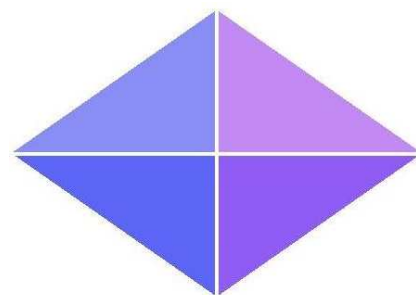


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Multiplicities of Galois representations
of weight one
(with an appendix by Niko Naumann)

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Abstract

In this article we consider mod p modular Galois representations which are unramified at p such that the Frobenius element at p acts through a scalar matrix. The principal result states that the multiplicity of any such representation is bigger than 1.

MSC Classification: 11F80 (primary), 11F33, 11F25 (secondary).

1 Introduction

A continuous odd irreducible Galois representation $\rho : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_2(\overline{\mathbb{F}}_p)$ is said to be of weight one if it is unramified at p . According to Serre's conjecture (with the minimal weight as defined in [4]), all such representations should arise from Katz modular forms of weight 1 over $\overline{\mathbb{F}}_p$ for the group $\Gamma_1(N)$ with N the (prime to p) conductor of ρ . Assuming the modularity of ρ , this is known if $p > 2$ or if $p = 2$ and the restriction of ρ to a decomposition group at 2 is not an extension of twice the same character. A weight 1 Katz modular form over $\overline{\mathbb{F}}_p$ can be embedded into weight p and the same level in two different ways: by multiplication by the Hasse invariant of weight $p - 1$ and by applying the Frobenius (see [5], Section 4). Hence, the corresponding eigenform(s) in weight p should be considered as old forms; they lie in the ordinary part.

A modular Galois representation $\rho : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_2(\overline{\mathbb{F}}_p)$ of conductor N can be realised with a certain multiplicity (see Proposition 4.1) on the p -torsion of $J_1(Np)$ or $J_1(N)$ (for $p = 2$). In this article we prove that this multiplicity is bigger than 1 if ρ is of weight one and Frob_p acts by scalars. If $p = 2$, we also assume that the corresponding weight 1 form exists. Together with [2], Theorem 6.1, this completely settles the question of multiplicity one for modular Galois representations. Its study had been started by Mazur and continued among others by Ribet, Gross, Edixhoven and Buzzard. The first example of a modular Galois representation of not satisfying multiplicity one was found by Kilford in [9]. See [10] for a more detailed exposition.

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A systematic computational study of the multiplicity of Galois representations of weight one has been carried out in [10]. The data gathered suggest that the multiplicity always seems to be 2 if it is not 1. Moreover, the local factors of the Hecke algebras are becoming astonishingly large.

Overview

We give a short overview over the article with an outline of the proof. In Section 2 an isomorphism between a certain part of the p -torsion of a Jacobian of a modular curve with a local factor of a mod p Hecke algebra is established (Proposition 2.2). As an application one obtains a mod p version of the Eichler-Shimura isomorphism (Corollary 2.3). Together with a variant of a well-known theorem by Boston, Lenstra and Ribet (Proposition 4.1) one also gets an isomorphism between a certain kernel in the local mod p Hecke algebra and a part of the corresponding Galois representation. This gives for instance a precise link between multiplicities and properties of the Hecke algebra (Corollary 4.2). In Section 3 it is proved (Theorem 3.1) that under the identification of Section 2, the Frobenius at p on the part of the Galois representation corresponds to the Hecke operator T_p in the Hecke algebra. This relation is exploited in Section 4 to obtain the principal result (Theorem 4.3) and a couple of corollaries.

Notations

For integers $N \geq 1$ and $k \geq 1$, we let $S_k(\Gamma_1(N))$ be the $\mathbb{C}$ -vector space of holomorphic cusp forms and $S_k(\Gamma_1(N), \mathbb{F}_p)$ the $\mathbb{F}_p$ -vector space of Katz cusp forms on $\Gamma_1(N)$ of weight k . Whenever $S \subseteq R$ are rings, m is an integer and M is an R -module on which the Hecke and diamond operators act, we let $\mathbb{T}_S^{(m)}(M)$ be the S -subalgebra inside the R -endomorphism ring of M generated by the Hecke operators T_n with $(n, m) = 1$ and the diamond operators. If $\phi : S \rightarrow S'$ is a ring homomorphism, we let $\mathbb{T}_\phi^{(m)}(M) := \mathbb{T}_S^{(m)}(M) \otimes_S S'$ or with ϕ understood $\mathbb{T}_{S \rightarrow S'}^{(m)}(M)$. If $m = 1$, we drop the superscript.

Every maximal ideal $\overline{\mathfrak{m}} \subseteq \mathbb{T}_{\mathbb{Z} \rightarrow \mathbb{F}_p}(S_k(\Gamma_1(N)))$ corresponds to a Galois conjugacy class of cusp forms over $\overline{\mathbb{F}}_p$ of weight k on $\Gamma_1(N)$. One can attach to $\overline{\mathfrak{m}}$ by work of Shimura and Deligne a continuous odd semi-simple Galois representation $\rho_{\overline{\mathfrak{m}}} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_2(\overline{\mathbb{F}}_p)$ which is unramified outside Np and satisfies $\text{Tr}(\rho_{\overline{\mathfrak{m}}}(\text{Frob}_l)) \equiv T_l \pmod{\overline{\mathfrak{m}}}$ and $\text{Det}(\rho_{\overline{\mathfrak{m}}}(\text{Frob}_l)) \equiv \langle l \rangle l^{k-1} \pmod{\overline{\mathfrak{m}}}$ for all primes $l \nmid Np$ via an embedding $\mathbb{T}_{\mathbb{Z} \rightarrow \mathbb{F}_p}(S_k(\Gamma_1(N)))/\overline{\mathfrak{m}} \hookrightarrow \overline{\mathbb{F}}_p$. All Frobenius elements Frob_l are arithmetic ones.

For all the article we fix an isomorphism $\mathbb{C} \cong \overline{\mathbb{Q}}_p$ and a ring surjection $\overline{\mathbb{Z}}_p \rightarrow \overline{\mathbb{F}}_p$. If K is a field, we denote by $K(\epsilon) = K[\epsilon]/(\epsilon^2)$ the dual numbers. For a finite flat group scheme G , the Cartier dual is denoted by ${}^tG$. The maximal unramified extension of $\mathbb{Q}_p$ (inside $\overline{\mathbb{Q}}_p$) is denoted by $\mathbb{Q}_p^{\text{nr}}$ and its integer ring by $\mathbb{Z}_p^{\text{nr}}$.

Situations

We shall often assume one of the following two situations. In the applications, the second part will be taken for $p = 2$.

Situation 1.1 (I) Let p be an odd prime and N a positive integer not divisible by p . For $m \in \mathbb{N}$ write $\mathbb{T}_{\mathbb{Z}_p}^{(m)}$ for the Hecke algebra $\mathbb{T}_{\mathbb{Z} \rightarrow \mathbb{Z}_p}^{(m)}(S_2(\Gamma_1(Np)))$. Let $\mathfrak{m}$ be an ordinary (i.e. $T_p \notin \mathfrak{m}$) maximal ideal of $\mathbb{T}_{\mathbb{Z}_p}$. Denote the image of $\mathfrak{m}$ in $\mathbb{T}_{\mathbb{F}_p} := \mathbb{T}_{\mathbb{Z} \rightarrow \mathbb{F}_p}(S_2(\Gamma_1(Np)))$ by $\overline{\mathfrak{m}}$. Assume that $\rho_{\overline{\mathfrak{m}}}$ is irreducible. Let $\mathfrak{m}^{(m)} = \mathfrak{m} \cap \mathbb{T}_{\mathbb{Z}_p}^{(m)}$ and similarly for $\overline{\mathfrak{m}}^{(m)}$.

Let furthermore $K = \mathbb{Q}_p(\zeta_p)$ and $\mathcal{O} = \mathbb{Z}_p[\zeta_p]$ with a primitive p -th root of unity ζ_p . Also let $J := J_1(Np)_{\mathbb{Q}}$ be the Jacobian of $X_1(Np)$ over $\mathbb{Q}$.

(II) Let p be any prime and N a positive integer not divisible by p . For $m \in \mathbb{N}$ write $\mathbb{T}_{\mathbb{Z}_p}^{(m)}$ for the Hecke algebra $\mathbb{T}_{\mathbb{Z} \rightarrow \mathbb{Z}_p}^{(m)}(S_2(\Gamma_1(N)))$. Let $\mathfrak{m}$ be an ordinary (i.e. $T_p \notin \mathfrak{m}$) maximal ideal of $\mathbb{T}_{\mathbb{Z}_p}$. Denote the image of $\mathfrak{m}$ in $\mathbb{T}_{\mathbb{F}_p} := \mathbb{T}_{\mathbb{Z} \rightarrow \mathbb{F}_p}(S_2(\Gamma_1(N)))$ by $\overline{\mathfrak{m}}$. Assume that $\rho_{\overline{\mathfrak{m}}}$ is irreducible. Let $\mathfrak{m}^{(m)} = \mathfrak{m} \cap \mathbb{T}_{\mathbb{Z}_p}^{(m)}$ and similarly for $\overline{\mathfrak{m}}^{(m)}$.

Let furthermore $K = \mathbb{Q}_p$ and $\mathcal{O} = \mathbb{Z}_p$. Also let $J := J_1(N)_{\mathbb{Q}}$ be the Jacobian of $X_1(N)$ over $\mathbb{Q}$.

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2 Hecke algebras, Jacobians and p -divisible groups

Let us assume one of the two cases of Situation 1.1. The maximal ideal $\mathfrak{m}$ of $\mathbb{T}_{\mathbb{Z}_p}$ corresponds to an idempotent $e_{\mathfrak{m}} \in \mathbb{T}_{\mathbb{Z}_p}$, in the sense that applying $e_{\mathfrak{m}}$ to any $\mathbb{T}_{\mathbb{Z}_p}$ -module is the same as localising the module at $\mathfrak{m}$. Let $\mathcal{G}$ be the p -divisible group $J[p^{\infty}]_{\mathbb{Q}}$ over $\mathbb{Q}$. Consider the Tate module $T_p J = T_p \mathcal{G} = \varprojlim J[p^n](\overline{\mathbb{Q}})$. It is a $\mathbb{T}_{\mathbb{Z}_p}[\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})]$ -module. The idempotent $e_{\mathfrak{m}}$ gives rise to an endomorphism of $\mathbb{T}_{\mathbb{Z}_p}[\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})]$ -modules on $T_p J$. Such an endomorphism comes from an endomorphism $e_{\mathfrak{m}}$ of the p -divisible group $\mathcal{G}$, which is also an idempotent. We put $G = e_{\mathfrak{m}} \mathcal{G}$ and say that this is the p -divisible group over $\mathbb{Q}$ attached to $\mathfrak{m}$. We shall mainly be interested in the p -torsion of G . However, making the detour via p -divisible groups allows us to quote the following theorem by Gross.

Theorem 2.1 (Gross) Assume any of the two cases of Situation 1.1. Let G be the p -divisible group over $\mathbb{Q}$ attached to $\mathfrak{m}$, as explained above. Let $h = \text{rk}_{\mathbb{Z}_p} \mathbb{T}_{\mathbb{Z}_p, \mathfrak{m}}$, where $\mathbb{T}_{\mathbb{Z}_p, \mathfrak{m}}$ denotes the localisation of $\mathbb{T}_{\mathbb{Z}_p}$ at $\mathfrak{m}$.

(a) The p -divisible group G acquires good reduction over $\mathcal{O}$. We write $G_{\mathcal{O}}$ for the corresponding p -divisible group over $\mathcal{O}$. It sits in the exact sequence

$$0 \rightarrow G_{\mathcal{O}}^0 \rightarrow G_{\mathcal{O}} \rightarrow G_{\mathcal{O}}^e \rightarrow 0,$$

where $G_{\mathcal{O}}^e$ is étale and $G_{\mathcal{O}}^0$ is of multiplicative type, i.e. its Cartier dual is étale. The exact sequence is preserved by the action of the Hecke correspondences.

(b) Over $\mathcal{O}[\zeta_N]$ the group $G_{\mathcal{O}[\zeta_N]}$ is isomorphic to its Cartier dual ${}^tG_{\mathcal{O}[\zeta_N]}$. This gives isomorphisms of p -divisible groups over $\mathcal{O}[\zeta_N]$

$$G_{\mathcal{O}[\zeta_N]}^e \cong {}^tG_{\mathcal{O}[\zeta_N]}^0 \quad \text{and} \quad G_{\mathcal{O}[\zeta_N]}^0 \cong {}^tG_{\mathcal{O}[\zeta_N]}^e.$$

(c) We have $G_{\mathbb{F}_p}^e[p] \cong (\mathbb{Z}/p\mathbb{Z})_{\mathbb{F}_p}^h$ and $G_{\mathbb{F}_p}^0[p] \cong \mu_{p, \mathbb{F}_p}^h$.

Proof. (a) The statement on the good reduction is [8], Prop. 12.8 (1) and 12.9 (1). The exact sequence is proved in [8], Prop. 12.8 (4) and 12.9 (3). That it is preserved by the Hecke correspondences is a consequence of the fact that there are no non-trivial morphisms from a connected group scheme to an étale one, whence any Hecke correspondence on G restricts to G^0 .

(b) The Cartier self-duality of G over $K(\zeta_N)$ is also proved in [8], Prop. 12.8 (1) and 12.9 (1). It extends to a self-duality over $\mathcal{O}[\zeta_N]$. The second statement follows as in (a) from the non-existence of non-trivial morphisms from G^0 to G^e over $\mathcal{O}[\zeta_N]$; this argument gives $G^0 \cong {}^tG^e$. Applying Cartier duality to this, we also get $G^e \cong {}^tG^0$.

(c) By Part (b), G^e and G^0 have equal height. That height is equal to h by [8], Prop. 12.8 (1) and 12.9 (1). The statement is now due to the fact that up to isomorphism the given group schemes are the only ones of rank p^h which are killed by p and which are étale or of multiplicative type, respectively. $\square$

The last point makes the ordinarity of $\overline{\mathfrak{m}}$ look like the ordinarity of an abelian variety.

Proposition 2.2 Assume any of the two cases of Situation 1.1 and let G be the p -divisible group attached to $\mathfrak{m}$. Then we have the isomorphism $G^0[p](\overline{\mathbb{Q}}_p) \cong \mathbb{T}_{\mathbb{F}_p, \overline{\mathfrak{m}}}$ of $\mathbb{T}_{\mathbb{F}_p, \overline{\mathfrak{m}}}$ -modules.

Proof. Taking the p -torsion of the p -divisible groups in Thm. 2.1 (a), one obtains the exact sequence

$$0 \rightarrow G_{\mathcal{O}}^0[p](\overline{\mathbb{Q}}_p) \rightarrow G_{\mathcal{O}}[p](\overline{\mathbb{Q}}_p) \rightarrow G_{\mathcal{O}}^e[p](\overline{\mathbb{Q}}_p) \rightarrow 0 \quad (2.1)$$

of $\mathbb{T}_{\mathbb{F}_p, \overline{\mathfrak{m}}}$ -modules with Galois action. We also spell out the dualities in Thm. 2.1 (b) restricted to the p -torsion:

$$\begin{aligned} G_{\mathcal{O}[\zeta_N]}^0[p] &\cong \text{Hom}_{\text{gr.sch.}/\mathcal{O}[\zeta_N]}(G^e[p]_{\mathcal{O}[\zeta_N]}, \mu_{p, \mathcal{O}[\zeta_N]}) \quad \text{and} \\ G_{\mathcal{O}[\zeta_N]}^e[p] &\cong \text{Hom}_{\text{gr.sch.}/\mathcal{O}[\zeta_N]}(G^0[p]_{\mathcal{O}[\zeta_N]}, \mu_{p, \mathcal{O}[\zeta_N]}). \end{aligned} \quad (2.2)$$

When taking $\overline{\mathbb{Q}}_p$ -points, these give isomorphisms of $\mathbb{T}_{\mathbb{F}_p, \overline{\mathfrak{m}}}$ -modules, i.e. in particular of $\mathbb{F}_p$ -vector spaces. We will from now on identify $\mu_{p, \mathcal{O}[\zeta_N]}(\overline{\mathbb{Q}}_p)$ with $\mathbb{F}_p$ and the group homomorphisms on $\overline{\mathbb{Q}}_p$ -points above with $\mathbb{F}_p$ -linear ones.

The final ingredient in the proof is the fact that $G^e(\overline{\mathbb{Q}}_p)[\mathfrak{m}] = G^e[p](\overline{\mathbb{Q}}_p)[\mathfrak{m}]$ is a 1-dimensional $L := \mathbb{T}_{\mathbb{F}_p}/\overline{\mathfrak{m}}$ -vector space by [8], Prop. 12.8 (5) and 12.9 (4). We now quotient the first isomorphism of Equation 2.2 (on $\overline{\mathbb{Q}}_p$ -points) by $\overline{\mathfrak{m}}$ and obtain

$$G^0[p](\overline{\mathbb{Q}}_p)/\overline{\mathfrak{m}} \cong \mathrm{Hom}_{\mathbb{F}_p}(G^e[p](\overline{\mathbb{Q}}_p)[\mathfrak{m}], \mathbb{F}_p) \cong \mathrm{Hom}_{\mathbb{F}_p}(L, \mathbb{F}_p),$$

which is a 1-dimensional L -vector space. Consequently, Nakayama's Lemma applied to the finitely generated $\mathbb{T}_{\mathbb{F}_p, \overline{\mathfrak{m}}}$ -module $G^0[p](\overline{\mathbb{Q}}_p)$ yields a surjection $\mathbb{T}_{\mathbb{F}_p, \overline{\mathfrak{m}}} \twoheadrightarrow G^0[p](\overline{\mathbb{Q}}_p)$. From [10], Prop 4.7, it follows that $2 \dim_{\mathbb{F}_p} \mathbb{T}_{\mathbb{F}_p, \overline{\mathfrak{m}}} = \dim_{\mathbb{F}_p} H_{\mathrm{par}}^1(\Gamma_1(Np), \mathbb{F}_p)_{\overline{\mathfrak{m}}}$. As we also have

$$H_{\mathrm{par}}^1(\Gamma_1(Np), \mathbb{F}_p)_{\overline{\mathfrak{m}}} \cong J_1(Np)(\mathbb{C})[p]_{\overline{\mathfrak{m}}} \cong G[p](\overline{\mathbb{Q}}_p),$$

we obtain $\dim_{\mathbb{F}_p} \mathbb{T}_{\mathbb{F}_p, \overline{\mathfrak{m}}} = \dim_{\mathbb{F}_p} G^0[p](\overline{\mathbb{Q}}_p)$ and, thus, $\mathbb{T}_{\mathbb{F}_p, \overline{\mathfrak{m}}} \cong G^0[p](\overline{\mathbb{Q}}_p)$, as desired. $\square$

The following result together with very helpful hints on its proof (i.e. the preceding proposition) was suggested by Kevin Buzzard. See also the discussion before [6], Proposition 6.3, and [11].

Corollary 2.3 *Assume any of the two cases of Situation 1.1 and let G be the p -divisible group attached to $\mathfrak{m}$. Then there is the exact sequence*

$$0 \rightarrow \mathbb{T}_{\mathbb{F}_p, \overline{\mathfrak{m}}} \rightarrow G[p](\overline{\mathbb{Q}}) \rightarrow \mathbb{T}_{\mathbb{F}_p, \overline{\mathfrak{m}}}^{\vee} \rightarrow 0$$

of $\mathbb{T}_{\mathbb{F}_p, \overline{\mathfrak{m}}}$ -modules, where the dual is the $\mathbb{F}_p$ -linear dual.

Proof. Substituting the isomorphism of Prop. 2.2 into the second isomorphism of Equation 2.2 (on $\overline{\mathbb{Q}}_p$ -points) gives

$$G^e[p](\overline{\mathbb{Q}}_p) \cong \mathrm{Hom}(\mathbb{T}_{\mathbb{F}_p, \overline{\mathfrak{m}}}, \mathbb{F}_p)$$

as $\mathbb{T}_{\mathbb{F}_p, \overline{\mathfrak{m}}}$ -modules, whence the corollary follows from Equation 2.1. $\square$

The following proposition is similar in spirit to Proposition 2.2. It will not be needed in the sequel.

Proposition 2.4 *Assume any of the two cases of Situation 1.1 and let G be the p -divisible group attached to $\mathfrak{m}$. Then $G^0[p](\mathbb{F}_p(\epsilon))$ and $\mathbb{T}_{\mathbb{F}_p, \overline{\mathfrak{m}}}$ are isomorphic as $\mathbb{T}_{\mathbb{F}_p, \overline{\mathfrak{m}}}$ -modules.*

Proof. We only give a sketch. Since $G^0[p](\mathbb{F}_p)$ consists of the origin as unique point, $G^0[p](\mathbb{F}_p(\epsilon))$ coincides with the tangent space at 0 of $G_{\mathbb{F}_p}^0[p]$. The latter, however, is equal to the tangent space at 0 of $G_{\mathbb{F}_p}[p]$. On the other hand, its dual, the cotangent space at 0 of $G_{\mathbb{F}_p}[p]$, is isomorphic to $S_p(\Gamma_1(N), \mathbb{F}_p)_{\overline{\mathfrak{m}}}$. For Situation (II) this is well-known. In Situation (I) we quote [4], Eq. 6.7.1 and 6.7.2, as well as [8], Prop. 8.13 (note that the ordinarity assumption kills the second summand in that proposition). Consequently, $G^0[p](\mathbb{F}_p(\epsilon))$ is isomorphic to the Hecke algebra on $S_p(\Gamma_1(N), \mathbb{F}_p)_{\overline{\mathfrak{m}}}$ as a Hecke module. In [10], Prop. 2.3, it is shown that this algebra is $\mathbb{T}_{\mathbb{F}_p, \overline{\mathfrak{m}}}$. $\square$

From Prop. 2.2 and the reduction of points used in the direct proof of Theorem 3.1 we can also conclude an isomorphism $G^0[p](\overline{\mathbb{F}}_p(\epsilon)) \cong \mathbb{T}_{\mathbb{F}_p, \overline{\mathfrak{m}}}$.

3 Comparing Frobenius and the Hecke operator T_p

The aim of this section is to prove that the Hecke operator T_p and the Frobenius at p coincide on the unramified $\overline{\mathbb{Q}}_p$ -points of $G^0[p]$.

Theorem 3.1 *Assume any of the two cases of Situation 1.1 and let $G_{\mathcal{O}}^0$ be the p -divisible group of Thm. 2.1. The action of the geometric Frobenius on the points $G_{\mathcal{O}}^0[p](\mathbb{Q}_p^{\text{nr}}(\zeta_p))$ is the same as the action of the Hecke operator T_p .*

Using the Eichler-Shimura congruence relation in Situation (II) and the reduction of a well-known semi-stable model of the modular curve in Situation (I), the proof is quickly reduced to comparing the geometric Frobenius and Verschiebung on the special fibre of $G^0[p]$. This comparison has been worked out conceptually by Niko Naumann in Appendix A using Fontaine's theory of Honda systems in a general setting. We also give a direct elementary proof. The idea of that proof is to work with the tangent space at 0 over $\overline{\mathbb{F}}_p$, in order to have an injective reduction map from characteristic zero to the finite field. On the special fibre elementary computations then suffice.

Direct proof. We know that ${}^tG^0[p] = \text{Spec}(A)$ is a finite étale group scheme over $\mathcal{O}$ such that ${}^tG^0[p] \times_{\mathcal{O}} \mathbb{Z}_p^{\text{nr}}[\zeta_p] \cong (\mathbb{Z}/p\mathbb{Z})_{\mathbb{Z}_p^{\text{nr}}[\zeta_p]}^h$, i.e. $A \otimes_{\mathcal{O}} \mathbb{Z}_p^{\text{nr}}[\zeta_p] \cong \prod \mathbb{Z}_p^{\text{nr}}[\zeta_p]$. If $p = 2$, we put $\zeta_2 = -1$. We obtain a reduction map

$$\text{Hom}_{\text{gr.sch.}/\mathbb{Z}_p^{\text{nr}}[\zeta_p]}({}^tG^0[p] \times_{\mathcal{O}} \mathbb{Z}_p^{\text{nr}}[\zeta_p], \mu_{p, \mathbb{Z}_p^{\text{nr}}[\zeta_p]}) \rightarrow \text{Hom}_{\text{gr.sch.}/\overline{\mathbb{F}}_p(\epsilon)}({}^tG^0[p] \times_{\mathcal{O}} \overline{\mathbb{F}}_p(\epsilon), \mu_{p, \overline{\mathbb{F}}_p(\epsilon)}) \quad (3.3)$$

from the commutative diagram

$$\begin{array}{ccccc} \mathbb{Z}_p^{\text{nr}}[\zeta_p][X]/(X^p - 1) & \xleftarrow{\zeta_p \mapsto Y} & \mathbb{Z}_p^{\text{nr}}[X, Y]/(X^p - 1, Y^p - 1) & \xrightarrow{Y \mapsto 1 + \epsilon} & \overline{\mathbb{F}}_p(\epsilon)[X]/(X^p - 1) \\ \downarrow & & \downarrow & & \downarrow \\ \prod \mathbb{Z}_p^{\text{nr}}[\zeta_p] & \xleftarrow{\zeta_p \mapsto Y} & \prod \mathbb{Z}_p^{\text{nr}}[Y]/(Y^p - 1) & \xrightarrow{Y \mapsto 1 + \epsilon} & \prod \overline{\mathbb{F}}_p(\epsilon). \end{array}$$

Any morphism of group schemes ${}^tG^0[p] \times_{\mathcal{O}} \mathbb{Z}_p^{\text{nr}}[\zeta_p] \rightarrow \mu_{p, \mathbb{Z}_p^{\text{nr}}[\zeta_p]}$ corresponds to a Hopf algebra homomorphism as in the left column. It is easy to see that it has a unique lifting to a homomorphism as in the central column, so that it gives a homomorphism in the right column. Explicitly, a map in the left column is uniquely determined by the image of X , which is of the form $(\zeta_p^{i_1}, \dots, \zeta_p^{i_{hp}})$ for some $i_j \in \{0, \dots, p-1\}$. The corresponding map in the right column sends X to $(1 + i_1\epsilon, \dots, 1 + i_{hp}\epsilon)$. Hence, the reduction map 3.3 is injective. It is also compatible for the action induced by the Hecke correspondences. In fact, for $p > 2$, one can pass directly from the left hand side column to the right hand side via the map $\mathbb{Z}_p^{\text{nr}}[\zeta_p] \rightarrow \overline{\mathbb{F}}_p(\epsilon)$, sending ζ_p to $1 + \epsilon$.

Next, we describe the geometric Frobenius on the points ${}^tG[p](\mathbb{Q}_p^{\text{nr}}(\zeta_p))$ and ${}^tG[p](\overline{\mathbb{F}}_p(\epsilon))$. We consider the commutative diagram

$$\begin{array}{ccccc} \text{Hom}_{\text{gr.sch.}/\mathbb{Z}_p^{\text{nr}}[\zeta_p]}({}^tG^0[p] \times \mathbb{Z}_p^{\text{nr}}[\zeta_p], \mu_{p, \mathbb{Z}_p^{\text{nr}}[\zeta_p]}) & \xrightarrow{\sim} & (A \otimes \mathbb{Z}_p^{\text{nr}}[\zeta_p])^{\text{gl}} & \xrightarrow{\sim} & {}^tG[p](\mathbb{Z}_p^{\text{nr}}[\zeta_p]) \\ \downarrow \sim & & \downarrow & & \downarrow \\ \text{Hom}_{\mathbb{Z}_p^{\text{nr}}[\zeta_p] - \text{HA}}(\mathbb{Z}_p^{\text{nr}}[\zeta_p][X]/(X^p - 1), A \otimes \mathbb{Z}_p^{\text{nr}}[\zeta_p]) & \xrightarrow{\sim} & A \otimes \mathbb{Z}_p^{\text{nr}}[\zeta_p] & \xrightarrow{\sim \text{ev}} & \text{Hom}_{\mathcal{O}}({}^tA, \mathbb{Z}_p^{\text{nr}}[\zeta_p]). \end{array}$$

It is well-known that a Hopf algebra homomorphism $\psi : \mathbb{Z}_p^{\text{nr}}[\zeta_p][X]/(X^p - 1) \rightarrow A \otimes_{\mathcal{O}} \mathbb{Z}_p^{\text{nr}}[\zeta_p]$ is uniquely given by the “group-like element” $\psi(X) = \sum a_i \otimes s_i$, giving the upper left bijection. On the bottom right, we have the evaluation isomorphism $A \otimes_{\mathcal{O}} \mathbb{Z}_p^{\text{nr}}[\zeta_p] \rightarrow \text{Hom}_{\mathcal{O}}(\text{Hom}_{\mathcal{O}}(A, \mathcal{O}), \mathbb{Z}_p^{\text{nr}}[\zeta_p])$ which is defined by $\text{ev}(a \otimes s)(\varphi) = \varphi(a)s$. We use that as $\mathcal{O}$ -modules ${}^tA = \text{Hom}_{\mathcal{O}}(A, \mathcal{O})$ with $G^0[p] = \text{Spec}({}^tA)$, as well as the freeness of A . It is also well-known that the evaluation map gives rise to the upper right bijection.

Let now ϕ be the geometric Frobenius in $\text{Gal}(\mathbb{Q}_p^{\text{nr}}(\zeta_p)/\mathbb{Q}_p(\zeta_p))$. Its action on $\text{Hom}_{\mathcal{O}}({}^tA, \mathbb{Z}_p^{\text{nr}}[\zeta_p])$ is by composition. Via the evaluation map it is clear that ϕ acts on an element $a \otimes s \in A \otimes_{\mathcal{O}} \mathbb{Z}_p^{\text{nr}}[\zeta_p]$ by sending it to $a \otimes \phi(s)$. Consequently, the morphism ψ^ϕ on the left which is obtained by applying ϕ to ψ is uniquely determined by $\psi^\phi(X) = \sum a_i \otimes \phi(s_i)$. A similar statement holds for the reduction. We note that this implies the compatibility of the reduction map with the ϕ -action.

Next we show that the action of geometric Frobenius on the tangent space $G^0[p](\overline{\mathbb{F}}_p(\epsilon))$ coincides with the action induced by Verschiebung on $G_{\mathbb{F}_p}^0[p]$. The étale algebra $A \otimes_{\mathcal{O}} \mathbb{F}_p$ can be written as a product of algebras of the form $\mathbb{F}_p[X]/(f)$ with f an irreducible polynomial. An elementary calculation on the underlying rings gives the commutativity of the diagram

$$\begin{array}{ccc} \mathbb{F}_p[X]/(f) \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p(\epsilon) & \xrightarrow{F \otimes 1} & \mathbb{F}_p[X]/(f) \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p(\epsilon) \xrightarrow{1 \otimes \phi^{-1}} \mathbb{F}_p[X]/(f) \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p(\epsilon) \\ \downarrow \alpha & & \downarrow \alpha \\ \prod_{i=1}^d \overline{\mathbb{F}}_p(\epsilon) & \xrightarrow{\prod \phi^{-1}} & \prod_{i=1}^d \overline{\mathbb{F}}_p(\epsilon), \end{array} \quad (3.4)$$

where F denotes the absolute Frobenius on ${}^tG_{\mathbb{F}_p}^0[p]$ (defined by $X \mapsto X^p$), which by duality gives the Verschiebung on $G_{\mathbb{F}_p}^0[p]$. We point out that ϕ leaves ϵ invariant. Any $\overline{\mathbb{F}}_p(\epsilon)$ -Hopf algebra homomorphism $\psi : \overline{\mathbb{F}}_p(\epsilon)[X]/(X^p - 1) \rightarrow A \otimes_{\mathcal{O}} \overline{\mathbb{F}}_p(\epsilon)$ is uniquely given by $\psi(X) = \sum_i a_i \otimes s_i$, and under the identification $A \otimes_{\mathcal{O}} \overline{\mathbb{F}}_p(\epsilon) \cong \prod_{j=1}^{hp} \overline{\mathbb{F}}_p(\epsilon)$ we get $\psi(X) = (1 + i_1\epsilon, \dots, 1 + i_{hp}\epsilon)$, which is invariant under the arithmetic Frobenius of the bottom row of 3.4. Hence, $\phi^{-1}(F(\sum_i a_i \otimes s_i)) = \sum_i a_i \otimes s_i$, so that $F(\sum_i a_i \otimes s_i) = \sum_i a_i \otimes \phi(s_i)$. This proves that the geometric Frobenius and Verschiebung coincide.

We now finish the proof. In Situation (II) for $p = 2$, the Eichler-Shimura relation $T_p = \langle p \rangle F + V$ holds on the special fibre of $G[p]$ (see [8], proof of Prop. 12.8 (2)). Since F is zero on $G_{\mathbb{F}_p}^0[p]$, we get $T_p = V$ on it. As we have seen right above that V coincides with ϕ on $G_{\mathbb{F}_p}^0(\overline{\mathbb{F}}_p(\epsilon))$, we obtain the theorem for $p = 2$.

In Situation (I) we know that $G_{\mathbb{F}_p}^0[p]$ is naturally part of the p -torsion of the Jacobian of the Igusa curve $I_1(N)_{\mathbb{F}_p}$; but on the Igusa curve Verschiebung acts as T_p (see the proof of [8], 12.9 (2), for both these facts). Hence, we can argue as above and get the theorem also for $p > 2$. $\square$

More conceptual proof. In both situations, Theorem A.1 of Naumann gives an isomorphism between $G^0[p](\mathbb{Q}_p^{\text{nr}}(\zeta_p))$ and the Dieudonné module M attached to the special fibre $G_{\mathbb{F}_p}^0[p]$. Under this isomorphism the geometric Frobenius $\phi \in \text{Gal}(\mathbb{Q}_p^{\text{nr}}(\zeta_p)/\mathbb{Q}_p(\zeta_p))$ on $G^0[p](\mathbb{Q}_p^{\text{nr}}(\zeta_p))$ is identified with Verschiebung on the Dieudonné module. The isomorphism is compatible with the Hecke action. Using the same citations as at the end of the direct proof one immediately concludes that the equality $T_p = V$ holds on the Dieudonné module M , finishing the proof. $\square$

Remark 3.2 (a) Conceptually, taking $\mathbb{Z}_p^{\text{nr}}[\zeta_p]$ -points is the same as taking $\mathbb{Z}_p^{\text{nr}}$ -points of the Weil restriction from $\mathcal{O}$ to $\mathbb{Z}_p$.

(b) For a representation $\rho_{\overline{\mathfrak{m}}}$ which is unramified at p one knows that the arithmetic Frobenius Frob_p satisfies $X^2 - T_p X + \langle p \rangle = 0$. This is in accordance with Theorem 3.1. For, it gives that Frob_p acts on $G^0[p](\overline{\mathbb{Q}}_p)$ as a_p^{-1} . Due to his conventions, Gross must still twist his representation by the determinant character ϵ , so that Frob_p acts as $\epsilon(p)/a_p$. This coincides with Deligne's description of the restriction of $\rho_{\overline{\mathfrak{m}}}$ to a decomposition group at p (see, for instance, [4], Thm. 2.5, or [8], Prop. 12.1).

4 Application to multiplicities

We first state a slight strengthening of a well-known theorem by Boston, Lenstra and Ribet.

Proposition 4.1 (Boston, Lenstra, Ribet) Assume any of the two cases of Situation 1.1. Let m be an integer and $\mathbb{F} = \mathbb{T}_{\mathbb{F}_p, \overline{\mathfrak{m}}}/\overline{\mathfrak{m}}$. Then the $\mathbb{F}[\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})]$ -module $J(\overline{\mathbb{Q}})[\mathfrak{m}^{(m)}]$ is the direct sum of r copies of $\rho_{\overline{\mathfrak{m}}} \otimes \epsilon^{-1}$ for some $r \geq 1$ and Dirichlet character $\epsilon = \det(\rho_{\overline{\mathfrak{m}}})$.

The integer r is called the multiplicity of $\rho_{\overline{\mathfrak{m}}}$ on $J(\overline{\mathbb{Q}})[\mathfrak{m}^{(m)}]$. If $m = 1$, it is just called the multiplicity of $\rho_{\overline{\mathfrak{m}}}$.

Proof. The same proof as in the original proposition works. More precisely, one considers the two representations $\rho_{\overline{\mathfrak{m}}} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{GL}_2(\mathbb{F})$ and $\sigma : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow \text{Aut}(J(\overline{\mathbb{Q}})[\mathfrak{m}^{(m)}])$. By Chebotarev's density theorem we know that every conjugacy class of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})/\ker(\sigma \otimes \epsilon)$ is hit by a Frobenius element Frob_l for some $l \nmid Npm$.

The Eichler-Shimura congruence relation $T_l = \langle l \rangle F + V$ holds on $J_{\mathbb{F}_l}$ (taking J here over $\mathbb{Z}[\frac{1}{Np}])$ for all primes $l \nmid Npm$. Hence, the minimal polynomial of Frob_l on the Jacobian divides $X^2 - T_l/\langle l \rangle \cdot X + l/\langle l \rangle$. But T_l acts as a_l on $J(\overline{\mathbb{Q}})[\mathfrak{m}^{(m)}]$ and $X^2 - a_l X + \epsilon(l)l$ (with $T_l \equiv a_l \pmod{\overline{\mathfrak{m}}}$) is the characteristic polynomial of $\rho_{\overline{\mathfrak{m}}}(\text{Frob}_l)$. Consequently, $(\sigma \otimes \epsilon)(g)$ is annihilated by the characteristic polynomial of $\rho_{\overline{\mathfrak{m}}}(g)$ for all $g \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Hence, Theorem 1 of [1] gives the result. $\square$

The next corollary says that one can read off multiplicities from properties of Hecke algebras.

Corollary 4.2 *Assume any of the two cases of Situation 1.1. Let r be the multiplicity of $\rho_{\overline{\mathfrak{m}}}$. Then the relation*

$$r = \frac{1}{2}(\dim_{\mathbb{F}} \mathbb{T}_{\mathbb{F}_p, \overline{\mathfrak{m}}}[\overline{\mathfrak{m}}] + 1)$$

holds, where $\mathbb{F} = \mathbb{T}_{\mathbb{F}_p, \overline{\mathfrak{m}}} / \overline{\mathfrak{m}}$.

Proof. We note that in [2] Buzzard explains the exactness of the sequence

$$0 \rightarrow G^0(\overline{\mathbb{Q}}_p)[\mathfrak{m}] \rightarrow G(\overline{\mathbb{Q}}_p)[\mathfrak{m}] \rightarrow G^e(\overline{\mathbb{Q}}_p)[\mathfrak{m}] \rightarrow 0.$$

Via Corollary 2.3 we obtain the exact sequence

$$0 \rightarrow \mathbb{T}_{\mathbb{F}_p, \overline{\mathfrak{m}}}[\overline{\mathfrak{m}}] \rightarrow J_1(Np)(\overline{\mathbb{Q}}_p)[\mathfrak{m}] \rightarrow (\mathbb{T}_{\mathbb{F}_p, \overline{\mathfrak{m}}} / \overline{\mathfrak{m}})^\vee \rightarrow 0,$$

from which one reads off the claim by counting dimensions. $\square$

Theorem 4.3 *Assume any of the two cases of Situation 1.1 and that $\rho_{\overline{\mathfrak{m}}}$ is of weight one. Then the following statements are equivalent.*

- (a) *The representation $\rho_{\overline{\mathfrak{m}}}$ comes from a Katz cusp form of weight 1 on $\Gamma_1(N)$ over $\mathbb{F}_p$ and the multiplicity of $\rho_{\overline{\mathfrak{m}}}$ is 1.*
- (b) *$\mathbb{T}_{\mathbb{F}_p, \overline{\mathfrak{m}}}[\overline{\mathfrak{m}}] \subsetneq \mathbb{T}_{\mathbb{F}_p, \overline{\mathfrak{m}}}[\overline{\mathfrak{m}}^{(p)}]$*
- (c) *T_p does not act as scalars on $\mathbb{T}_{\mathbb{F}_p, \overline{\mathfrak{m}}}[\overline{\mathfrak{m}}^{(p)}]$.*
- (d) *The multiplicity of $\rho_{\overline{\mathfrak{m}}}$ is 1, its multiplicity on $J(\overline{\mathbb{Q}})[\overline{\mathfrak{m}}^{(p)}]$ is 2, and $\rho_{\overline{\mathfrak{m}}}(\text{Frob}_p)$ is non-scalar.*

Proof. (a) $\Rightarrow$ (b) : By Cor. 4.2 and Nakayama's Lemma $\mathbb{T}_{\mathbb{F}_p, \overline{\mathfrak{m}}}$ is Gorenstein, i.e. it is isomorphic to its dual as a module over itself. By the q -expansion principle, the dual is $S_p(\Gamma_1(N), \mathbb{F}_p)_{\overline{\mathfrak{m}}}$. By [5], Prop. 6.2, the existence of a corresponding weight 1 form is equivalent to $S_p(\Gamma_1(N), \mathbb{F}_p)_{\overline{\mathfrak{m}}}[\overline{\mathfrak{m}}^{(p)}]$ being 2-dimensional. This establishes (b), since by the q -expansion principle $S_p(\Gamma_1(N), \mathbb{F}_p)_{\overline{\mathfrak{m}}}[\overline{\mathfrak{m}}]$ is 1-dimensional.

(b) $\Rightarrow$ (c) : This is evident.

(c) $\Rightarrow$ (d) : First of all, $\mathbb{T}_{\mathbb{F}_p, \overline{\mathfrak{m}}}[\overline{\mathfrak{m}}^{(p)}]$ is at least 2-dimensional (as $\mathbb{T}_{\mathbb{F}_p, \overline{\mathfrak{m}}} / \overline{\mathfrak{m}}$ -vector space). From Theorem 3.1 we know that T_p acts as the inverse of Frob_p on $G^0[p](\overline{\mathbb{Q}})$. We conclude that $\rho_{\overline{\mathfrak{m}}}(\text{Frob}_p)$ cannot be scalar. On $\mathbb{T}_{\mathbb{F}_p, \overline{\mathfrak{m}}}[\overline{\mathfrak{m}}]$ the action of T_p is by scalars. If the multiplicity r of $\rho_{\overline{\mathfrak{m}}}$ were not 1, then $\mathbb{T}[\overline{\mathfrak{m}}] = G^0[p](\overline{\mathbb{Q}})[\overline{\mathfrak{m}}]$ would have dimension $2r - 1 > 1$ (by the proof of Cor. 4.2). From Theorem 3.1 we obtain a contradiction. We note that this argument, showing that $\rho_{\overline{\mathfrak{m}}}(\text{Frob}_p)$ being non-scalar implies that the multiplicity of $\rho_{\overline{\mathfrak{m}}}$ is 1, did not use the statement of (c). If the multiplicity s of $\rho_{\overline{\mathfrak{m}}}$ on $J(\overline{\mathbb{Q}})[\overline{\mathfrak{m}}^{(p)}]$ were bigger than 2, then $\mathbb{T}_{\mathbb{F}_p, \overline{\mathfrak{m}}}[\overline{\mathfrak{m}}^{(p)}]$ would be at least 4-dimensional. Then it follows that it must contain at least two linearly independent eigenvectors for T_p , contradicting the fact that $\mathbb{T}_{\mathbb{F}_p, \overline{\mathfrak{m}}}[\overline{\mathfrak{m}}]$ is 1-dimensional.

(d) $\Rightarrow$ (a) : Clearly, $\overline{\mathfrak{m}} \neq \overline{\mathfrak{m}}^{(p)}$. Hence, $\mathbb{T}_{\mathbb{F}_p, \overline{\mathfrak{m}}}/\overline{\mathfrak{m}} \neq \mathbb{T}_{\mathbb{F}_p, \overline{\mathfrak{m}}}/\overline{\mathfrak{m}}^{(p)}$ and, dually,

$$S_p(\Gamma_1(N), \mathbb{F}_p)_{\overline{\mathfrak{m}}}[\overline{\mathfrak{m}}] \subsetneq S_p(\Gamma_1(N), \mathbb{F}_p)_{\overline{\mathfrak{m}}}[\overline{\mathfrak{m}}^{(p)}],$$

which implies the existence of a corresponding weight 1 form, again by [5], Prop. 6.2. $\square$

In [2] Buzzard proved that the multiplicity of $\rho_{\overline{\mathfrak{m}}}$ is 1 if $\rho_{\overline{\mathfrak{m}}}(\text{Frob}_p)$ is non-scalar. We obtain that this is in fact an equivalence (under a standard assumption in the case $p = 2$).

Corollary 4.4 *Assume any of the two cases of Situation 1.1 and that $\rho_{\overline{\mathfrak{m}}}$ is of weight one. If $p = 2$, also assume that a weight 1 Katz form of level N exists which gives rise to $\rho_{\overline{\mathfrak{m}}}$.*

Then the multiplicity of $\rho_{\overline{\mathfrak{m}}}$ is 1 if and only if $\rho_{\overline{\mathfrak{m}}}(\text{Frob}_p)$ is non-scalar.

Proof. By [4], Theorem 4.5, together with the remark at the end of the introduction to that article, the existence of the corresponding weight 1 form is also guaranteed for $p > 2$. If the multiplicity is 1, Theorem 4.3 gives that $\rho_{\overline{\mathfrak{m}}}(\text{Frob}_p)$ is non-scalar. On the other hand, if $\rho_{\overline{\mathfrak{m}}}(\text{Frob}_p)$ is non-scalar, the argument used in the implication (c) $\Rightarrow$ (d) of Theorem 4.3 shows that the multiplicity is 1. $\square$

Corollary 4.5 *Assume any of the two cases of Situation 1.1. If $p = 2$, also assume that if $\rho_{\overline{\mathfrak{m}}}$ is of weight one, then a weight 1 Katz form of level N exists which gives rise to $\rho_{\overline{\mathfrak{m}}}$.*

Then the multiplicity of $\rho_{\overline{\mathfrak{m}}}$ on $J[\overline{\mathfrak{m}}^{(p)}]$ is 1 if and only if $\rho_{\overline{\mathfrak{m}}}$ is ramified at p .

Proof. If $\rho_{\overline{\mathfrak{m}}}$ is ramified at p , the result is Theorem 6.1 of [2]. Suppose now that $\rho_{\overline{\mathfrak{m}}}$ is unramified at p . If $\rho_{\overline{\mathfrak{m}}}(\text{Frob}_p)$ is scalar, the corollary follows from Corollary 4.4. If $\rho_{\overline{\mathfrak{m}}}(\text{Frob}_p)$ is non-scalar, then the result follows from Corollary 4.4 together with the implication (a) $\Rightarrow$ (d) of Theorem 4.3. $\square$

Corollary 4.6 *Assume any of the two cases of Situation 1.1 and that $\rho_{\overline{\mathfrak{m}}}$ is of weight one. Assume also that the multiplicity of $\rho_{\overline{\mathfrak{m}}}$ on $J(\overline{\mathbb{Q}})[\overline{\mathfrak{m}}^{(p)}]$ is 2. Then the following statements are equivalent.*

(a) *The multiplicity of $\rho_{\overline{\mathfrak{m}}}$ is 1 and a weight 1 Katz form of level N exists which gives rise to $\rho_{\overline{\mathfrak{m}}}$.*

(b) *$\rho_{\overline{\mathfrak{m}}}(\text{Frob}_p)$ is non-scalar.*

Proof. We have seen the implication (a) $\Rightarrow$ (b) above. As in the proof of Thm. 4.3, we obtain from $\rho_{\overline{\mathfrak{m}}}(\text{Frob}_p)$ being non-scalar that the multiplicity of $\rho_{\overline{\mathfrak{m}}}$ is 1. From the assumption the inequality $\overline{\mathfrak{m}} \neq \overline{\mathfrak{m}}^{(p)}$ follows, implying the existence of the weight 1 form as above by [5], Prop. 6.2. $\square$

If one could prove that the multiplicity of $\rho_{\overline{\mathfrak{m}}}$ on $J(\overline{\mathbb{Q}})[\overline{\mathfrak{m}}^{(p)}]$ is always equal to 2 in the unramified situation, Corollary 4.6 would extend weight lowering for $p = 2$ to $\rho_{\overline{\mathfrak{m}}}(\text{Frob}_p)$ being non-scalar.

A Appendix

By Niko Naumann¹

Let p be a prime, $A := \mathbb{Z}_p$, $A' := \mathbb{Z}_p[\zeta_p]$, $K := \mathbb{Q}_p$, $K' := \mathbb{Q}_p(\zeta_p)$ and $K' \subseteq \overline{K}$ an algebraic closure. We have the inertia sub-group $I \subseteq G_{K'} := \text{Gal}(\overline{K}/K')$ and for a $G_{K'}$ -module V we denote by τ the geometric Frobenius acting on the inertia invariants V^I . If G/A' is a finite flat group-scheme, always assumed to be commutative, we denote by M the Dieudonné-module of its special fiber and by $V : M \rightarrow M$ the Verschiebung.

Theorem A.1 *Let G/A' be a finite flat group-scheme which is connected with étale Cartier-dual and annihilated by multiplication with p . Then $G(\overline{K})^I = G(\overline{K})$ and there is an isomorphism $\phi : G(\overline{K})^I \rightarrow M$ of $\mathbb{F}_p$ -vector spaces such that $\phi \circ \tau = V \circ \phi$.*

The assumption that $pG = 0$ cannot be dropped in Theorem A.1:

Proposition A.2 *For every $n \geq 2$ there is a finite flat group-scheme G/A' of order p^n which is connected with an étale dual and such that $G(\overline{K})^I \simeq \mathbb{Z}/p\mathbb{Z}$ with τ acting trivially and $V \neq 1$ on the Dieudonné-module of the special fiber of G .*

Proof of Theorem A.1. Denoting by G' the Cartier-dual of G/A' we have an isomorphism of $G_{K'}$ -modules

$$G(\overline{K}) \simeq \text{Hom}(G'(\overline{K}), \mu_{p^\infty}(\overline{K})) \stackrel{(pG'=0)}{=} \text{Hom}(G'(\overline{K}), \mu_p(\overline{K})).$$

Since $G'(\overline{K})$ is unramified because G/A' is étale and $\mu_p(\overline{K})$ is unramified because $\zeta_p \in K'$ we see that $G(\overline{K})^I = G(\overline{K})$. Letting p^n denote the order of G we have

$$\dim_{\mathbb{F}_p}(G(\overline{K})^I) = \dim_{\mathbb{F}_p}(G(\overline{K})) = n = \dim_{\mathbb{F}_p}(M).$$

In the rest of the proof we use the explicit quasi-inverse to J.-M. Fontaine's functor associating with G a finite Honda system in order to determine the action of τ on $G(\overline{K})^I$ [7],[3].

Let (M, L) be the finite Honda system over A' associated with G/A' . Recall that M is the Dieudonné-module of the special fiber of G and $L \subseteq M_{A'}$ is an A' -sub-module where $M_{A'}$ is an A' -module functorially associated with M [7, Ch. IV, §2].

We claim that $L = M_{A'}$: Let $\mathfrak{m} \subseteq A'$ denote the maximal ideal. Using the notation of [3, Section 2], the defining epimorphism of A' -modules $M_{A'} \rightarrow \text{coker}(\mathcal{F}_M)$ factors through an epimorphism $M_{A'}/\mathfrak{m}M_{A'} \rightarrow \text{coker}(\mathcal{F}_M)$ because $\mathfrak{m} \cdot \text{coker}(\mathcal{F}_M) = 0$ [3, Lemma 2.4]. Denoting by l the length of a module we have

$$l_{A'}(\text{coker}(\mathcal{F}_M)) \stackrel{[3,2.4]}{=} l_A(\ker F) = l_A(\ker(p : M \rightarrow M)) = l_A(M) = n$$

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because $\ker F = \ker(p : M \rightarrow M)$ since V is bijective, and $pM = 0$. On the other hand, the canonical morphism of A' -modules $\iota_M : M \otimes_A A' \rightarrow M_{A'}$ is an isomorphism by [7, Ch. IV, Proposition 2.5] using again that V is bijective. Thus

$$l_{A'}(M_{A'}/\mathfrak{m}M_{A'}) = l_{A'}(M \otimes_A A'/\mathfrak{m}) = l_{A'}(M/pM) = l_A(M) = n$$

and $M_{A'}/\mathfrak{m}M_{A'} \xrightarrow{\sim} \text{coker}(\mathcal{F}_M)$. Since $L/\mathfrak{m}L \xrightarrow{\sim} \text{coker}(\mathcal{F}_M)$ holds for every finite Honda system we see that the inclusion $L \subseteq M_{A'}$ induces an isomorphism $L/\mathfrak{m}L \xrightarrow{\sim} M_{A'}/\mathfrak{m}M_{A'}$ and Nakayama's lemma implies that $L = M_{A'}$.

Fix $\pi \in \overline{K}$ with $\pi^{p-1} = -p$, then $K' = K(\pi)$: This is obvious for $p = 2$ and for $p \neq 2$ it follows from local class field theory and the norm computations $N_K^{K'}(\zeta_p - 1) = N_K^{K(\pi)}(\pi) = p$. Note that $\pi \in A'$ is a local uniformizer. Let K'^{ur} denote the completion of the maximal unramified extension of K' inside $\overline{K}$ and $\mathcal{O} \subseteq K'^{ur}$ its ring of integers.

By [7, Remarque on p. 218] and the fact that $L = M_{A'}$ we see that reduction induces an isomorphism

$$G(\overline{K})^I = G(K'^{ur}) = G(\mathcal{O}) \xrightarrow{\sim} \left\{ \phi \in \text{Hom}_{D_{\mathbb{F}_p}}(M, CW_{\mathbb{F}_p}(\pi\mathcal{O}/\pi^2\mathcal{O})) \mid w'^c \circ \phi_{A'} = 0 \right\} \quad (1.5)$$

where $D_{\mathbb{F}_p} = \mathbb{F}_p[F, V]$ is the Dieudonné-ring, CW denote Witt-covectors [7, Ch. II, §1],

$$w'^c : CW_{\mathbb{F}_p}(\pi\mathcal{O}/\pi^2\mathcal{O})_{A'} \rightarrow K'^{ur}/\pi^2\mathcal{O}$$

is as in [7, Ch. IV, §3] and $\phi_{A'} : M_{A'} \rightarrow CW_{\mathbb{F}_p}(\pi\mathcal{O}/\pi^2\mathcal{O})_{A'}$ is induced by ϕ . By construction of w'^c we have, for every $\phi \in \text{Hom}_{D_{\mathbb{F}_p}}(M, CW_{\mathbb{F}_p}(\pi\mathcal{O}/\pi^2\mathcal{O}))$, a commutative diagram

$$\begin{array}{ccccc} M_{A'} & \xrightarrow{\phi_{A'}} & CW_{\mathbb{F}_p}(\pi\mathcal{O}/\pi^2\mathcal{O})_{A'} & \xrightarrow{w'^c} & K'^{ur}/\pi^2\mathcal{O} \\ & \nearrow \iota_{CW_{\mathbb{F}_p}(\pi\mathcal{O}/\pi^2\mathcal{O})} & \uparrow & \nearrow \tilde{w} & \uparrow w^c \\ \iota_M \simeq \uparrow & & CW_{\mathbb{F}_p}(\pi\mathcal{O}/\pi^2\mathcal{O}) \otimes_A A' & \xleftarrow{\phi} & CW_{\mathbb{F}_p}(\pi\mathcal{O}/\pi^2\mathcal{O}) \\ & \nearrow \phi \otimes 1 & \nearrow \phi & & \\ M \otimes_A A' & \xleftarrow{\quad} & M & & \end{array}$$

in which $w^c((x_{-n})_{n \geq 0}) = \sum_{n=0}^{\infty} p^{-n} \hat{x}_{-n}^p$ with $\hat{x}_{-n} \in \pi\mathcal{O}$ lifting x_{-n} , $\tilde{w} = w^c \otimes 1$ is the A' -linear extension of w^c and $\iota_{CW_{\mathbb{F}_p}(\pi\mathcal{O}/\pi^2\mathcal{O})}$ is surjective by [7, Ch. IV, Proposition 2.5] since $CW_{\mathbb{F}_p}(\pi\mathcal{O}/\pi^2\mathcal{O})$ is V -divisible. It is easy to see that we have

$$w'^c \circ \phi_{A'} = 0 \Leftrightarrow w^c \circ \phi = 0. \quad (1.6)$$

Combining (1.6) and (1.5) we obtain an isomorphism

$$G(\overline{K})^I \xrightarrow{\sim} \{ \phi \in \text{Hom}_{D_{\mathbb{F}_p}}(M, CW_{\mathbb{F}_p}(\pi\mathcal{O}/\pi^2\mathcal{O})) \mid w^c \circ \phi = 0 \}. \quad (1.7)$$

Now we need to study $\ker(w^c)$. We will use the isomorphism of $\overline{\mathbb{F}}_p$ - vector spaces

$$\pi\mathcal{O}/\pi^2\mathcal{O} \xrightarrow{\cdot\pi} \mathcal{O}/\pi\mathcal{O} \simeq \overline{\mathbb{F}}_p \quad (1.8)$$

to describe elements of $CW_{\mathbb{F}_p}(\pi\mathcal{O}/\pi^2\mathcal{O})$ as covectors $(y_{-n})_{n \geq 0}$ with $y_{-n} \in \overline{\mathbb{F}}_p$. Of course, since (1.8) is not multiplicative, some care has to be taken with this. We denote by $\sigma : \overline{\mathbb{F}}_p \rightarrow \overline{\mathbb{F}}_p$, $\sigma(x) = x^p$ the absolute Frobenius and claim that

$$\ker(w^c) = \{(y_{-n})_n \mid y_{-n} \in \overline{\mathbb{F}}_p, y_{-1} = y_0^{\sigma^{-1}}\}. \quad (1.9)$$

To see this, let $(x_{-n})_n \in CW_{\mathbb{F}_p}(\pi\mathcal{O}/\pi^2\mathcal{O})$ be given, choose $\hat{x}_{-n} \in \pi\mathcal{O}$ lifting x_{-n} and write $\hat{x}_{-n} = \pi\hat{y}_{-n}$ with $\hat{y}_{-n} \in \mathcal{O}$. Then we compute in $K'^{ur}/\pi^2\mathcal{O}$:

$$w^c((x_{-n})) = \sum_{n=0}^{\infty} p^{-n} (\pi\hat{y}_{-n})^{p^n} \stackrel{(\pi^{p-1} = -p)}{=} \sum_{n=0}^{\infty} (-1)^n \pi^{p^n - n(p-1)} \hat{y}_{-n}^{p^n} = \pi(\hat{y}_0 - \hat{y}_{-1}^p),$$

using that $p^n - n(p-1) \geq 2$ for all $n \geq 2$. Now (1.9) is obvious.

Next, we claim that the subset

$$CW_{\mathbb{F}_p}(\pi\mathcal{O}/\pi^2\mathcal{O}) \supseteq \mathcal{M} := \{(y_0^{\sigma^{-n}})_{n \geq 0} \mid y_0 \in \overline{\mathbb{F}}_p\} \quad (1.10)$$

is a $D_{\mathbb{F}_p}$ -sub-module. First note that $F = 0$ on $CW_{\mathbb{F}_p}(\pi\mathcal{O}/\pi^2\mathcal{O})$ so we will consider it as a $D_{\mathbb{F}_p}/F = \mathbb{F}_p[V]$ -module in the following. Since all products in $\pi\mathcal{O}/\pi^2\mathcal{O}$ are zero we have

$$(x_{-n}) + (y_{-n}) = (x_{-n} + y_{-n})$$

in $CW_{\mathbb{F}_p}(\pi\mathcal{O}/\pi^2\mathcal{O})$ and $\mathcal{M}$ is indeed a $\mathbb{F}_p$ -sub-module, visibly stable under V .

We claim that the inclusion (1.10) induces an isomorphism

$$\mathrm{Hom}_{\mathbb{F}_p[V]}(M, \mathcal{M}) \xrightarrow{\cong} \{\phi \in \mathrm{Hom}_{D_{\mathbb{F}_p}}(M, CW_{\mathbb{F}_p}(\pi\mathcal{O}/\pi^2\mathcal{O})) \mid w^c \circ \phi = 0\}. \quad (1.11)$$

Since $\mathcal{M} \subseteq \ker(w^c)$ by (1.9) we only need to see that a $\mathbb{F}_p[V]$ -linear morphism

$$\phi : M \rightarrow CW_{\mathbb{F}_p}(\pi\mathcal{O}/\pi^2\mathcal{O})$$

with $\phi(M) \subseteq \ker(w^c)$ factors through $\mathcal{M}$: For every $m \in M$ and $n \geq 0$ we have, writing $\phi(m) =: (y_{-n})$ with $y_{-n} \in \overline{\mathbb{F}}_p$,

$$0 = w^c(\phi(V^n m)) = w^c(V^n(\phi(m))) = w^c((\dots, y_{-n-1}, y_{-n})),$$

thus $y_{-n-1} = y_{-n}^{\sigma^{-1}}$ by (1.9) and as this is true for every $n \geq 0$ we get $\phi(m) \in \mathcal{M}$.

To proceed, note that

$$\mathcal{M} \rightarrow \overline{\mathbb{F}}_p, (y_0^{\sigma^{-n}}) \mapsto y_0 \quad (1.12)$$

is an isomorphism of $\mathbb{F}_p[V]$ -modules if one defines $V(\alpha) := \alpha^{\sigma^{-1}}$ for $\alpha \in \overline{\mathbb{F}_p}$. Denoting by $\Phi : G(\overline{K})^I \xrightarrow{\sim} \text{Hom}_{\mathbb{F}_p[V]}(M, \overline{\mathbb{F}_p})$ the isomorphism obtained by combining (1.7), (1.11) and (1.12), by construction we have a commutative diagram

$$\begin{array}{ccc} G(\overline{K})^I & \xrightarrow{\Phi} & \text{Hom}_{\mathbb{F}_p[V]}(M, \overline{\mathbb{F}_p}) \\ \downarrow \tau & & \downarrow \text{Hom}(V, \overline{\mathbb{F}_p}) \\ G(\overline{K})^I & \xrightarrow{\Phi} & \text{Hom}_{\mathbb{F}_p[V]}(M, \overline{\mathbb{F}_p}). \end{array} \quad (1.13)$$

Let e_i (resp. ϕ_i) ($1 \leq i \leq n$) be an $\mathbb{F}_p$ -basis of M (resp. $\text{Hom}_{\mathbb{F}_p[V]}(M, \overline{\mathbb{F}_p})$) and define $Ve_i := \sum_j a_{ij}e_j$, hence $A := (a_{ij}) \in \text{Gl}_n(\mathbb{F}_p)$, $\psi_i := \text{Hom}(V, \overline{\mathbb{F}_p})(\phi_i) =: \sum_j b_{ij}\phi_j$, hence $B := (b_{ij}) \in \text{Gl}_n(\mathbb{F}_p)$ and $C := (\phi_i(e_j)) \in \text{Gl}_n(\overline{\mathbb{F}_p})$. By definition, A is a representing matrix of $V : M \rightarrow M$ and by (1.13) B is a representing matrix for τ . So we will be done if we can show that A and B are conjugate over $\mathbb{F}_p$.

From the computation

$$\psi_i(e_j) = \phi_i(Ve_j) = \sum_k a_{jk}\phi_i(e_k) = \sum_k b_{ik}\phi_k(e_j)$$

we obtain ${}^tA = C^{-1}BC$. Now recall that over every field κ two square matrices with coefficients in κ which are conjugate over an algebraic closure of κ are conjugate over κ and, furthermore, that every square matrix with coefficient in κ is conjugate, over κ , to its transpose. Hence A is indeed conjugate to B over $\mathbb{F}_p$. $\square$

Remark A.3 *Inspecting the above proof we see that for G/A' connected with étale dual (not necessarily annihilated by p) we have a commutative diagram*

$$\begin{array}{ccc} G(\overline{K})^I & \xrightarrow[\simeq]{\Phi} & \text{Hom}_{\mathbb{F}_p[V]}(M/FM, \overline{\mathbb{F}_p}) \\ \downarrow \tau & & \downarrow \text{Hom}(V, \overline{\mathbb{F}_p}) \\ G(\overline{K})^I & \xrightarrow[\simeq]{\Phi} & \text{Hom}_{\mathbb{F}_p[V]}(M/FM, \overline{\mathbb{F}_p}). \end{array}$$

Proof of Proposition A.2. Define a finite Honda system over A' by

$$M := \mathbb{Z}/p^n\mathbb{Z}, \quad 1 \neq V \in 1 + p(\mathbb{Z}/p^n\mathbb{Z}) \subseteq (\mathbb{Z}/p^n\mathbb{Z})^* = \text{Aut}_{\mathbb{Z}_p}(M), \quad F := pV^{-1}, \quad L := M_{A'}.$$

It is easy to see that this is indeed a finite Honda system. For the corresponding group G/A' we have by Remark A.3

$$G(\overline{K})^I \simeq \text{Hom}_{\mathbb{F}_p[V]}(M/FM, \overline{\mathbb{F}_p}) = \overline{\mathbb{F}_p}^{V=1} = \mathbb{F}_p$$

with trivial geometric Frobenius, note that V is the identity on M/FM , but $V \neq 1$. $\square$

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