.e. the unstable K_1 -groups.

5.3 Example: Chevalley Groups and Torsors

Using the local-global principles for the elementary subgroups from the last chapter, we will show in this section how to prove analytic patching and therefore the affine Brown-Gersten property for $\text{Sing}_{\bullet}^{\mathbb{A}^1}(G(\Phi))$. Here, $G(\Phi)$ denotes the universal Chevalley group scheme for the root system Φ over an excellent Dedekind ring. This result follows from a combination of the local-global principle Proposition 5.2.10 together with the fact that the difference between $G(\Phi)$ and its elementary subgroup $E(\Phi)$, i.e. the unstable $K_1(\Phi)$ -functor, has $\mathbb{A}^1$ -invariance.

We start by recalling the definition of Chevalley groups.

Chevalley Groups: Our main reference for the construction of Chevalley groups is [Hum78, Chapter VII].

We fix a semisimple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$. Fixing further a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ yields the Cartan decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_{\alpha}.$$

Here Φ is a *root system* of $\mathfrak{g}$ with respect to $\mathfrak{h}$. We also fix a base $\Delta = \{\alpha_1, \dots, \alpha_l\}$ of *simple roots* for Φ , meaning that every $\alpha \in \Phi$ is either a positive or negative linear combination of the α_i . A *Chevalley basis* $\{h_i \mid i = 1, \dots, r\} \cup \{x_\alpha \mid \alpha \in \Phi\}$ for $\mathfrak{g}$, with $r = \dim_{\mathbb{C}} \mathfrak{h}$, is a basis of $\mathfrak{h}$ resp. of the root eigenspaces $\mathfrak{g}_\alpha$, such that the structure constants of the Lie algebra $\mathfrak{g}$ lie in $\mathbb{Z}$. This makes it possible to define the Lie algebra already over $\mathbb{Z}$.

For a faithful representation $\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ and an admissible lattice M for this representation, we can consider $GL(M \otimes_{\mathbb{Z}} R)$ for any commutative ring R . For any $t \in R$ and $\alpha \in \Phi$, we can define an element $x_\alpha(t) \in GL(M \otimes_{\mathbb{Z}} R)$, which is the specialization of

$$\sum_{k=0}^{\infty} \frac{(T\rho(x_\alpha))^k}{k!} \in GL(M \otimes_{\mathbb{Z}} \mathbb{Z}[T])$$

along $\mathbb{Z}[T] \rightarrow R : T \mapsto t$. In case R is a field, the Chevalley group associated to the above data is the subgroup $G_\rho(k)$ of $GL(M \otimes_{\mathbb{Z}} k)$ generated by the $x_\alpha(t)$. The algebra of functions on this algebraic group $G_\rho(k)$ is already defined over $\mathbb{Z}$, yielding a group scheme G_ρ over $\mathbb{Z}$. For a general commutative ring R , the Chevalley group $G_\rho(R)$ is then defined as the group of R -valued points of the $\mathbb{Z}$ -group scheme G_ρ . We are interested in Chevalley groups of *universal type*, where the admissible lattice is the lattice of all fundamental weights in V . Up to isomorphism, the group $G_\rho(R)$ depends only on the admissible lattice and the root system, so we denote the universal Chevalley group over R for a root system Φ by $G(\Phi, R)$.

The elements $x_\alpha(t)$ are called *elementary matrices* of $G_\rho(k)$. The subgroups generated by the elementary matrices are called *elementary subgroups*, and are denoted by $E(\Phi, R)$. Note also, that the assignment $R \rightarrow G_\rho(R) : t \mapsto x_\alpha(t)$ gives a morphism $\mathbb{A}^1 \rightarrow G_\rho(R)$, hence elementary matrices yield naive $\mathbb{A}^1$ -homotopies.

REMARK 5.3.1 *The Bruhat decomposition for Chevalley groups over a field implies that $G(\Phi, R) = E(\Phi, R)$, i.e. $G(\Phi, R)$ is generated by its unipotent elements. Gromov's statement in [Gro89] is that groups which are generated by their unipotent elements have a spray, which is exactly given by the elementary subgroups. This however does not work over more general schemes, and the failure of the group $G(\Phi, A)$ to be generated by unipotent elements is exactly determined by the quotient $G(\Phi, A)/E(\Phi, A)$. In the cases we will deal with, this is an unstable K_1 -group.*

Factorization of Matrices: We are now giving the proof that $\text{Sing}_{\bullet}^{\mathbb{A}^1}(G(\Phi))$ for a Chevalley group $G(\Phi)$ of universal type has the affine Brown-Gersten property. The most important references we use are [Lin81, Sus77, Vor81] resp. [Abe83]. Having described the local-global principles for $\text{Sing}_{\bullet}^{\mathbb{A}^1}(G(\Phi))$ in the previous section, all we need is a factorization of matrices, showing that all matrices σ over $R[t_1, \dots, t_n]$ whose constant part satisfies $\sigma(0, \dots, 0) = I$ are indeed elementary.

The first step in establishing this factorization result is the case of the base scheme, which in our setting is assumed to be a Dedekind ring.

PROPOSITION 5.3.2 *For any reduced irreducible root system Φ of rank ≥ 2 , and any Dedekind ring S , we have the following factorization:*

$$G(\Phi, S[t_1, \dots, t_n]) = G(\Phi, S) \cdot E(\Phi, S[t_1, \dots, t_n]).$$

PROOF: The proof structure is the same as [Abe83, Theorem 3.5] resp. [GMV91, Theorem 1.2]: One uses the local-global principle Theorem 5.2.5 to reduce to the assertion $G(\Phi, S[t_1, \dots, t_n]) = E(\Phi, S[t_1, \dots, t_n])$ for any discrete valuation ring S . This assertion is then proved by induction, the case $n = 1$ is a computation in the Steinberg group which we will not reproduce. The induction proof of [GMV91, Theorem 1.2] uses localizations at monic polynomials which goes through directly. The only additional ingredient needed is a K_1 -version of Horrocks theorem, which for an arbitrary root system is proved in [Abe83, Proposition 3.3]. The condition (P') in the work of Abe holds for general Chevalley groups as the work on A_2 -structure proofs shows, cf. [GNV07]. ■

REMARK 5.3.3 *The computations in the proof above are done via the Steinberg relations. The group generated by the elementary subgroup symbols with the Steinberg relations is $K_2(R)$, which is exactly the fundamental group*

$$K_2(R) = \pi_1(\mathrm{Sing}_\bullet^{\mathbb{A}^1}(G(\Phi), I)(R)).$$

It seems that to prove the affine Brown-Gersten property for SL_2 resp. $\mathbb{P}^1$, it is necessary to generalize the computations with Steinberg relations to elements in the group K_2^{MW} .

The following result extends the factorization of matrices from rings to arbitrary localizations, and is therefore some kind of converse of the local-global principle. It has been proven in [Vor81, Lemma 2.1] for the case of GL_n , and generalized in [Abe83, Lemma 3.6].

PROPOSITION 5.3.4 *Let A be a commutative ring and S be a multiplicative subset. If $G(A[t_1, \dots, t_n]) = G(A)E(A[t_1, \dots, t_n])$, then also $G(A_S[t_1, \dots, t_n]) = G(A_S)E(A_S[t_1, \dots, t_n])$.*

Using the local-global principle, we obtain the above matrix factorization for general smooth finite type S -algebras. The proof is basically the one of [Vor81, Theorem 3.3].

PROPOSITION 5.3.5 *Let Φ be a reduced and irreducible root system of rank $\mathrm{rk} \Phi > 1$, let R be an excellent Dedekind ring, and let B be a smooth R -algebra essentially of finite type. If $\sigma(t_1, \dots, t_n) \in G(\Phi, B[t_1, \dots, t_n])$ is such that $\sigma(0) = I$, then $\sigma(t_1, \dots, t_n) \in E(\Phi, B[t_1, \dots, t_n])$.*

PROOF: The proof is by induction on the Krull dimension of B , where the base case of the induction is given by Proposition 5.3.2.

By the local-global principle Theorem 5.2.5, we can assume that R and B are local rings, in particular R is a field or a discrete valuation ring. We concentrate on the dvr case. By Theorem 5.2.8, B is an étale neighbourhood of a localization of a polynomial algebra over a discrete valuation ring, i.e. there is an étale discrete

valuation ring extension $R \hookrightarrow W$, a localization A of $W[t_1, \dots, t_{\dim B-1}]$ at a maximal ideal, and an element $f \in A$ such that $Bf + A = B$ and $Bf \cap A = Af$. We are therefore exactly in the situation of Proposition 5.2.10, as f is not a zero-divisor in $B[t_1, \dots, t_n]$ since B is regular and local. Since B is local, we also have $\dim B_f < \dim B$, and by induction hypothesis, $\sigma_f(t_1, \dots, t_n) \in E(\Phi, B_f[t_1, \dots, t_n])$. By part (ii) of Proposition 5.2.10, there is a factorization

$$\sigma(t_1, \dots, t_n) = \gamma(t_1, \dots, t_n)\beta(t_1, \dots, t_n)$$

with $\gamma(t_1, \dots, t_n) \in E(\Phi, B[t_1, \dots, t_n])$ and $\beta(t_1, \dots, t_n) \in G(\Phi, A[t_1, \dots, t_n])$. From this we also obtain

$$\sigma(t_1, \dots, t_n) = \gamma(t_1, \dots, t_n)\gamma(0, \dots, 0)^{-1}\beta(0, \dots, 0)^{-1}\beta(t_1, \dots, t_n),$$

since $I = \alpha(0, \dots, 0) = \gamma(0, \dots, 0)\beta(0, \dots, 0)$. From the induction assumption, the first two factors are in $E(\Phi, B[t_1, \dots, t_n])$. By the base case, we know that the result holds for polynomial rings $R[t_1, \dots, t_n]$, and via Proposition 5.3.4 it also holds for localizations of polynomial rings, in particular for A . Therefore, $\beta(0, \dots, 0)^{-1}\beta(t_1, \dots, t_n) \in E(\Phi, A[t_1, \dots, t_n]) \hookrightarrow E(\Phi, B[t_1, \dots, t_n])$. ■

$\mathbb{A}^1$ -Representability of Unstable K-Groups: The proof structure of the following result is similar to [Lam06, Theorem VI.1.18]. In fact, the affine Brown-Gersten property is a generalization of the weak pullback property for the unstable K_1 -functors.

THEOREM 5.3.6 *Let Φ be a root system of rank $\text{rk } \Phi > 1$, and let R be an excellent Dedekind ring. Then the simplicial presheaf $\text{Sing}_{\bullet}^{\mathbb{A}^1}(G(\Phi))$ on the site Sms_S with $S = \text{Spec } R$ has the affine Brown-Gersten property for the Nisnevich topology.*

PROOF: This follows from Proposition 5.2.10 and Proposition 5.3.5: Let A be a smooth R -algebra, B an étale A -algebra and $f \in A$ an element such that $A/f \rightarrow B/f$ is an isomorphism.

Recall that in this situation, $Bf + A = B$ and $Bf \cap A = Af$. The same also holds for the corresponding polynomial rings, i.e. $B[t_1, \dots, t_n]f + A[t_1, \dots, t_n] = B[t_1, \dots, t_n]$ and $B[t_1, \dots, t_n]f \cap A[t_1, \dots, t_n] = A[t_1, \dots, t_n]f$.

A general analytic patching situation for the square

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A_f & \longrightarrow & B_f \end{array}$$

as in Definition 5.1.2 consists of elements

$$\alpha \in G(\Phi, A_f[t_1, \dots, t_n]) \text{ and } \beta \in G(\Phi, B[t_1, \dots, t_n])$$

which are homotopic over $B_f[t_1, \dots, t_n]$, i.e. there is $H \in G(\Phi, B_f[t_1, \dots, t_n][t])$ such that $H(t=0) = \alpha$ and $H(t=1) = \beta$.

Consider the element $\sigma = \beta\alpha^{-1} \in G(\Phi, B_f[t_1, \dots, t_n])$. Since H is a homotopy between α and β , $(H \cdot \alpha^{-1})(t=0) = I$ and $(H \cdot \alpha^{-1})(t=1) = \sigma$. This implies that $\sigma \in E(\Phi, B_f[t_1, \dots, t_n])$, by Proposition 5.3.5.

By (i) of Proposition 5.2.10, we can decompose $\sigma = \beta_0^{-1}\alpha_0$ with

$$\alpha_0 \in E(\Phi, A_f[t_1, \dots, t_n]) \text{ and } \beta_0 \in E(\Phi, B[t_1, \dots, t_n]).$$

Then $I = \sigma^{-1}\sigma = \alpha_0^{-1}\beta_0\beta\alpha^{-1}$, and we therefore have the equation $\alpha_0\alpha = \beta_0\beta$ in $G(\Phi, B_f[t_1, \dots, t_n])$. Since $G(\Phi)$ is a sheaf in the Nisnevich topology, there is a global $\gamma \in G(\Phi, A[t_1, \dots, t_n])$ such that $\gamma = \alpha_0\alpha$ in $G(\Phi, A_f[t_1, \dots, t_n])$ and $\gamma = \beta_0\beta$ in $G(\Phi, B[t_1, \dots, t_n])$. Because $\alpha_0 \in E(\Phi, A_f[t_1, \dots, t_n])$, there is an element $H_\alpha \in G(\Phi, A_f[t_1, \dots, t_n][t])$ with $H_\alpha(t=0) = I$ and $H_\alpha(t=1) = \alpha_0$. Then $H \cdot \alpha \in G(\Phi, A_f[t_1, \dots, t_n][t])$ provides a homotopy $\alpha \sim \alpha_0\alpha$. The same argument works for $\beta \sim \beta_0\beta$. Therefore, the sheaf $G(\Phi, -)$ satisfies analytic patching.

Finally, we apply Proposition 5.1.3. Since the sheaf $G(\Phi, -)$ on Sm_S with $S = \text{Spec } R$ satisfies analytic patching, we conclude that $\text{Sing}_\bullet^{\mathbb{A}^1}(G(\Phi))$ has the affine Brown-Gersten property. $\blacksquare$

REMARK 5.3.7 *The proof above basically follows the same lines as the proof of [Mor07, Theorem 1.3.4]. We use the same main geometric results, i.e. the factorization of elementary matrices from Proposition 5.2.10 and the characterization of the elementary subgroups from Proposition 5.3.5. However, our proof seems more suitable for generalizations beyond group schemes than the proof in [Mor07].*

Karoubi and Villamayor [KV69] used the definition of K_1 as the quotient of GL_n modulo the elementary matrices E_n together with a definition of loop rings to define higher algebraic K-groups. We recall the definition from [Ger73].

DEFINITION 5.3.8 (KAROUBI-VILLAMAYOR K-THEORY) *Let A be a commutative ring, and let Φ be an irreducible and reduced root system. Then we can define a $K_1(\Phi, A) = G(\Phi, A)/E(\Phi, A)$.*

The path ring of A is defined to be $EA = \ker(A[t] \rightarrow A : t \mapsto 0)$. It comes with an augmentation $\epsilon : EA \rightarrow A : \sum a_i t^i \mapsto \sum a_i$. Then the loop ring of A is defined to be $\Omega A = \ker \epsilon$. The functor E admits the structure of a cotriple on the category of rings, yielding a simplicial ring $E^\bullet A$. We then apply the Chevalley group functor $G(\Phi, -)$ to obtain a simplicial group $G(\Phi, E^\bullet A)$.

The Karoubi-Villamayor K-groups of R and Φ are then given by

$$KV_{n+2}(\Phi, A) = \hat{\pi}_n(G(\Phi, E^\bullet A)), i \geq -1$$

where the $\hat{\pi}$ is the usual homotopy groups for $i \geq 1$, and some slight modification of the usual homotopy groups for $i = -1, 0$. For more details, cf. [Ger73, Definition 3.1] or [Jar83, p. 194].

Then a result of Rector [Ger73, Theorem 3.13] resp. [Jar83, Theorem 3.8] shows that there are isomorphisms of groups

$$KV_{n+1}(\Phi, A) \cong \pi_n \text{Sing}_\bullet^{\mathbb{A}^1}(G(\Phi))(\text{Spec } A).$$

Therefore, putting together the above results, we find that Karoubi-Villamayor K-theory is $\mathbb{A}^1$ -representable. Reading this result in the other direction, we find that we can determine the $\mathbb{A}^1$ -homotopy groups of Chevalley groups as Karoubi-Villamayor K-theory.

COROLLARY 5.3.9 *Let Φ be a root system of rank $\text{rk } \Phi > 1$, let R be an excellent Dedekind ring and let $S = \text{Spec } R$. Then the group schemes $G(\Phi)$ represent the unstable Karoubi-Villamayor K-groups $KV_n(\Phi, -)$ in the $\mathbb{A}^1$ -local model structure on $\Delta^{\text{op}}\text{Shv}(\text{Sm}_S)$. The corresponding simplicial resolutions $\text{Sing}_{\bullet}^{\mathbb{A}^1}(G(\Phi))$ represent the unstable Karoubi-Villamayor K-groups in the simplicial model structure on $\Delta^{\text{op}}\text{Shv}(\text{Sm}_S)$.*

Therefore, for any smooth affine S -scheme U , we obtain isomorphisms

$$\pi_n^{\mathbb{A}^1}(G(\Phi), I)(U) \cong KV_{n+1}(\Phi, U).$$

PROOF: From Proposition 5.1.4, we conclude from the affine Brown-Gersten property proved above that the affine replacement of $\text{Sing}_{\bullet}^{\mathbb{A}^1}(G(\Phi))$ has the Brown-Gersten property for quasi-projective schemes in the Zariski topology. Then for any smooth affine scheme U , we obtain a weak equivalence $\text{Sing}_{\bullet}^{\mathbb{A}^1}(G(\Phi))(U) \rightarrow \text{Ex}_{\mathbb{A}^1}^{\infty} \text{Sing}_{\bullet}^{\mathbb{A}^1}(G(\Phi))(U)$ where $\text{Ex}_{\mathbb{A}^1}^{\infty}$ denotes the fibrant replacement in the $\mathbb{A}^1$ -local model structure on Sm_S with the Zariski-topology. ■

REMARK 5.3.10 *Over a base field k , we can use Morel's result [Mor07, Theorem A.4.2] to conclude that Karoubi-Villamayor K-theory is representable in the $\mathbb{A}^1$ -local model structure also for the Nisnevich topology. It seems reasonable to conjecture that the same result can also be obtained for Dedekind rings.*

Note that combining this result with the discussion in Section 4.4, this completes the description of the homotopy groups of split algebraic groups over a perfect field k . In particular, all the homotopy groups $\pi_n^{\mathbb{A}^1}(G)$ for $n \geq 2$ are always unstable Karoubi-Villamayor K-groups for the root system of the semisimple part of G .

Patching for Torsors: We finally want to note that affine $\mathbb{A}^1$ -invariance for $H_{\text{Zar}}^1(-, GL_n)$ also holds over a Dedekind ring. This is a result of Popescu [Pop89, Theorem 4.1]:

PROPOSITION 5.3.11 *Let R be a Dedekind ring, and let A be an R -algebra, smooth and of finite type over R . Then every finitely generated projective module P over $A[t_1, \dots, t_n]$ is extended from R , i.e. there is an isomorphism*

$$H_{\text{Zar}}^1(\text{Spec } A[t_1, \dots, t_n], GL_n) \cong H_{\text{Zar}}^1(\text{Spec } A, GL_n)$$

This follows, since for smooth R -algebras the conditions of [Pop89, Theorem 4.1] are satisfied. These conditions require that for all primes $\mathfrak{p} \subseteq R$ with $\mathfrak{p}A \neq A$, (i) the residue field of A at $\mathfrak{p}$ is perfect, and (ii) the algebra $A/\mathfrak{p}A$ is regular. Condition (ii) follows, since A is smooth over R , therefore $A/\mathfrak{p}A$ is smooth over $R/\mathfrak{p}R$ and therefore regular. The condition on perfectness of the residue field is only needed to make sure the field extensions of $R/\mathfrak{p}R$ appearing at points of A are separable. But this also follows from the fact that A is smooth over R .

The above fact also implies that the infinite Grassmannians resp. the associated simplicial presheaves $\text{Sing}_{\bullet}^{\mathbb{A}^1}(Gr_n)$ have the affine Brown-Gersten property. To obtain the affine Brown-Gersten property for the finite Grassmannians $Gr_{n,r}$, it is

necessary to obtain a patching result for projective modules with explicitly given generators, for which I could find no reference in the literature.

Looking again at the analytic intuition from the h-principle, where every homogeneous space G/H under a complex Lie group satisfies the homotopy principle, it seems worthwhile investigating which homogeneous spaces G/H satisfy the affine Brown-Gersten property. This is again related to the question of triviality of torsors: Considering G as H -torsor over G/H , homotopies $\mathrm{Spec} R[t_1, \dots, t_n] \rightarrow G/H$ can be lifted if $H_{\mathrm{Nis}}^1(-, H)$ is homotopy invariant, because then we have the isomorphism $H^1(R[t_1, \dots, t_n], H) \cong H^1(R, H)$, and the latter is trivial for R local. Morphisms $\mathrm{Spec} R \rightarrow G/H$ can locally be lifted to G , and the same follows for homotopies if $H_{\mathrm{Nis}}^1(-, H)$ is homotopy invariant. This allows to glue together the locally homotopic morphisms in G , provided G has the affine Brown-Gersten property. Therefore a homogeneous space G/H should have the affine Brown-Gersten property, if G has the affine Brown-Gersten property and $H_{\mathrm{Nis}}^1(-, H)$ is homotopy invariant. Note also that under the above conditions,

$$\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(H) \rightarrow \mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(G) \rightarrow \mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(G/H)$$

is a fibre sequence, as will be apparent from the results in the next section.

5.4 More Examples of Fibre Sequences

Once we know that $\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(F) \rightarrow \mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(E) \rightarrow \mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(B)$ is a simplicial fibre sequence, the locality criterion from Theorem 4.3.10 can be applied more easily: The non-local part of the base $\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(B)$ is under better control than the non-local part of B , and also the classifying space of $\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(F)$ -fibre sequences has a better chance of being local. Moreover, Proposition 3.1.11 allows to check for simplicial fibre sequences by looking at the points of the topos. We explain how to go through such a proof for the example of principal GL_n -bundles.

PROPOSITION 5.4.1 *Let S be an excellent Dedekind ring. Then for any GL_n -torsor $GL_n \rightarrow E \rightarrow B$, the corresponding sequence*

$$\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(GL_n) \rightarrow \mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(E) \rightarrow \mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(B)$$

is a fibre sequence of simplicial presheaves.

PROOF: The proof is structured as follows: In (i), we reduce the assertion to showing that for local rings R , the space $\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(B)(R)$ is the quotient of a free action of $\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(GL_n)(R)$ on $\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(E)(R)$. The quotient assertion is proved in (ii) and (iii), and the freeness is proved in (iv).

(i) By Proposition 3.1.11, it suffices to check that for any henselian local ring R , the following is a fibre sequence:

$$\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(GL_n)(R) \rightarrow \mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(E)(R) \rightarrow \mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(B)(R).$$

By [GJ99, Corollary V.2.7], it suffices to show that the action of $\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(GL_n)(R)$ on $\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(E)(R)$ is free, and that

$$\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(E)(R) / \mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(GL_n)(R) = \mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(B)(R).$$

Note that the latter is a nontrivial assertion, since the higher simplices are global objects. This assertion is similar to statements proved in [Mor07, Theorem 12(2)].

(ii) Let $\sigma : \operatorname{Spec} R[t_1, \dots, t_m] \rightarrow B$ be a simplex of B , and consider the pullback of $p : \operatorname{Sing}_{\bullet}^{\mathbb{A}^1}(E) \rightarrow \operatorname{Sing}_{\bullet}^{\mathbb{A}^1}(B)$ along σ . This yields a $\operatorname{Sing}_{\bullet}^{\mathbb{A}^1}(GL_n)$ -torsor $E' = \operatorname{Sing}_{\bullet}^{\mathbb{A}^1}(E) \times_{\operatorname{Sing}_{\bullet}^{\mathbb{A}^1}(B)} \operatorname{Spec} R[t_1, \dots, t_m]$ over $\operatorname{Spec} R[t_1, \dots, t_m]$. From Proposition 5.3.11, we get the following isomorphisms

$$\begin{aligned} H_{\operatorname{Zar}}^1(\operatorname{Spec} R[t_1, \dots, t_m], \operatorname{Sing}_{\bullet}^{\mathbb{A}^1}(GL_n)) &\cong H_{\operatorname{Zar}}^1(\operatorname{Spec} R[t_1, \dots, t_m], GL_n) \\ &\cong H_{\operatorname{Zar}}^1(\operatorname{Spec} R, GL_n) \\ &= 0, \end{aligned}$$

where we use that the above cohomology groups are over an affine scheme, and that R is local. The torsor $E' \rightarrow \operatorname{Spec} R[t_1, \dots, t_m]$ is therefore trivial and has a section, whence we have a morphism $\operatorname{Spec} R[t_1, \dots, t_m] \rightarrow \operatorname{Sing}_{\bullet}^{\mathbb{A}^1}(E)$ lifting σ . Note that a priori we only get a continuous section, but the affine Brown-Gersten property implies that this can be patched to some regular section.

(iii) The preimage of $\sigma : \operatorname{Spec} R[t_1, \dots, t_m] \rightarrow B$ in the set of morphisms $\tau : \operatorname{Spec} R[t_1, \dots, t_m] \rightarrow E$ is isomorphic to $GL_n(R[t_1, \dots, t_m])$, since the set of commutative triangles

$$\begin{array}{ccc} & & E \\ & \nearrow \tau & \downarrow p \\ \operatorname{Spec} R[t_1, \dots, t_m] & \xrightarrow{\sigma} & B \end{array}$$

coincides with the set of sections of $E' \rightarrow \operatorname{Spec} R[t_1, \dots, t_m]$, which is isomorphic to $GL_n(R[t_1, \dots, t_m])$.

(iv) It follows now from steps (ii) and (iii) that $\operatorname{Sing}_{\bullet}^{\mathbb{A}^1}(B)(R)$ is the quotient of $\operatorname{Sing}_{\bullet}^{\mathbb{A}^1}(E)(R)$ by the action of $\operatorname{Sing}_{\bullet}^{\mathbb{A}^1}(GL_n)(R)$. We only need to verify that this action is degreewise free, i.e. for any $\sigma \in GL_n(R[t_1, \dots, t_m])$, if $\sigma x = x$ for some $x \in E(\operatorname{Spec} R[t_1, \dots, t_m])$, then $\sigma = I$. We can again consider the trivialization over the simplex $p \circ \sigma : \operatorname{Spec} R[t_1, \dots, t_m] \rightarrow B$, since p is equivariant for the trivial action on B , i.e. $p \circ \sigma(x) = p(x)$. Because we can equivariantly identify the n -simplices in E with $GL_n(R[t_1, \dots, t_m])$ equipped with the multiplication action, the action of $GL_n(R[t_1, \dots, t_m])$ on $E(\operatorname{Spec} R[t_1, \dots, t_m])$ is free.

We have shown that $\operatorname{Sing}_{\bullet}^{\mathbb{A}^1}(B)(R)$ is the quotient of the simplicial group action of $\operatorname{Sing}_{\bullet}^{\mathbb{A}^1}(GL_n)(R)$ on $\operatorname{Sing}_{\bullet}^{\mathbb{A}^1}(E)(R)$, and therefore we have the following fibre sequence of simplicial presheaves:

$$\operatorname{Sing}_{\bullet}^{\mathbb{A}^1}(GL_n) \rightarrow \operatorname{Sing}_{\bullet}^{\mathbb{A}^1}(E) \rightarrow \operatorname{Sing}_{\bullet}^{\mathbb{A}^1}(B)$$

■

We note that we did not use many results specific to GL_n , therefore the proof of the above result generalizes to show the following:

PROPOSITION 5.4.2 *Let S be an excellent Dedekind ring, and let G be a sheaf of groups satisfying the following condition:*

Assume that $\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(G)$ satisfies the affine Brown-Gersten property and that for all smooth henselian local rings R over S and all n , the set of torsors $H_{\mathrm{Nis}}^1(\mathrm{Spec} R[t_1, \dots, t_n], \mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(G))$ is trivial.

Then for any Nisnevich locally trivial G -torsor $G \rightarrow E \rightarrow B$, the sequence

$$\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(G) \rightarrow \mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(E) \rightarrow \mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(B)$$

is a fibre sequence.

EXAMPLE 5.4.3 The following groups satisfy the above condition:

- (i) For an excellent Dedekind ring, the general linear groups GL_n satisfies the condition by the solution of the Serre conjecture due to Quillen and Suslin.
- (ii) The symplectic groups Sp_{2n} over an excellent Dedekind ring also satisfy the condition [Lam06, Theorem VII.5.8].
- (iii) We will see in this chapter that also the projective linear groups PGL_n satisfy the above condition over a field of characteristic zero.

Note that the situation is not clear for torsors under orthogonal groups. There do exist non-extended torsors over affine spaces which were constructed by Parimala, cf. [Lam06]. However, these torsors are probably not locally trivial in the Nisnevich topology. On the other hand, although there is a local-global principle for quadratic modules, their behaviour in Nisnevich neighbourhoods is not clear. $\square$

Therefore we have a condition on the sheaf of groups over a given base scheme, which allows to conclude that principal fibre bundles induce fibre sequences for the $\mathbb{A}^1$ -simplicial resolutions. From principal fibre bundles with fibre a sheaf of groups G , we can also construct further locally trivial morphisms, by glueing in a fibre F which is equipped with an action of G . We can then show that the resulting morphisms induce fibre sequences. This applies e.g. to homogeneous space bundles like projective bundles.

PROPOSITION 5.4.4 *Let S be an excellent Dedekind ring, and let $p : E \rightarrow B$ be a Nisnevich locally trivial morphism which is obtained from a G -torsor $E' \rightarrow B$ by change-of-fibre along an action $G \times F \rightarrow F$. Assume that the group G satisfies the conditions of Proposition 5.4.2. Then the following is a fibre sequence:*

$$\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(F) \rightarrow \mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(E) \rightarrow \mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(B).$$

This fibre sequence is $\mathbb{A}^1$ -local if the corresponding G -torsor is $\mathbb{A}^1$ -local.

PROOF: It again suffices via Proposition 3.1.11 to prove that for any henselian local ring R ,

$$\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(GL_n)(R) \rightarrow \mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(E)(R) \rightarrow \mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(B)(R)$$

is a fibre sequence of simplicial sets.

This can be proved via the same argument as [GJ99, Corollary V.2.7]. Replacing the space $\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(F)$ with a fibrant model of $\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(F)$ does not change the assertion about the fibre sequence above, therefore we can assume that $\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(F)(R)$ is a fibrant simplicial set for any R .

Now consider a lifting problem:

$$\begin{array}{ccc}
\Lambda_k^n & \longrightarrow & \mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(E)(R) \\
\downarrow & & \downarrow \\
\Delta^n & \xrightarrow{\sigma} & \mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(B)(R).
\end{array}$$

By the assumption on the group, the G -torsor is trivial over the simplex σ , and the sequence $F \rightarrow E \rightarrow B$ is obtained by change-of-fibre along the group action $G \times F \rightarrow F$. Therefore also $\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(E)(R) \rightarrow \mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(B)(R)$ is trivial over σ , and the lifting problem above is equivalent to the following lifting problem:

$$\begin{array}{ccc}
\Lambda_k^n & \longrightarrow & \mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(F)(R) \times \Delta^n \\
\downarrow & & \downarrow \pi \\
\Delta^n & \xrightarrow{=} & \Delta^n,
\end{array}$$

where π is the projection away from $\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(F)(R)$. Since $\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(F)(R)$ is fibrant, the lifting problem has a solution.

The locality assertion follows from Theorem 4.3.10: Since the $\mathbb{A}^1$ -localization functors commute with products and are continuous, $L_{\mathbb{A}^1}G$ acts on $L_{\mathbb{A}^1}F$, and we have a change-of-fibre morphism $BL_{\mathbb{A}^1}G \rightarrow B^f L_{\mathbb{A}^1}F$. By assumption, the composition

$$A_{L_{\mathbb{A}^1}}B \rightarrow B \rightarrow B^f F \rightarrow B^f L_{\mathbb{A}^1}F$$

factors through $B \rightarrow BL_{\mathbb{A}^1}G$, therefore null-homotopy of the morphism $A_{L_{\mathbb{A}^1}} \rightarrow BL_{\mathbb{A}^1}G$ implies the null-homotopy of $A_{L_{\mathbb{A}^1}}B \rightarrow B^f L_{\mathbb{A}^1}F$. $\blacksquare$

COROLLARY 5.4.5 *Let k be a field of characteristic zero, and let $\mathbb{P}^{n-1} \rightarrow E \rightarrow B$ be a projective space bundle, coming from a PGL_n -torsor over B . This is an $\mathbb{A}^1$ -local fibre sequence for $n \geq 3$.*

For a general excellent Dedekind ring, the conclusion obtains if the projective bundle has trivial class in $\mathrm{Br}(B)$.

PROOF: We prove affine $\mathbb{A}^1$ -invariance for PGL_n under the assumption that the base field has characteristic zero. This follows from the four lemma applied to the following diagram:

$$\begin{array}{ccccccc}
\mathrm{Pic}(X \times \mathbb{A}^1) & \longrightarrow & H^1(X \times \mathbb{A}^1, GL_n) & \longrightarrow & H^1(X \times \mathbb{A}^1, PGL_n) & \longrightarrow & \mathrm{Br}(X \times \mathbb{A}^1) \\
\cong \downarrow p & & \cong \downarrow g & & \downarrow i & & b \downarrow \cong \\
\mathrm{Pic}(X) & \longrightarrow & H^1(X, GL_n) & \longrightarrow & H^1(X, PGL_n) & \longrightarrow & \mathrm{Br}(X)
\end{array}$$

The rows are given by the exact sequence for cohomology associated to the sequence of groups $\mathbb{G}_m \rightarrow GL_n \rightarrow PGL_n$. Note that X is an affine smooth scheme, which implies that g is an isomorphism. The morphism p is always an isomorphism for regular schemes. The morphism b on the Brauer group is an isomorphism since X is affine smooth over a base of characteristic zero, [CTO92, Theorem 4.2]. Therefore i is injective. Surjectivity follows, since the composition

$$H^1(X, PGL_n) \rightarrow H^1(X \times \mathbb{A}^1, PGL_n) \xrightarrow{i} H^1(X, PGL_n)$$

is an isomorphism.

A similar argument shows that $H^1(-, \text{Sing}_{\bullet}^{\mathbb{A}^1}(PGL_n))$ has homotopy invariance, by using the fibre sequence

$$\text{Sing}_{\bullet}^{\mathbb{A}^1}(\mathbb{G}_m) \cong \mathbb{G}_m \rightarrow \text{Sing}_{\bullet}^{\mathbb{A}^1}(GL_n) \rightarrow \text{Sing}_{\bullet}^{\mathbb{A}^1}(PGL_n).$$

Moreover, we get the identification $H^1(X, PGL_n) \rightarrow H^1(X, \text{Sing}_{\bullet}^{\mathbb{A}^1}(PGL_n))$ from the corresponding identifications for GL_n , cf. Proposition 5.4.1, and $\mathbb{G}_m$. This is where we use $n \geq 3$.

Then the conditions of Proposition 5.4.2 are satisfied, and we see that every PGL_n -torsor induces a fibre sequence

$$\text{Sing}_{\bullet}^{\mathbb{A}^1}(PGL_n) \rightarrow \text{Sing}_{\bullet}^{\mathbb{A}^1}(E) \rightarrow \text{Sing}_{\bullet}^{\mathbb{A}^1}(B).$$

Finally, we use the change-of-fibre along the action $PGL_n \times \mathbb{P}^{n-1} \rightarrow \mathbb{P}^{n-1}$, and the argument from Proposition 5.4.4 shows that the projective space bundle induces a fibre sequence

$$\text{Sing}_{\bullet}^{\mathbb{A}^1}(\mathbb{P}^{n-1}) \rightarrow \text{Sing}_{\bullet}^{\mathbb{A}^1}(E) \rightarrow \text{Sing}_{\bullet}^{\mathbb{A}^1}(B).$$

The statement over Dedekind rings can be obtained as follows: If the class of the PGL_n -torsor maps to zero under $H^1(X, PGL_n) \rightarrow H^2(X, \mathbb{G}_m) \cong \text{Br}(X)$, then it has a reduction to a vector bundle. Then we use Proposition 5.4.1 for GL_n , and the corresponding change of fibre along $GL_n \times \mathbb{P}^{n-1} \rightarrow PGL_n \times \mathbb{P}^{n-1} \rightarrow \mathbb{P}^{n-1}$, cf. Proposition 5.4.4. $\blacksquare$

More generally, the above proposition holds for homogeneous spaces under groups satisfying the assumptions of Proposition 5.4.2.

Locality of Fibre Sequences: In this paragraph we discuss which fibre sequences produced above are $\mathbb{A}^1$ -local.

Indeed, from [Mor07, Remark 2.1.1], it follows that the classifying space of $\text{Sing}_{\bullet}^{\mathbb{A}^1}(GL_n)$ is $\mathbb{A}^1$ -local over the base $\text{Spec } k$, and moreover is weakly equivalent to $\text{Sing}_{\bullet}^{\mathbb{A}^1}(Gr_n)$, the resolution of the Grassmannian. The pullback stability Proposition 4.3.14 then implies, that all fibre sequences with fibre GL_n which are locally trivial in the Nisnevich topology actually are $\mathbb{A}^1$ -local fibre sequences.

I would however like to use Theorem 4.3.10 more directly to deduce the following, but I do not know how to do this. Therefore the argument is rather close to [Mor07, Theorem A.4.2].

We shortly recall the spaces involved in the next result: The Grassmann variety $Gr_n(\mathbb{A}^k)$ is the variety of n -planes in affine k -space. The Stiefel variety $V_n(\mathbb{A}^k)$ is the variety of all ordered n -tuples of linearly independent points in $\mathbb{A}^k$. Sending the n -tuple of points to the n -plane they span induces a morphism $V_n(\mathbb{A}^k) \rightarrow Gr_n(\mathbb{A}^k)$. These objects exist as schemes over $\mathbb{Z}$ and therefore over Dedekind rings in general.

One can also pass to the limit of these varieties via the inclusions $V_n(\mathbb{A}^k) \rightarrow V_n(\mathbb{A}^{k+1})$ resp. $Gr_n(\mathbb{A}^k) \rightarrow Gr_n(\mathbb{A}^{k+1})$, whence we obtain sheaves $V_n = V_n(\mathbb{A}^\infty)$ resp. $Gr_n = Gr_n(\mathbb{A}^\infty)$.

COROLLARY 5.4.6 *Let S be an excellent Dedekind ring. The fibre sequence*

$$\text{Sing}_{\bullet}^{\mathbb{A}^1}(GL_n) \rightarrow \text{Sing}_{\bullet}^{\mathbb{A}^1}(V_n) \rightarrow \text{Sing}_{\bullet}^{\mathbb{A}^1}(Gr_n)$$

is an $\mathbb{A}^1$ -local fibre sequence provided the fibrant replacement of $\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(GL_n)$ is $\mathbb{A}^1$ -local. Hence any principal GL_n -bundle is an $\mathbb{A}^1$ -local fibre sequence.

PROOF: From Proposition 5.4.1, we have a simplicial fibre sequence.

(i) Assume that $\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(GL_n)$ and $\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(V_n)$ are $\mathbb{A}^1$ -local and additionally $\pi_0(\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(Gr_n))$ is $\mathbb{A}^1$ -invariant. By an argument similar to one used in the proof of Theorem 4.3.7, we conclude that $\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(Gr_n)$ is local: We have a ladder diagram of fibre sequences

$$\begin{array}{ccccc} \mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(GL_n) & \longrightarrow & \mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(V_n) & \longrightarrow & \mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(Gr_n) \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{Hom}(\mathbb{A}^1, \mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(GL_n)) & \longrightarrow & \mathbf{Hom}(\mathbb{A}^1, \mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(V_n)) & \longrightarrow & \mathbf{Hom}(\mathbb{A}^1, \mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(Gr_n)). \end{array}$$

By assumptions, the morphisms $\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(GL_n) \rightarrow \mathbf{Hom}(\mathbb{A}^1, \mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(GL_n))$ as well as $\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(V_n) \rightarrow \mathbf{Hom}(\mathbb{A}^1, \mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(V_n))$ are weak equivalences. We want to show the same for $\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(Gr_n) \rightarrow \mathbf{Hom}(\mathbb{A}^1, \mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(Gr_n))$. The morphisms induced on homotopy group sheaves $\pi_i \mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(Gr_n) \rightarrow \pi_i \mathbf{Hom}(\mathbb{A}^1, \mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(Gr_n))$ are isomorphisms for $i \geq 1$ by the five lemma. Since $\pi_0 \mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(Gr_n)$ is $\mathbb{A}^1$ -invariant by assumption, the same also holds for π_0 , and we find that $\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(Gr_n)$ is indeed $\mathbb{A}^1$ -local.

(ii) The $\mathbb{A}^1$ -invariance of $\pi_0 \mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(Gr_n)$ follows from Proposition 5.3.11. Indeed, we only need that the morphism

$$\pi_0 \mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(Gr_n)(R) \rightarrow \pi_0 \mathbf{Hom}(\mathbb{A}^1, \mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(Gr_n))(R)$$

is a bijection for any local ring R . Then we identify the corresponding elements in π_0 with morphisms $\mathrm{Spec} R \rightarrow Gr_n$ resp. $\mathrm{Spec} R[t] \rightarrow Gr_n$, and Proposition 5.3.11 yields the required bijection.

(iii) The space $\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(V_n)$ is contractible and therefore $\mathbb{A}^1$ -local. To see this, it suffices to check that for a local ring R , the simplicial set $\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(V_n(\mathbb{A}^\infty))(R)$ is contractible. We will show that any simplex in $\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(V_n(\mathbb{A}^k))$ becomes homotopic to a constant simplex after passing to some $\mathbb{A}^l$. Note that we can interpret the m -simplices as morphisms $\mathrm{Spec} R[t_1, \dots, t_m] \rightarrow V_n(\mathbb{A}^k)$ and therefore as projective modules over $\mathrm{Spec} R[t_1, \dots, t_m]$ of rank n with k generating sections. Let any such projective module be given: $\sigma : R[t_1, \dots, t_m]^{\oplus k} \twoheadrightarrow P$. By Proposition 5.3.11, P is extended from R , therefore there is a presentation $\rho : R[t_1, \dots, t_m]^{\oplus k'} \twoheadrightarrow P$ which is extended from R . We pass to the Stiefel variety $V_n(\mathbb{A}^{k+k'})$ by considering the presentation $\sigma \oplus \rho : R[t_1, \dots, t_m]^{\oplus (k+k')} \twoheadrightarrow P$. Keeping the presentation ρ fixed we can move the sections from the presentation σ to the zero sections, without destroying the fact that the sections of $\sigma \oplus \rho$ generate the projective module. This implies that any homotopy class in $V_n(\mathbb{A}^k)$ becomes null-homotopic after passing to some $V_n(\mathbb{A}^k)$, therefore $\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(V_n(\mathbb{A}^\infty))(R)$ is contractible. $\blacksquare$

It would be interesting to know if $V_k(\mathbb{A}^n)$ is $(n - k - 1)$ -connected, which would be the connectivity of the *real* realization.

REMARK 5.4.7 *Note that the above proposition has the unpleasant proviso that the fibrant replacement of $\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(GL_n)$ is still local. To get rid of this proviso, we*

would need statements similar to [Mor07, Theorem A.4.2], which we however have not yet been able to prove in the case of Dedekind rings.

As remarked earlier, this implies that all other GL_n -torsors induce $\mathbb{A}^1$ -local fibre sequences. One particularly important is the fibre sequence

$$\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(GL_n) \rightarrow \mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(GL_{n+1}) \rightarrow \mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(\mathbb{A}^{n+1} \setminus \{0\}),$$

whose existence is the $\mathbb{A}^1$ -homotopy incarnation of the stabilization theorems for the unstable K-theories of the general linear groups. The stabilization results translate into connectivity assertions for the motivic spheres $\mathrm{Sing}_{\bullet}^{\mathbb{A}^1}(\mathbb{A}^{n+1} \setminus \{0\})$, and obstructions to stabilization lie in respective homotopy groups of spheres. Indeed, the main applications of unstable $\mathbb{A}^1$ -homotopy at the moment seem to be stabilization results for Karoubi-Villamayor K-theory, or results on the classification of vector bundles, cf. [Mor07].

Similar results can be obtained for the symplectic groups, and also for the projective linear groups over a field of characteristic zero. Using Proposition 5.4.4, we also get many $\mathbb{A}^1$ -local fibre sequences with homogeneous space fibres. In particular, projective space bundles over a field of characteristic zero induce $\mathbb{A}^1$ -local fibre sequences, and all those spherical fibrations which arise from a vector bundle via the Thom construction also are $\mathbb{A}^1$ -local spherical fibrations. Hopefully such “geometric” fibre sequences can be used to widen our knowledge on $\mathbb{A}^1$ -homotopy groups of schemes over Dedekind rings.

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