



**On Maximal Proper Subgroups of
Field Automorphism Groups**

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ON MAXIMAL PROPER SUBGROUPS OF FIELD AUTOMORPHISM GROUPS

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ABSTRACT. Let G be the automorphism group of an extension F/k of algebraically closed fields of characteristic zero of transcendence degree n , $1 \leq n \leq \infty$. In this paper we

- construct some maximal closed non-open, and some (all, in the case of countable transcendence degree) maximal open subgroups of G ;
- describe, in the case of countable transcendence degree, the automorphism subgroups over the intermediate subfields (a question of Krull, [K2]);
- construct a subfunctor Γ of the identity functor on the category of smooth representations of G , coincident (via the forgetful functor) with the functor Γ on the category of smooth admissible semi-linear representations of G constructed in [R2] in the case $n = \infty$ and $k = \overline{\mathbb{Q}}$.

1. STRUCTURE OF G

Let F/k be an extension of countable or finite transcendence degree n , $1 \leq n \leq \infty$, of algebraically closed fields of characteristic zero, and $G = G_{F/k}$ be its automorphism group.

Following [Jac, IIII-III, Sh, I] (and generalizing the case [K1] of algebraic extension), consider G as a topological group with the base of open subgroups given by the stabilizers of finite subsets of F . Then G is a totally disconnected Hausdorff group, and for any intermediate subfield $k \subseteq K \subseteq F$ the topology on $G_{F/K}$ coincides with the restriction of the topology on G . G is locally compact if and only if $n < \infty$.

The classical morphism $\beta : \{\text{subfields } F \text{ over } k\} \hookrightarrow \{\text{closed subgroups of } G\}$, given by $K \mapsto G_{F/K}$, is injective, inverts the inclusions, transforms the compositum of subfields to the intersection of subgroups, and respects the units: $k \mapsto G$. The image of β is stable under the passages to sup-/sub- groups with compact quotients; β identifies the subfields over which F is algebraic with the compact subgroups of G ([Jac, IIII-III, Sh, I]).

In particular, the proper subgroups in the image of β are the compact subgroups in the case $n = 1$.

The map $H \mapsto F^H$, left inverse of β , preserves the order, but it is not a morphism.

Let $A\Pi$ be the set of algebraically closed subfields of F of finite transcendence degree over k . There is a morphism of partially ordered commutative associative unitary monoids (transforming the intersection of subgroups to the algebraic closure of the compositum of subfields) $\alpha : \{\text{open subgroups of } G\} \twoheadrightarrow A\Pi$, uniquely determined by the condition $G_{F/\alpha(U)} \subseteq U$ and the transcendence degree of $\alpha(U)$ over k is minimal. Details are in Proposition 2.6.

Theorem 1.1 ([R1]). 1. *The subgroup G° of G , generated by the compact subgroups, is open and topologically simple, if $n < \infty$. If $n = \infty$ then G° is dense in G .*
 2. *Any closed normal proper subgroup of G is trivial, if $n = \infty$.*

If $n < \infty$, the left G -action on the one-dimensional oriented $\mathbb{Q}$ -vector space of right-invariant measures on G gives rise to a surjective homomorphism, the modulus, $\chi : G \rightarrow \mathbb{Q}_+^\times$, which is trivial on G° . However, I know nothing about the discrete group $\ker \chi / G^\circ$.

Then one can characterize the image of β in the case $n = \infty$ in the following 4 steps.

1. The normalizers $G_{\{F,L\}/k}$ of $G_{F/L}$ in G for all $L \in A\Pi \setminus \{k\}$ are the maximal open proper subgroups of G . (This follows from Proposition 2.6.)
2. The subgroups $G_{F/L}$ of G for all $L \in A\Pi \setminus \{k\}$ are minimal closed non-trivial normal subgroups in $G_{\{F,L\}/k}$ from (1). (This follows from Theorem 1.1 (2).)
3. The subgroups $G_{F/L}$ of G for all non-trivial extensions $L \subset F$ of k of finite type are the open subgroups containing normal co-compact subgroups of type $G_{F/\bar{L}}$ from (2). (This follows from the above Galois theory for $\bar{L}/k$.)
4. The proper subgroups in the image of β are intersections of subgroups from (3).

Remark. The subgroups $G_{F/L}$ of G for all extensions $L \subset F$ of k of finite type and transcendence degree one are the subgroups $G_{F/L}$ from (3) with the only maximal proper subgroup of G containing them.

2. MAXIMAL OPEN SUBGROUPS

In this section we assume that $\text{tr.deg}(F/k) = n \leq \infty$.

Lemma 2.1. *If $\bar{L}_1, \bar{L}_2$ are proper subfields of F and $\text{tr.deg}(\bar{L}_1/\bar{L}_1 \cap \bar{L}_2) > 1$ then the group H , generated by $G_{F/\bar{L}_1}$ and $G_{F/\bar{L}_2}$, contains $G_{F/\bar{L}}$, where $\bar{L}$ is a subfield of F , not containing L_1 , and such that $\bar{L}_1 \cap \bar{L}$ contains $\bar{L}_1 \cap \bar{L}_2$ as a proper subfield.*

Proof. Set $k := \bar{L}_1 \cap \bar{L}_2$. Clearly, H contains $G_{F/\sigma(\bar{L}_1)}$ for any $\sigma \in G_{F/\bar{L}_2}$. As $\sigma(\bar{L}_1)$ is algebraically closed and $\bar{L}_1 \cap \sigma(\bar{L}_1) \supseteq k$, it remains to choose $\sigma \in G_{F/\bar{L}_2}$ such that $\sigma^{-1}(\bar{L}_1) \not\subseteq \bar{L}_1$ and $\bar{L}_1 \cap \sigma(\bar{L}_1) \neq k$, and to set $L := \sigma(\bar{L}_1)$.

Let $x \in \bar{L}_1 - k$ and $y \in \bar{L}_1 - k(x)$. We need $\sigma \in G_{F/\bar{L}_2}$ such that $\sigma x \in \bar{L}_1$ and $\sigma^{-1}y \notin \bar{L}_1$. If y is not in $\bar{L}_2(x)$ then such $\sigma \in G_{F/\bar{L}_2}$, clearly, exists.

Let now y be algebraic over $L_2(x)$. Then x and y are related by a polynomial $P(x, y) = 0$ with coefficients in $\bar{L}_2$. Let us enumerate the non-zero coefficients of P by a set $\{0, \dots, N\}$. We shall assume that the polynomial $P(X, Y) = \sum_{s=0}^N p_s X^{i_s} Y^{j_s}$ is irreducible over $\bar{L}_2$, and $p_0 = 1$. Suppose that $\sigma^{-1}(\bar{L}_1) \subseteq \bar{L}_1$ for any $\sigma \in G_{F/\bar{L}_2}$ such that $\sigma x \in \bar{L}_1$. Then $-(dP)(\sigma x, \sigma y) = P_I(\sigma x, \sigma y)d(\sigma x) + P_{II}(\sigma x, \sigma y)d(\sigma y) \in F \otimes_{L_1} \Omega_{L_1/k}^1$, which can be rewritten as $\sum_{s=1}^N \sigma x^{i_s} \sigma y^{j_s} dp_s \in F \otimes_{L_1} \Omega_{L_1/k}^1$. It remains to show that the whole space F^N is the linear envelope of the vectors $(x^{i_1} \sigma y^{j_1}, \dots, x^{i_N} \sigma y^{j_N})$ for $\sigma \in G_{F/\bar{L}_2}$ and $\sigma x \in \bar{L}_1$ (since then $dp_s \in F \otimes_{L_1} \Omega_{L_1/k}^1$ for all s , i.e. $p_s \in k$).

If $\sum_{s=1}^N \lambda_s \sigma x^{i_s} \sigma y^{j_s} = 0$ for $\sigma \in G_{F/\bar{L}_2}$ as above, then we may assume that $\lambda_1 = 1$ and the number M of non-zero λ_s is minimal: $\sum_{s=1}^M \lambda_s \sigma x^{i_s} \sigma y^{j_s} = 0$. Subtracting from this equality its image under the action of an appropriate σ' , we get that $\lambda_s \in \bigcap_{z \in \bar{L}_1 - k} \bar{L}_2(z) \subseteq \bigcap_{b=1}^\infty \bar{L}_2(x^b) = \bar{L}_2$, contradicting the minimality of the polynomial P . $\square$

Corollary 2.2. *If $\text{tr.deg}(L_1/L_1 \cap L_2) < \infty$, L_1, L_2 are proper algebraically closed subfields in F and L_1 is not contained in L_2 then $H \supseteq G_{F/L}$ for some algebraically closed subfield L such that $\text{tr.deg}(L_1/L_1 \cap L) = 1$.*

Proof. The set of algebraically closed subfields of L , not containing L_1 and such that $H \supseteq G_{F/L}$, is non-empty, since contains L_2 . Let $\text{tr.deg}(L_1/L_1 \cap L)$ be minimal. According to Lemma 2.1, $\text{tr.deg}(L_1/L_1 \cap L) = 1$. $\square$

Lemma 2.3. *If L_1, L_2 are algebraically closed subfields in F and $\text{tr.deg}(L_1/L_1 \cap L_2) = 1$ then the subgroup H generated by G_{F/L_1} and G_{F/L_2} is dense in $G_{F/L_1 \cap L_2}$.*

Proof. This is a version of Lemma 2.16 from [R1]. We may assume that L_2 is a proper subfield of F not contained in L_1 . Thus, L_1 is an algebraic closure of $k(x)$ for some $x \in F - L_2$, where $k := L_1 \cap L_2$. Then for any $z \in F - L_2$, $y \in F - L_1$ there exist $\sigma \in G_{F/L_2}$, $\tau \in G_{F/L_1}$ such that $\sigma x = z$, $\tau x = y$. Then for any $y \in F - k$ there exists $\sigma \in G_{F/L_1} G_{F/L_2}$ such that $\sigma x = y$. It follows that H is a normal subgroup of G . If $n = \infty$ then the topological group G is simple (Theorem 2.9, [R1]), i.e., H is dense in G . If $n < \infty$ then, according to the same Theorem 2.9, [R1], the topological group G° is also simple, and thus, the closure of H contains G° . It is known (Lemma 2.15, [R1]) that $G = G_{F/L_1} G^\circ$, i.e., the closure of H again coincides with G . $\square$

Corollary 2.4. *If L_1, L_2 are proper algebraically closed subfields of F , L_1 is not contained in L_2 and $\text{tr.deg}(L_1/L_1 \cap L_2) < \infty$ then the closure of H contains $G_{F/L}$ for some proper algebraically closed subfield L in L_1 .*

Proof. According to Corollary 2.2, there exists an algebraically closed subfield L' in F such that $\text{tr.deg}(L_1/L_1 \cap L') = 1$ and $H \supseteq G_{F/L'}$. It follows from Lemma 2.3 that the subgroup generated by G_{F/L_1} and $G_{F/L'}$ is dense in $G_{F/L}$, where $L := L_1 \cap L'$. We deduce from this that the closure of H contains some $G_{F/L}$ of the desired type. $\square$

Proposition 2.5. *The subgroup $H = \langle G_{F/\overline{L_1}}, G_{F/\overline{L_2}} \rangle$ is dense in $G_{F/\overline{L_1 \cap L_2}}$ for any subfields L_1 and L_2 in F such that $\text{tr.deg}(\overline{L_1}/\overline{L_1} \cap \overline{L_2}) < \infty$.*

Proof. We may assume that $\overline{L_2}$ does not contain L_1 , and $\overline{L_1} \neq F$. Set $k := \overline{L_1} \cap \overline{L_2}$. Let $L = \overline{L} \supseteq k$ be such that $H \supseteq G_{F/L}$ and let $\text{tr.deg}(L/k) (\leq \text{tr.deg}(\overline{L_1}/k))$ be minimal. If $L \not\subseteq \overline{L_j}$ ($j = 1$ or $j = 2$) then, according to Corollary 2.4, $(\text{tr.deg}(L/L \cap \overline{L_j}) \leq \text{tr.deg}(L/k) < \infty)$, the closure of the subgroup generated by $G_{F/L}$ and $G_{F/\overline{L_j}}$, (and thus, the closure of H as well) contains $G_{F/L'}$, where L' is a proper algebraically closed subfield in L , contradicting the minimality of $\text{tr.deg}(L/k)$. Thus, $L = k$. $\square$

Proposition 2.6. *There is a morphism of commutative associative unital monoids inverting inclusions (transforming the intersection of subgroups to the algebraic closure of the compositum of subfields, and the identity G to the identity k)*

$$\left\{ \begin{array}{c} \text{open} \\ \text{subgroups of } G \end{array} \right\} \xrightarrow{\alpha} \left\{ \begin{array}{c} \text{algebraically closed subfields of } F \\ \text{of finite transcendence degree over } k \end{array} \right\} =: \text{API},$$

uniquely determined by the following equivalent conditions:

- $G_{F/\alpha(H)}$ is a normal subgroup of H and, if possible, $\alpha(H) \neq F$;
- $G_{F/\alpha(H)} \subseteq H$ and $\text{tr.deg}(\alpha(H)/k)$ is minimal.

In particular, for any $L \in \text{API}$ distinct from F and k , the normalizer $G_{\{F, L\}/k}$ of $G_{F/L}$ is maximal among proper subgroups of G .

If $n = \infty$ then any proper open subgroup of G is contained in a maximal proper subgroup of G , and any maximal proper open subgroup of G is of type $G_{\{F, L\}/k}$ for some $L \in \text{API}$, $L \neq k$.

Proof. If a subgroup H of G is open then there exists $L \in A\Pi$ such that $G_{F/L} \subseteq H$. Let L be of minimal possible transcendence degree. For any $\sigma \in H$ the group H contains the closure of the subgroup generated by $G_{F/L}$ and $G_{F/\sigma(L)}$. Then according to Proposition 2.5, H contains $G_{F/L \cap \sigma(L)}$, and therefore, $\sigma(L) \supseteq L$. Thus, $H \subseteq G_{\{F,L\}/k}$. In other words, $G_{F/L} \triangleleft H$.

Let us check the maximality of $G_{\{F,L\}/k}$ for $L \neq k, F$. If $G_{\{F,L\}/k} \subseteq U$ then $G_{F/K} \subseteq U \subseteq G_{\{F,K\}/k}$ for some $K = \overline{K}$ such that $\text{tr.deg}(K/k) \leq \text{tr.deg}(L/k)$, since $G_{F/L} \subseteq U$. The inclusion $G_{\{F,L\}/k} \subseteq G_{\{F,K\}/k}$ takes place only if either $K = L$, or $K = k$. In the second case $U = G$.

If, moreover, $G_{F/L'} \subseteq H$ for another $L' \in A\Pi$ then either $L \subseteq L'$, or $L = F$. If, in addition, $H \subseteq G_{\{F,L'\}/k}$ then either $L = L'$, or $L = F$, or $L' = F$. This shows that α is well-defined.

Let us check that α is a morphism. If $G_{F/\overline{K}} \subseteq U \subseteq G_{\{F,\overline{K}\}/k}$ and $G_{F/\overline{L}} \subseteq V \subseteq G_{\{F,\overline{L}\}/k}$ then $G_{F/\overline{K}\overline{L}} \subseteq U \cap V \subseteq G_{\{F,\overline{K},\overline{L}\}/k}$. As $G_{F/\overline{K}\overline{L}} \supseteq G_{F/\overline{K}\overline{L}}$ and $G_{\{F,\overline{K},\overline{L}\}/k} \subseteq G_{\{F,\overline{K}\overline{L}\}/k}$, it remains to show that $\alpha(U \cap V) = F$ in the case $\overline{K}\overline{L} = F$. If $\alpha(U) = F$ then evidently $\alpha(U \cap V) = F$. If $\overline{K}$ and $\overline{L}$ are distinct from F then $\alpha(G_{\{F,\overline{K},\overline{L}\}/k}) = \overline{K}\overline{L}$, i.e. again $\alpha(U \cap V) = F$. $\square$

Remark. If $H \subset G$ is contained in a group of type $G_{\{F,L\}/k}$ for neither $L \in A\Pi \setminus \{k, F\}$ then, for any $x \in F - k$, F is algebraic over the subfield generated by the H -orbit of x .

Corollary 2.7. *G is not the union of its proper open subgroups if $n = \infty$. In particular, there exist $\sigma \in G$ such that $W^{(\sigma)} = W^G$ for any smooth (i.e. with open stabilizers) representation W of G .*

Proof. For instance, if $\{x_i \mid i \in \mathbb{Z}\}$ is a transcendence basis of F over k and $\sigma x_i = x_{i+1}$ for any $i \in \mathbb{Z}$ then G is the only open subgroup containing σ . $\square$

Questions. 1. The preimage of any subgroup of a prime index in $\mathbb{Q}_+^\times$ under the modulus character gives, if $n < \infty$, an example of a maximal open proper subgroup, not encountered by Proposition 2.6. Are there any other?

2. Do there exist closed subgroups not contained in maximal proper ones?

3. VALUATION SUBGROUPS

Let $\mathcal{O}_v$ be a valuation ring in F , $\mathfrak{m}_v = \mathcal{O}_v - \mathcal{O}_v^\times$ be the maximal ideal, and $\kappa(v)$ be the residue field. If $k \subseteq \mathcal{O}_v$, fix a subfield $k \subseteq F' \subseteq \mathcal{O}_v$ identified with $\kappa(v)$ by the reduction modulo $\mathfrak{m}_v$. In this case $\kappa(v)$ is of characteristic zero (and algebraically closed).

Set $G_v := \{\sigma \in G \mid \sigma(\mathcal{O}_v) = \mathcal{O}_v\}$. This is a closed subgroup in G .

The group $\Gamma := F^\times / \mathcal{O}_v^\times \cong \mathbb{Q}$ is totally ordered: $v(x) \geq v(y)$ if and only if $xy^{-1} \in \mathcal{O}_v$, where $v : F^\times \rightarrow \Gamma$ is the natural projection. The rank of v is $r = \dim_{\mathbb{Q}} \Gamma$.

Let us call the transcendence degree of F over F' the codimension of v .

The basic example of $\mathcal{O}_v$ is, after a choice of $x_1, \dots, x_n$ algebraically independent over k and of an embedding $F_j \hookrightarrow \lim_{N \rightarrow \infty} F_{j-1}((x_j^{1/N}))$ over $F_{j-1}(x_j)$, where $F_j = \overline{F_{j-1}(x_j)}$, preimage in F of the ring $\widehat{\mathcal{O}}_n = k \oplus \widehat{\mathfrak{p}}_1 = \lim_{N \rightarrow \infty} k[[x_1^{1/N}]] \oplus \widehat{\mathfrak{p}}_2 = \widehat{\mathcal{O}}_{j-1} \oplus \widehat{\mathfrak{p}}_j$, where $\widehat{\mathfrak{p}}_{n+1} = 0$ and $\widehat{\mathfrak{p}}_j = \lim_{N \rightarrow \infty} x_j^{1/N} k((x_1^{1/N})) \dots ((x_{j-1}^{1/N}))[[x_j^{1/N}]] \oplus \widehat{\mathfrak{p}}_{j+1}$ are prime ideals for all $1 \leq j \leq n+1$. In this case $v(x_1^{m_1}) < \dots < v(x_n^{m_n})$ for all $m_1, \dots, m_n > 0$.

If $r < \infty$ and $\sigma(\mathcal{O}_v) \subseteq \mathcal{O}_v$ for some $\sigma \in G$ then $\sigma \in G_v$, since σ induces surjective endomorphism Γ , i.e. an automorphism.

It is well-known, Exercise 32, Chapter 5, [AM], that $v(\mathcal{O}_p - \wp)$ is the set of all non-negative elements in some isolated subgroup, and

$$\mathbf{Spec} \mathcal{O}_p \xrightarrow{\sim} \{\text{isolated subgroups in } L^\times / \mathcal{O}_p^\times\},$$

$\wp \mapsto \langle v(\mathcal{O}_p - \wp) \rangle$, so there are exactly $r + 1$ prime ideals in $\mathcal{O}_v$.

REMARKS. 1. If $\wp \neq 0$ is a non-maximal prime ideal of finite codimension in $\mathcal{O}_v$ and $\mathcal{O}_{v'} := (\mathcal{O}_v)_\wp$ then $G_v \subseteq G_{v'}$ (since any element $\sigma \in G_v$ preserves $\wp$, thus also $\mathcal{O}_v - \wp$, i.e. induces an automorphism $\mathcal{O}_{v'}$).

2. There are proper inclusions $G_{\{F, \mathcal{O}_v[x_1^{-1}]\}/\overline{k(x_1)}} \subset G_v \subset G_{\{F, \mathcal{O}_v[x_1^{-1}]\}/k}$ (since $x_j x_1^{\mathbb{Z}} \subset \mathcal{O}_v$ for any $j \geq 2$, and ...) for $r > 1$, i.e. G_v is not maximal.

Let $\mathcal{P}_L^r$ is the set of valuation rings or rank and codimension r in L , containing k , admitting also the following description. Let $\mathcal{C}_X^r$ be the set of chains of irreducible normal subvarieties up to codimension r on an irreducible proper normal variety X over k . Any birational morphism $X' \xrightarrow{\pi} X$ induces an embedding $\mathcal{C}_X^r \hookrightarrow \mathcal{C}_{X'}^r$, $(Z^1 \supset \dots \supset Z^r) \mapsto (W^1 \supset \dots \supset W^r)$, where $W^0 := X'$ and $W^j := (\pi|_{W^{j-1}})^{-1}_{\text{prop}}(Z^j)$ for $1 \leq j \leq r$ (and $\pi|_{W_1} : W_1 \xrightarrow{\sim} Z_1$). If L is of finite type over k then $\mathcal{P}_L^r \cong \varinjlim_{X \rightarrow L} \mathcal{C}_X^r$, where X runs over the models of L/k , and $\mathcal{P}_F^r = \varprojlim_L \mathcal{P}_L^r$. In particular, $\mathcal{P}_L^1 = C(k)$, if $n = 1$, where C is a smooth proper model of the field L over k .

Lemma 3.1. *If $0 \leq r < n + 1 \leq \infty$ then the G -action on $\mathcal{P}_F^r$ is transitive.*

Proof. Let $v : F^\times \twoheadrightarrow \Gamma \cong \mathbb{Q}^r$ be an element of $\mathcal{P}_F^r$.

Let $0 = \Gamma_0 \subset \Gamma_1 \subset \dots \subset \Gamma_{r-1} \subset \Gamma_r = \Gamma$ be the isolated subgroups in Γ . Choose $x_1, x_2, \dots \in \mathcal{O}_v$ such that $v(x_j) \in \Gamma_j - \Gamma_{j-1}$ for $1 \leq j \leq r$ and $x_{r+1}, x_{r+2}, \dots$ modulo $\mathfrak{m}_v$ form a transcendence basis of $\mathcal{O}_v/\mathfrak{m}_v$ over k . Clearly, $x_1, x_2, \dots$ are algebraically independent over k . Set $k' = \overline{k(X_{r+1}, X_{r+2}, \dots)}$ and $\widehat{F}_r := \varinjlim_N k'((X_1^{1/N})) \dots ((X_r^{1/N}))$.

Consider the set S of embeddings into $\widehat{F}_r$ of subfields in F containing $k(x_1, \dots, x_n)$, extending φ and respecting the valuation. The set S is partially ordered. Clearly, S contains maximal elements. Let $L' \xrightarrow{\xi} \widehat{F}_r$ be a maximal element of S . If $\sum_s a_s y^s = 0$ is a minimal polynomial of $y \in F$ over L' then there exist $i < j$ such that $v(y) = \frac{v(a_i/a_j)}{j-i}$. There exists $z \in \widehat{F}_r$ such that $\sum_s \xi(a_s) z^s = 0$ and $v(z) = \frac{v(\xi(a_i/a_j))}{j-i}$. Therefore, ξ extends to $L'(z)$, and thus, $L' = F$.

This shows that the embedding $k(x_1, x_2, \dots) \xrightarrow{\varphi} k'((X_1)) \dots ((X_r))$, $x_j \mapsto X_j$, respects the valuation, and extends to an embedding $F \hookrightarrow \widehat{F}_r$, respecting the valuation.

Thus, any element of $\mathcal{P}_F^r$ is determined by a choice of $x_1, x_2, \dots \in F$ algebraically independent over k and by an embedding of F into $\widehat{F}_r$ over $k(x_1, x_2, \dots)$. Clearly, the G -action is transitive on these data. $\square$

The G_v -action on $\kappa(v)$ induces a homomorphism $G_v \twoheadrightarrow G_{\kappa(v)/k}$.

Let $G_v^\dagger := \{\sigma \in G_v \mid \sigma x \equiv x \pmod{\mathfrak{m}_v} \text{ for any } x \in \mathcal{O}_v\}$ be its kernel.

A continuous¹ section of $G_{\kappa(v)/k} \hookrightarrow G_v$ is determined by a subfield $F' \subseteq \mathcal{O}_v$, identified with $\kappa(v)$ by the reduction modulo $\mathfrak{m}_v$, and by an embedding of F' into the field of iterated Puiseux series $\lim_{N \rightarrow \infty} F'((t_1^{1/N})) \dots ((t_r^{1/N}))$ over F' compatible with valuations (i.e. by a choice of a section $\Gamma \hookrightarrow F^\times$ of the valuation v).

As the centralizer of $G_v^\dagger$ is trivial, such sections form a $G_v^\dagger$ -torsor. One has a decomposition $G_v = G_v^\dagger(G_v \cap G_{\{F, F'\}/k})$, where $G_v^\dagger \cap G_{\{F, F'\}/k} = G_v \cap G_{F/F'}$.

Let B_+ be the group of linear transformations of Γ (isomorphic to the group of upper-triangular rational $r \times r$ matrices with positive diagonal entries), respecting the order, i.e. the flag of isolated subgroups $0 = \Gamma_0 \subset \Gamma_1 \subset \dots \subset \Gamma_r = \Gamma$ and the orientation of each $\Gamma_j/\Gamma_{j-1} \cong \mathbb{Q}$, $1 \leq j \leq r$. Then $\mathcal{P}_L^\times$ consists of (some) valuations $(L^\times/k^\times) \otimes \mathbb{Q} \twoheadrightarrow \mathbb{Q}$ modulo the B_+ -action. The G_v -action on $F^\times$ induces a linear action on Γ , preserving the order, i.e. a homomorphism $G_v \rightarrow B_+$, surjective and splitting, if the rank of v coincides with its codimension.

Let $G_v^\circ := \{\sigma \in G_v \mid \sigma x/x \in \mathcal{O}_v^\times \text{ for any } x \in F^\times\}$ be its kernel.

There is a homomorphism $G_v^\dagger \cap G_v^\circ \rightarrow \text{Hom}(\Gamma, \kappa(v)^\times)$,² $\sigma \mapsto (v(x) \mapsto \sigma x/x \bmod \mathfrak{m}_v)$, surjective and splitting, if the rank of v coincides with its codimension.

Lemma 3.2. *Its kernel $G_v^1 := \{\sigma \in G_v \mid \sigma x/x \in 1 + \mathfrak{m}_v \text{ for any } x \in F^\times\}$ is a discrete subgroup if $n < \infty$. The functors $H^0(G, -)$ and $H^0(G_v^1, -)$ coincide on the category $\mathcal{S}m_G$ of smooth representations of G , if $n = \infty$ and the rank of v is finite, non-zero and coincides with its codimension.*

Proof. The G_v -action on F extends to a continuous G_v -action on the completion F_v of F , and the continuous G_v^1 -action on F_v is determined by the action on some r -tuple of elements, representing each of isolated subgroups of Γ and by the action on some maximal subfield with the trivial valuation. Clearly, the latter action is determined by its restriction to any transcendence basis over k . The second assertion follows from the fact that any open subgroup containing G_v^1 coincides with G , which follows from Proposition 2.6. $\square$

Proposition 3.3. *The set $\{L \xrightarrow{/k} F\}$ of embeddings into F over k of the function field L of a d -dimensional variety over k is a disjoint union of (non-empty) G_v -orbits $O_{p,v} := \{L \xrightarrow{\sigma} F \mid \sigma^{-1}(\mathcal{O}_v) = \mathcal{O}_p\}$, where $v \in \mathcal{P}_F^r$, $p \in \mathcal{P}_L^j$ and $m := \max(0, r + d - n) \leq j \leq M := \min(d, r)$. In particular, $\mathbb{Q}[\{L \xrightarrow{/k} F\}]_{G_v^\dagger} = \bigoplus_{j=m}^M \bigoplus_{p \in \mathcal{P}_L^j} \mathbb{Q}[\{\kappa(p) \xrightarrow{/k} \kappa(v)\}]$ and $G_v \backslash \{L \xrightarrow{/k} F\} = \coprod_{j=m}^M \mathcal{P}_L^j$. If $d = n$ then the stabilizers are compact and isomorphic to $\widehat{\mathbb{Z}}^r$.*

Proof. Any element $x \in \sigma(L^\times)$ is either contained in $\mathcal{O}_v$, or its inverse is contained in $\mathcal{O}_v$, i.e., $\mathcal{O}_p := \sigma^{-1}(\mathcal{O}_v)$ is a valuation ring.³

Clearly, $\text{rk}(L^\times/\mathcal{O}_p^\times) \leq M$. Let us check that $\text{rk}(L^\times/\mathcal{O}_p^\times) \geq m$, and in particular, the inclusion $(L^\times/\mathcal{O}_p^\times) \otimes \mathbb{Q} \xrightarrow{\sigma} F^\times/\mathcal{O}_v^\times$ is bijective for $d = n$. Let $x \in F^\times$ and $\sum_{j=0}^m a_j x^j = 0$

¹One can apply the well-known fact that the coefficients of any algebraic (over $k(t)$) element of $k((t))$ generate in k a subfield of finite type over $\mathbb{Q}$.

²Clearly, $\text{Hom}(\Gamma, \kappa(v)^\times) \cong (\varprojlim_N M_N)^r$, where $M_{NN'} := \kappa(v)^\times \twoheadrightarrow M_{N'} := \kappa(v)^\times$ is the raising to the N -th power.

³Note, that the henselian property does not suffice. For instance, if $k(X) = k(x_1, \dots, x_d)$ is embedded into the field of fractions of $\mathcal{O} := k[[T_1, \dots, T_N]]$, $x_j \mapsto T^{\alpha_j}$, $|\alpha_j| = 0$ then $k(X) \cap \mathcal{O} = k$, since it consists of homogeneous functions in $T_1, \dots, T_N$ of degree 0.

be a minimal polynomial over $\sigma(\mathcal{O}_p)$. Then there exist $0 \leq i < j \leq m$ such that $a_i a_j \neq 0$ and $v(a_i x^i) = v(a_j x^j)$, so $v(x) = \frac{v(a_i/a_j)}{j-i}$, i.e., the element $v(x) \in F^\times/\mathcal{O}_v^\times$ is in the image of $(L^\times/\mathcal{O}_p^\times) \otimes \frac{1}{m!}\mathbb{Z}$.

We may assume that $a_j = 1$ for some j . Thus, if $x \in \mathcal{O}_v$ then its image in $\mathcal{O}_v/\mathfrak{m}_v$ is algebraic over $\sigma(\mathcal{O}_p)/\sigma(\mathfrak{m}_p)$, i.e. $\mathcal{O}_v/\mathfrak{m}_v \cong \overline{\mathcal{O}_p/\mathfrak{m}_p}$, and therefore, $p \in \mathcal{P}_L^r$.

To check that $\mathcal{O}_{p,v}$ is a G_v -orbit, we have to show that for any pair of embeddings $\sigma, \tau : L \xrightarrow{/k} F$ such that $\sigma(\mathcal{O}_p), \tau(\mathcal{O}_p) \subset \mathcal{O}_v$ there exists $\xi \in G_v$, extending $\tau\sigma^{-1} : \sigma(\mathcal{O}_p) \xrightarrow{\sim} \tau(\mathcal{O}_p)$. According to Zorn's lemma, there are maximal elements in the set

$$S := \left\{ A \xrightarrow{\varphi} \mathcal{O}_v \mid \begin{array}{l} v(\varphi(x)) \geq v(\varphi(y)), \text{ if } v(x) \geq v(y) \text{ for } x, y \in A, \\ \sigma(\mathcal{O}_p) \subseteq A \subseteq \mathcal{O}_v, \quad \varphi|_{\sigma(\mathcal{O}_p)} = \tau\sigma^{-1} \end{array} \right\} \ni (\sigma(\mathcal{O}_p) \xrightarrow{\tau\sigma^{-1}} \mathcal{O}_v),$$

for instance, $B \xrightarrow{\psi} \mathcal{O}_v$. Then B is integrally closed in $\mathcal{O}_v$, since for any $x \in \mathcal{O}_v$ integral over B an embedding ψ extends to $B[x] \subseteq \mathcal{O}_v$, even respecting the order,⁴ since $\mathcal{O}_v$ is integrally closed in F . Thus, any element of $\mathcal{O}_v$ can be presented as a/b , where $a, b \in B$ and $v(a) \geq v(b)$. As ψ respects the order, it maps $\mathcal{O}_v$ into $\mathcal{O}_v$, i.e., $\mathcal{O}_v = B$.

The set $\mathcal{O}_{p,v}$ is non-empty, since the valuation p extends to $\overline{L} \cong F$, and the group G permutes the elements of $\mathcal{P}_F^r$ (Lemma 3.1). Then $\mathcal{O}_{p,v} \cong G_v/St_{p,v}$, where $St_{p,v} = G_v \cap G_{F/\sigma(L)}$ is an open compact subgroup. The compact subgroups of G_v are contained in G_v° and they project to $\text{Hom}(\Gamma, k^\times)$ injectively (Lemma 3.2), so $St_{p,v} \cong \widehat{\mathbb{Z}}(1)^r$. $\square$

3.1. Valuations and maximal subgroups. **EXAMPLE.** If $n = 1$ then to any valuation v the decomposition $\{k(C) \xrightarrow{/k} F\} = C(F) - C(k) = \coprod_{C(k)} (\mathfrak{m}_v - \{0\})$ is associated.

Proposition 3.4. *If $n = 1$ then $\mathcal{O}_{p,v} \cap \mathcal{O}_{q,v'}$ is non-empty for any pair of distinct $v, v' \in \mathcal{P}_F^1$ and any pair of distinct points $p, q \in C(k)$ on a smooth proper curve C over k , i.e. there is an embedding $\sigma : k(C) \xrightarrow{/k} F$ such that $\sigma(\mathcal{O}_p) \subset \mathcal{O}_v$ and $\sigma(\mathcal{O}_q) \subset \mathcal{O}_{v'}$.*

Proof. We consider v and v' as a compatible system of points on smooth proper curves over k with the function fields embedded into F : if $C_\beta \rightarrow C_\alpha$ then $v_\beta \mapsto v_\alpha$ and $v'_\beta \mapsto v'_\alpha$. One needs to check that there exist β and a map $C_\beta \rightarrow C$ such that $v_\beta \mapsto p$ and $v'_\beta \mapsto q$.

Choose a non-constant function $x \in \mathcal{O}(C \setminus \{p\})$, i.e. a surjective morphism $C \rightarrow \mathbb{P}^1$ sending to ∞ only p . Let $C' \rightarrow C$ be a cover such that the composition $C' \rightarrow C \rightarrow \mathbb{P}^1$ is a Galois cover with the group A . For some α such that $v_\alpha \neq v'_\alpha$ choose a surjection $C_\alpha \rightarrow \mathbb{P}^1$ such that $v_\alpha \mapsto \infty$ and $v'_\alpha \mapsto x(q)$. Consider the normalization D of an irreducible component of $C' \times_{\mathbb{P}^1} C_\alpha$. The surjection $D \rightarrow C_\alpha$ is isomorphic to the surjection $C_\beta \rightarrow C_\alpha$ for some β . Let $\pi : C_\beta \rightarrow C'$ be the projection. As A acts transitively on the fibres of the composition $C' \rightarrow C \rightarrow \mathbb{P}^1$, there is an element $\gamma \in A$ such that $\gamma\pi(v'_\beta)$ belongs to the preimage of q . Then the composition $C_\beta \xrightarrow{\gamma\pi} C' \rightarrow C$ maps v_β to p , and v'_β to q . $\square$

Proposition 3.5. *If $r = 1$ then the subgroup H , generated by G_v and $G_{v'}$, acts transitively on $\{L \xrightarrow{/k} F\}$, i.e. H is dense in G .*

Proof. Let L be an extension of k and $v, v' : L^\times/k^\times \rightarrow \mathbb{Q}$ be a pair of discrete valuations. If $\mathcal{O}_v^\times \subseteq \mathcal{O}_{v'}^\times$ in L then $\mathcal{O}_{v'}^\times/\mathcal{O}_v^\times$ is the kernel of $\mathbb{Q} \supseteq L^\times/\mathcal{O}_v^\times \twoheadrightarrow L^\times/\mathcal{O}_{v'}^\times \subseteq \mathbb{Q}$, which is

⁴i.e., if $\sum_{s=0}^m a_s x^s = 0$, $a_s \in B$ and $v(x) = \frac{v(a_i/a_j)}{j-i}$ for some $i < j$ then one can choose $y \in \mathcal{O}_v$ such that $\sum_{s=0}^m \psi(a_s) y^s = 0$ and $v(y) = \frac{v(\psi(a_i/a_j))}{j-i}$.

evidently injective, so $v = v'$. Let $v \neq v'$. Then for any $x \in \mathcal{O}_v^\times$ one has either $x \in \mathcal{O}_{v'}$, or $x^{-1} \in \mathcal{O}_{v'}$, that is there is some $t \in \mathcal{O}_v^\times \cap \mathcal{O}_{v'}$ such that $t \notin \mathcal{O}_{v'}^\times$. Fix such t and $t^{1/N}$ for all integers $N \geq 1$. Let $x_1, x_2, \dots \in \mathcal{O}_v^\times$ be a lifting of a transcendence basis of $\kappa(v)$ over $k(t)$. Set $k_0 := k$ and define inductively a strictly increasing sequence of algebraically closed subfields in $\{0\} \cup (\mathcal{O}_v^\times \cap \mathcal{O}_{v'}^\times) \subset L$ as follows. For any $i \geq 1$ there exist $P \in k_{i-1}[T]$, an integer $N \geq 1$ and $M \in \mathbb{Q}_+^\times$ such that $y_i := t^{-M}(x_i^{v'(t)} - t^{v'(x_i)}P(t^{1/N})) \in \mathcal{O}_{v'}^\times$ and $y_i \notin k_{i-1} + \mathfrak{m}_{v'}$. Then $y_1, y_2, \dots$ is another lifting of a transcendence basis of $\kappa(v)$ over $k(t)$ in $\mathcal{O}_v^\times$. Set $k_i := \overline{k_{i-1}(y_i)}$ and $k' = \bigcup_{i \geq 1} k_i$. Then $\kappa(v)$ is algebraic over the reduction of k' modulo $\mathfrak{m}_v$ and $\kappa(v')$ is algebraic over the reduction of k' modulo $\mathfrak{m}_{v'}$. This shows that we are reduced to the case of $n = 1$.

By Proposition 3.4, for any $\xi : L \xrightarrow{k} F$ the map $G_v \times G_{v'} \rightarrow \{L \xrightarrow{k} F\}$, given by $(\sigma, \tau) \mapsto \sigma\tau\xi$, is surjective. $\square$

If $n = 1$ define $\varphi : G_v^1 \rightarrow \Gamma$ by $\sigma \mapsto v(\sigma x/x - 1)$ for any $x \in \mathfrak{m} - \{0\}$, or $x \in F - \mathcal{O}_v$. Independence of x : if $y = \sum_{j \geq 1} a_j x^{j/N}$ and $\sigma x/x = 1 + ax^\alpha + o(x^\alpha)$, where $a \in k^\times$, then $\sigma y - y = \sum_{j \geq 1} a_j x^{j/N} ((1 + ax^\alpha + o(x^\alpha))^{j/N} - 1)$, i.e.

$$(1) \quad \sigma y/y - 1 = \frac{v(y)}{v(x)} ax^\alpha + o(x^\alpha).$$

Thus, $v(\sigma y - y) = v(y) + [x^\alpha]$, and therefore, $v(\sigma y/y - 1) = v(\sigma x/x - 1)$.

From the equality $\sigma\tau x/x - 1 = (\sigma(\tau x)/\tau x - 1)\tau x/x + \tau x/x - 1$ we get that $\varphi(\sigma\tau) \geq \min(\varphi(\sigma), \varphi(\tau))$, and $\varphi(\sigma) = \varphi(\sigma^{-1})$: $v(\sigma^{-1}x/x - 1) = v(x/\sigma x - 1) = \varphi(\sigma)$.

Let $G_v^1(\beta) := \{\sigma \in G_v^1 \mid \varphi(\sigma) \geq \beta\}$, where $\beta \in \Gamma \otimes \mathbb{R}$. This is a normal subgroup in G_v° . Then $G_v^1 = G_v^1(0) = G_v^1(0)^+$, where $G_v^1(\beta)^+ := \bigcup_{\gamma > \beta} G_v^1(\gamma) = \{\sigma \in G_v^1 \mid \varphi(\sigma) > \beta\}$. Clearly, $G_v^1(\beta) \neq G_v^1(\gamma)$, if $\beta \neq \gamma$.

Lemma 3.6. *There is a canonical isomorphism $G_v^1(\beta)/G_v^1(\beta)^+ \xrightarrow{\sim} \text{Hom}(\Gamma, \mathfrak{m}^{[\beta]})$, where*

$$\mathfrak{m}^{[\beta]} = \{x \in \mathfrak{m} \mid v(x) \geq \beta\} / \{x \in \mathfrak{m} \mid v(x) > \beta\} \cong \begin{cases} k, & \text{if } \beta \in \Gamma \text{ and } \beta > 0 \\ 0, & \text{otherwise} \end{cases}$$

Proof. It is clear from the formula (1) that any element $\sigma \in G_v^1(\beta)$ induces a homomorphism $\Gamma \rightarrow \mathfrak{m}^{[\beta]}$, $[y] \mapsto \sigma y/y - 1$. This gives a surjective homomorphism $G_v^1(\beta)/G_v^1(\beta)^+ \rightarrow \text{Hom}(\Gamma, \mathfrak{m}^{[\beta]})$: $\sigma\xi \mapsto ([x] \mapsto \sigma\xi x/x - 1 = \sigma(\xi x/x - 1)\sigma x/x + (\sigma x/x - 1) \equiv (\xi x/x - 1) + (\sigma x/x - 1))$.

Let us check its injectivity: $\sigma x = x(1 + ax^\beta + x^\beta h)$, $h \in \mathfrak{m}$. Let $\tau[x(1 + ax^\beta + x^\beta h)] := x(1 + ax^\beta)$. Then $\tau\sigma x/x = 1 + ax^\beta$ and, if we set $y := x(1 + ax^\beta)$ then $x(1 + ax^\beta + x^\beta h) = y + o(y^{\beta+1})$, and thus, $\tau \in G_v^1(\beta)^+$. Finally, $G_v^1(\beta)/G_v^1(\beta)^+ = \text{Hom}(\Gamma, \mathfrak{m}^{[\beta]})$. $\square$

Lemma 3.7. *$G_v^1(\beta)$ is surjective over G/G° for any $\beta \in \Gamma \otimes \mathbb{R}$.*

Proof. As G_v is maximal and does not contain G° , it suffices to show that the subgroup generated by $G_v^1(\beta)$ and G° contains G_v . A choice of a section $s : \Gamma \rightarrow F^\times$ of the projection $v : F^\times \rightarrow \Gamma$ determines an additive section $\text{Hom}(\Gamma, k^\times) \rightarrow G_v^\circ \cap G^\circ$ of the projection $G_v^\circ \rightarrow \text{Hom}(\Gamma, k^\times)$. A choice of $x \in \mathfrak{m} \cap s(\Gamma)$ gives a bijection of sets G_v^1 and $\mathfrak{m}$: $\mathfrak{m} \ni m : x \mapsto x(1 + m)$. The elements $(x \mapsto (1 + x^\alpha)^{1/\alpha} - 1) \in G_v \cap G^\circ$, $\alpha \in \mathbb{Q}_+^\times$, determine a set-theoretic section of the projection $G_v \rightarrow \mathbb{Q}_+^\times$. It remains to check that the subgroup generated by $G_v^1(\beta)$ and $G_v^1 \cap G^\circ$ contains G_v^1 . Clearly, $x(1 + ax^\alpha)^{-1/\alpha} \in G_v^1 \cap G^\circ$. If an element of G_v^1 is presented by $h = x(1 + ax^\alpha + o(x^\alpha)) \in x(1 + \mathfrak{m}) \cap k[[x^{1/N}]]$ then

$h(1 + a\alpha h^\alpha)^{-1/\alpha} = x(1 + o(x^\alpha)) \in x(1 + \mathfrak{m}) \cap k[[x^{1/N}]]$. So the composition of h with an appropriate element of $G_v^1 \cap G^\circ$ will be in $G_v^1(\beta)$. $\square$

4. VALUATIONS AND ASSOCIATED FUNCTORS

4.1. The “globalization” functor. For a discrete valuation $v \in \mathcal{P}_F^r$ and for any $W \in \mathcal{S}m_G$ set $W_v := \sum_{\sigma \in G_v} W^{G_{F/\sigma(F')}} = \sum_{\sigma \in G_v^\dagger} W^{G_{F/\sigma(F')}} \subseteq W$. Clearly, the additive functor $\mathcal{S}m_G \rightarrow \mathcal{S}m_{G_v}$, $W \mapsto W_v$, is faithful and preserves surjections if $n = \infty$ (and injections in general). Set $\Gamma_r(W) := \bigcap_{v \in \mathcal{P}_F^r} W_v$ and $\Gamma := \Gamma_1$, so $\Gamma_r : \mathcal{S}m_G \rightarrow \mathcal{S}m_G$ are additive functors.

Denote by $\mathcal{I}_G$ is the full subcategory in $\mathcal{S}m_G$ consisting of representations W such that $W^{G_{F/M}} = W^{G_{F/M'}}$ for any extension M of k in F and any purely transcendental extension M' of M in F . Denote by $\mathcal{I} = \mathcal{I}_{/k} = \mathcal{I}_{F/k} : \mathcal{S}m_G \rightarrow \mathcal{I}_G$ the left adjoint to the inclusion functor $\mathcal{I}_G \hookrightarrow \mathcal{S}m_G$, and set $C_L := \mathcal{I}_{F/k} \mathbb{Q}[\{L \xrightarrow{/k} F\}]$ for any extension L of k of finite type, cf. [R1], §6.

Lemma 4.1. *If $n = \infty$ and $r = 1$ then the projection $\mathbb{Q}[\{k(X) \xrightarrow{/k} \mathcal{O}_v\}] \rightarrow C_{k(X)}$ is surjective for any irreducible variety X over k .*

Proof. Let $P : k(X) \xrightarrow{/k} F$ be a generic point. Choose a subfield $k' \subset \mathcal{O}_v \cap \overline{P(k(X))}$ over k projecting isomorphically onto $(\mathcal{O}_v \cap \overline{P(k(X))})/(\mathfrak{m}_v \cap \overline{P(k(X))}) \subset \kappa(v)$.

Let $k'(C) := k'P(k(X)) \subset F$. If P does not factor through $\mathcal{O}_v$ then $k'(C)$ is the function field of a smooth proper curve over k' . As the class of P in $C_{k(X)}$ belongs to the image of $\mathbb{Q}[\{k'(C) \xrightarrow{/k'} F\}] \rightarrow \mathcal{I}_{/k'} \mathbb{Q}[\{k'(C) \xrightarrow{/k'} F\}] = \text{Pic}(C_F)_{\mathbb{Q}} \rightarrow C_{k(X)}$, it remains only to check the surjectivity of $\mathbb{Q}[\{k'(C) \xrightarrow{/k'} \mathcal{O}_v\}] \rightarrow \text{Pic}(C_F)_{\mathbb{Q}}$.

By Corollary 3.5 from [R1], the generic points $k'(C) \xrightarrow{/k'} \kappa(v)$ generate $\text{Pic}(C_{\kappa(v)})$.

Let us check the transitivity of the $(G_v^\dagger \cap G_{F/k'})$ -action on the generic fibres of the specialization $\text{Pic}(C_F) \xrightarrow{s} \text{Pic}(C_{\kappa(v)})$ (i.e. over all $k'(\text{Pic}^j C) \xrightarrow{\sigma} \kappa(v)$). Let $x_1, \dots, x_g$ be a transcendence basis of $k'(\text{Pic}^j C)$ over k' , and $\xi_1, \xi_2 : k'(\text{Pic}^j C) \xrightarrow{/k} \mathcal{O}_v$ be a pair of liftings of σ . Then there exists an element $\tau \in G_v^\dagger \cap G_{F/k'}$ such that $\tau \xi_1 x_j = \xi_2 x_j$ for all $1 \leq j \leq g$. The field $k'(\text{Pic}^j C)$ is generated over $k'(x_1, \dots, x_g)$ by an algebraic element f . According to Hensel’s lemma, $\tau \xi_1 f = \xi_2 f$. (Note, that one can replace $\text{Pic} C$ by an arbitrary smooth variety over k' .)

Thus, the kernel of s coincides with $\{\tau \tilde{\sigma} - \tilde{\sigma} \mid \tau \in G_v^\dagger \cap G_{F/k'}\}$, and therefore, the generic points $k'(C) \xrightarrow{/k'} \mathcal{O}_v$ (i.e. whose specializations are also generic) generate $\text{Pic}(C_F)$. $\square$

Corollary 4.2. $\Gamma(W) = W_v = W$ for any $W \in \mathcal{I}_G$.

Proof. This follows from Lemma 4.1 in the case of $W = C_{k(X)}$. As $C_{k(X)}$ for all irreducible varieties X over k form a system of generators of $\mathcal{I}_G$, and the functor $W \mapsto W_v$ preserves the surjections, we get that $W_v = W$ for arbitrary $W \in \mathcal{I}_G$. $\square$

Lemma 4.3. *Suppose that $n = \infty$. Then $(W_1 \otimes W_2)_v \subseteq (W_1)_v \otimes (W_2)_v$ and $\Gamma(W_1 \otimes W_2) \subseteq \Gamma(W_1) \otimes \Gamma(W_2)$ for any $W_1, W_2 \in \mathcal{S}m_G$. However, $(W \otimes W)_v \neq W_v \otimes W_v$ if $W = \mathbb{Q}[F - k]$.*

If either W_1 is a quotient of $A(F)$ for a commutative algebraic k -group A , or $W_1 \in \mathcal{I}_G$ then $(W_1 \otimes W_2)_v = (W_1)_v \otimes (W_2)_v$ for any $W_2 \in \mathcal{S}m_G$.

Proof. The first follows from $(W_1 \otimes W_2)^{G_{F/F'}} = W_1^{G_{F/F'}} \otimes W_2^{G_{F/F'}}$, the second follows from the first. One has $(W \otimes W)_v \not\supseteq [x] \otimes [x'] \in W_v \otimes W_v$ for any pair of distinct $x, x' \in \mathcal{O}_v^\times$ such that $x \equiv x' \pmod{\mathfrak{m}_v}$.

Let $w \in W_2^{G_{F/F'}}$. Then $G_{F/F'} \subset G_{F/L} \subseteq \text{Stab}_w$, where $L \subset F'$ is of finite type over k .

If F'' is an algebraically closed subfield in $\mathcal{O}_v$ and either F'' and L are algebraically independent over k in $\kappa(v)$, or $L \subseteq F''$ then $\overline{LF''} \subset \mathcal{O}_v$, so $W_1^{G_{F/F''}} \otimes w \subseteq (W_1 \otimes W_2)^{G_{F/\overline{LF''}}} \subseteq (W_1 \otimes W_2)_v$. If W_1 is of type as above then $W_1^{G_{F/F''}}$, with F'' and L algebraically independent over k in $\kappa(v)$, generate $(W_1)_v$, i.e., $(W_1)_v \otimes w \subseteq (W_1 \otimes W_2)_v$, thus finally, $(W_1)_v \otimes (W_2)_v \subseteq (W_1 \otimes W_2)_v$. $\square$

REMARKS. 1. By Corollary 4.2 and Lemma 4.3, $\Gamma(W \otimes F) = W \otimes k$ for any $W \in \mathcal{I}_G$, so $\Gamma(V) \otimes F \twoheadrightarrow V$ for any semi-linear quotient V of $W \otimes F$.

2. If W is an F -vector space then the F -vector space structure $F \otimes W \rightarrow W$ on W induces an $\mathcal{O}_v$ -module structure $(F \otimes W)_v = \mathcal{O}_v \otimes W_v \rightarrow W_v$ on W_v . Clearly, $F \otimes_{\mathcal{O}_v} W_v \rightarrow W$ is injective, but not surjective, as shows the example of $W = F[\{L \xrightarrow{k} F\}]$.

3. Clearly, Γ_r preserves the injections, but not the surjections. Namely, let $W := \bigotimes_k^N F \rightarrow \Omega_{F/k}^{N-1}$ be given by $a_1 \otimes \cdots \otimes a_N \mapsto a_1 da_2 \wedge \cdots \wedge da_N$. Then $W_v = \bigotimes_k^N \mathcal{O}_v$ if $n \geq 2N$, so $(\bigotimes_F^{N-1} \Omega_{F/k_0}^1)_v = \bigotimes_F^{N-1} \Omega_{\mathcal{O}_v/k_0}^1$ for any $k_0 \subseteq k$; and $\Gamma(\bigotimes_k^N F) = k$, but $\Gamma_r(\Omega_{F/k}^\bullet) = \Omega_{F/k, \text{reg}}^\bullet$ for any $r \geq 1$.

Proof. Clearly, $W^{G_{F/F'}} = \bigotimes_k^N F'$, so $W_v \subseteq \bigotimes_k^N \mathcal{O}_v$. Let $\omega = x_1 \otimes \cdots \otimes x_N$ for some $x_1, \dots, x_N \in F'$ whose images in $\kappa(v)$ algebraically independent over $k(\overline{y_1}, \dots, \overline{y_N})$ for some $y_1, \dots, y_N \in \mathcal{O}_v$. Then for each $1 \leq i \leq N$ there exists an element $\sigma_i \in G_v \cap G_{F/k(x_1, \dots, \widehat{x_i}, \dots, x_N, y_1, \dots, y_N)}$ such that $\sigma_i x_i = x_i + y_i$. Then $\prod_{i=1}^N (\sigma_i - 1)\omega = y_1 \otimes \cdots \otimes y_N$. $\square$

In the case $n = \infty$ one can also apply Lemma 4.3.

$$4. \mathbb{Q}[\{L \xrightarrow{k} F\}]_v = \mathbb{Q}[\{L \xrightarrow{k} \mathcal{O}_v\}].$$

For an integral normal k -variety X with $k(X) \subset F$ let $\mathfrak{V}(X)$ be the set of all discrete valuations of F of rank one trivial on k such that their restrictions to $k(X)$ are either trivial, or correspond to divisors on X . Set $\mathcal{W}(X) := W^{G_{F/k(X)}} \cap \bigcap_{v \in \mathfrak{V}(X)} W_v \subseteq W$.

REMARK. $W^{G_{F/k(X)}} \cap W_v$ depends only on the restriction of v to $k(X)$, since the set of $G_{F/k(X)}$ -orbits $G_{F/k(X)} \backslash G/G_v$ of the valuations of F coincides, by Proposition 3.3, with the set of valuations of $k(X)$ of rank $\leq r$. E.g., if the restriction of v to $k(X)$ is trivial then $W^{G_{F/k(X)}} \subseteq W_v$.

Consider the following site $\mathfrak{H}$. Objects of $\mathfrak{H}$ are the smooth varieties over k . Morphisms in $\mathfrak{H}$ are the locally dominant morphisms, transforming non-dominant divisors to divisors. Coverings are smooth morphisms surjective over the generic point of any divisor on the target.

Lemma 4.4. *A choice of k -embeddings into F of all generic points of all smooth k -varieties defines a sheaf $\mathcal{W}$ on $\mathfrak{H}$ for any $W \in \mathcal{S}m_G$.*

Proof. Clearly, if a dominant morphism $f : U \rightarrow X$ transforms divisors on U , non-dominant over X , to divisors on X then $\mathfrak{V}(U) \subseteq \mathfrak{V}(X)$, so $\mathcal{W}(X) \subseteq \mathcal{W}(U)$.

If, moreover, the pull-back of any divisor on X is a divisor on U then $\mathfrak{V}(X) = \mathfrak{V}(U)$.

By Lemma 1.1 of [JR], the sequence $0 \rightarrow \prod_{x \in X^0} W^{G_{F/k(x)}} \rightarrow \prod_{x \in U^0} W^{G_{F/k(x)}} \rightarrow \prod_{x \in (U \times_X U)^0} W^{G_{F/k(x)}}$ is exact (in fact, $X \mapsto \prod_{x \in X^0} W^{G_{F/k(x)}}$ is a sheaf on a topology $\mathfrak{D}m_k$). As $\mathfrak{V}(X) = \mathfrak{V}(U) = \mathfrak{V}(U \times_X U)$, the sheaf property for the covering f amounts to the exactness of the above sequence restricted to $\prod_{x \in U^0} \bigcap_{v \in \mathfrak{V}(X)} W_v$. $\square$

4.2. The “specialization” functor.

Lemma 4.5. *If $r = 1$ and $n = \infty$ then $H_0(G_v^\dagger, -) = -_{G_v^\dagger}$ gives functors $Sm_G \rightarrow Sm_{G_{\kappa(v)/k}}$ and $\mathcal{I}_G \rightarrow \mathcal{I}_{G_{\kappa(v)/k}}$. More precisely, there are natural surjections $W^{G_{F/F'}} \twoheadrightarrow H_0(G_v^\dagger, W_v)$ and $\mathcal{I}_{\kappa(v)/k} H_0(G_v^\dagger, W) \twoheadrightarrow H_0(G_v^\dagger, \mathcal{I}W)$ for any $W \in Sm_G$.*

Proof. For any smooth representation W of G the stabilizer of any vector $\bar{w} \in H_0(G_v^\dagger, W)$ contains the stabilizer in G_v of its preimage $w \in W$, which implies the smoothness of $H_0(G_v^\dagger, W)$, since the projection $G_v \twoheadrightarrow G_{\kappa(v)/k}$ is open.

The first surjection is evident. Clearly, it is $G_{F'/k}$ -equivariant.

By Lemma 6.7 of [R1], the functor $H^0(G_{F/F'}, -)$ induces equivalences of categories $Sm_G \xrightarrow{\sim} Sm_{G_{F'/k}}$ and $\mathcal{I}_G \xrightarrow{\sim} \mathcal{I}_{G_{F'/k}}$, so $H_0(G_v^\dagger, W_v) \in Sm_{G_{\kappa(v)/k}}$ and, by Corollary 4.2, $H_0(G_v^\dagger, \mathcal{I}W) = H_0(G_v^\dagger, (\mathcal{I}W)_v) \in \mathcal{I}_{G_{\kappa(v)/k}}$. $\square$

Lemma 4.6. *Let X be a smooth proper variety over k and $q \geq 0$.*

Then $H_0(G_v^\dagger, CH^q(X_F)) = CH^q(X_{\kappa(v)})$ and $H_0(G_v^\dagger, \Omega_{F/k, \text{reg}}^\bullet) = \Omega_{\kappa(v)/k, \text{reg}}^\bullet$.

Proof. The specialization homomorphism $CH^q(X_F) \twoheadrightarrow CH^q(X_{\kappa(v)})$ (cf. [S]) is $G_v^\dagger$ -invariant, so it factors through the coinvariants $H_0(G_v^\dagger, CH^q(X_F))$. Clearly, the composition of the surjection $CH^q(X_{F'}) = CH^q(X_F)^{G_{F/F'}} \xrightarrow{\xi} H_0(G_v^\dagger, CH^q(X_F))$ from Lemma 4.5 with the specialization coincides with the natural isomorphism $CH^q(X_{F'}) \xrightarrow{\sim} CH^q(X_{\kappa(v)})$. The case of $\Omega_{F/k, \text{reg}}^\bullet$ is similar. $\square$

REMARK. Let us show that $CH^q(X \times_k \mathcal{O}_v) = CH^q(X_F)$. One has a short exact sequence of Gersten complexes of flasque sheaves on $X \times_k \mathcal{O}_v$:

$$0 \rightarrow \coprod_{x \in X^{\bullet-1}} K_{q-\bullet}(\kappa(x)) \rightarrow \coprod_{x \in (X \times_k \mathcal{O}_v)^\bullet} K_{q-\bullet}(\kappa(x)) \rightarrow \coprod_{x \in X_F^\bullet} K_{q-\bullet}(\kappa(x)) \rightarrow 0,$$

the end of whose long exact cohomological sequence looks as

$$H^{q-1}(X_F, \mathcal{K}_q) \xrightarrow{\partial} CH^{q-1}(X) \rightarrow CH^q(X \times_k \mathcal{O}_v) \rightarrow CH^q(X_F) \rightarrow 0.$$

The composition of the surjections $CH^{q-1}(X_F) \otimes F^\times \xrightarrow{id \cdot v} CH^{q-1}(X_F) \xrightarrow{\text{sp}} CH^{q-1}(X)$ factors through $CH^{q-1}(X_F) \otimes F^\times \rightarrow H^{q-1}(X_F, \mathcal{K}_q)$, which implies the surjectivity of ∂ , and therefore, that $CH^q(X \times_k \mathcal{O}_v) \rightarrow CH^q(X_F)$ is an isomorphism. $\square$

Let $\mathcal{I}_G^r$ be the maximal full subcategory of $\mathcal{I}_G$, for whose objects W the natural map $W^{G_{F/F'}} \rightarrow H_0(G_v^\dagger, W)$ is an isomorphism.

Corollary 4.7. *If $n = \infty$ then $\mathcal{I}_G^1$ is an abelian category closed under taking subquotients in $\mathcal{I}_G$.*

Proof. For any $W \in \mathcal{I}_G^1$ and any short exact sequence $0 \rightarrow W_1 \rightarrow W \rightarrow W_2 \rightarrow 0$ the rows in the commutative diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & W_1^{G_{F/F'}} & \rightarrow & W^{G_{F/F'}} & \rightarrow & W_2^{G_{F/F'}} \rightarrow 0 \\ & & \alpha_1 \downarrow & & \downarrow \cong & & \downarrow \alpha_2 \\ & & H_0(G_v^\dagger, W_1) & \rightarrow & H_0(G_v^\dagger, W) & \rightarrow & H_0(G_v^\dagger, W_2) \rightarrow 0 \end{array}$$

are exact (the upper – since, $H^0(G_{F/F'}, -) : \mathcal{S}m_G \rightarrow \mathcal{S}m_{G_{F'/k}}$ is an equivalence of categories, cf. [R1], Lemma 6.7). By Corollary 4.2 and Lemma 4.5, α_1 and α_2 are surjective, so they are isomorphisms. $\square$

REMARKS. 1. So far I do not know any smooth representation W with non-injective $W^{G_{F/F'}} \rightarrow H_0(G_v^\dagger, W_v)$. But first, I do not know, whether $W \mapsto W_v$ is exact.

2. The functor $H_0(G_v^\dagger, -) : \mathcal{S}m_G \rightarrow \mathcal{S}m_{G_{\kappa(v)/k}}$ is neither full, since $\mathbb{Q}[\{k(X) \xrightarrow{k} F\}]^G = 0$ for $r \geq d > 0$, but $H_0(G_v^\dagger, \mathbb{Q}[\{k(X) \xrightarrow{k} F\}])^{G_{\kappa(v)/k}} = \mathbb{Q}[\mathcal{P}_{k(X)}^d]$, nor faithful, since $(F^\times)^G = k^\times$ and $F^G = k$, but $H_0(G_v^\dagger \cap G_{F/F'}, F^\times) = H_0(G_v^\dagger \cap G_{F/F'}, F) = 0$.⁵ Also one has $H_0(G_v^\dagger, V) = 0$ for any semi-linear representation V of G over F , if $\text{tr.deg}(\kappa(v)/k) = \infty$.

Proof. Let $w \in V$ and $G_{F/L} \subseteq \text{Stab}_w$ for some L of finite type over k . Choose $t \in \mathfrak{m}_v \setminus \{0\}$ and $f \in F'$ transcendental over $L(t)$. Then there is $\sigma \in G_{F/L(t)} \cap G_v^\dagger$ such that $\sigma f - f = t$, i.e. $w = \sigma(t^{-1}fw) - t^{-1}fw$. $\square$

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⁵since for any $x \in F \setminus \mathcal{O}_v$ there are $\sigma, \tau \in G_v^\dagger \cap G_{F/F'}$ such that $\sigma t = tx^{-1}$ and $\tau x = 2x$, and $F \setminus \mathcal{O}_v$ generates both $F^\times$ and F .

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