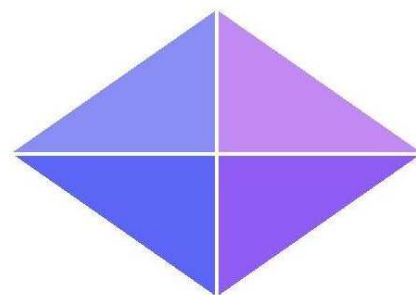


DFG-Forschergruppe
Regensburg/Leipzig
Algebraische Zyklen und
L-Funktionen



Admissible Semi-linear Representations

Marat Rovinsky

Preprint Nr. 01/2006

ADMISSIBLE SEMI-LINEAR REPRESENTATIONS

M.ROVINSKY

ABSTRACT. The category of admissible (in the appropriately modified sense of representation theory of totally disconnected groups) semi-linear representations of the automorphism group of an algebraically closed extension of infinite transcendence degree of the field of algebraic complex numbers is described.

Let k be a field of characteristic zero containing all ℓ -primary roots of unity for a prime ℓ , F be a universal domain over k , i.e., an algebraically closed extension of k of countable transcendence degree, and $G_{F/k}$ be the field automorphism group of F over k . We consider $G_{F/k}$ as a topological group with the base of open subgroups generated by $\{G_{F/k(x)} \mid x \in F\}$.

Denote by $\mathcal{C}$ the category of smooth (with open stabilizers) F -semi-linear representations of $G_{F/k}$, i.e., F -vector spaces V endowed with an additive semi-linear ($g(fv) = gf \cdot gv$ for any $f \in F$, $g \in G_{F/k}$ and $v \in V$) action $G_{F/k} \times V \rightarrow V$ of $G_{F/k}$.

Denote by $\mathcal{A}$ the full sub-category of $\mathcal{C}$ whose objects V are *admissible*: $\dim_{F^U} V^U < \infty$ for any open subgroup $U \subseteq G_{F/k}$. Clearly, $\mathcal{A}$ is an additive category and it is shown in [R] that it is a tensor (but not rigid) category. In the present paper one proves that the category $\mathcal{A}$ is abelian (Theorem 3.6), and F is its projective object (Proposition 3.4).

Let the ideal $\mathfrak{m} \subset F \otimes_{k_0} F$ be the kernel of the multiplication map $F \otimes_{k_0} F \xrightarrow{\times} F$, where $k_0 = k \cap \overline{\mathbb{Q}}$ is the number subfield of k . Consider the powers $\mathfrak{m}^s \subseteq F \otimes_{k_0} F$ of the ideal $\mathfrak{m}$ for all $s \geq 0$ as objects of $\mathcal{C}$ with the F -multiplication via $F \otimes_{k_0} k_0$.

In this paper we study the category $\mathcal{A}$ and describe it if k is a number field. Namely, in the case $k = \overline{\mathbb{Q}}$ we prove the following:

- The sum of the images of the F -tensor powers $\bigotimes_F^{\geq \bullet} \mathfrak{m}$ under all morphisms in $\mathcal{C}$ defines a decreasing filtration $W^\bullet$ on the objects of $\mathcal{A}$ such that its graded quotients gr_W^q are finite direct sums of direct summands of $\bigotimes_F^q \Omega_F^1$ (cf. §4.1, p.17 and Theorem 4.10). This filtration is evidently functorial and multiplicative: $(W^p V_1) \otimes_F (W^q V_2) \subseteq W^{p+q}(V_1 \otimes_F V_2)$ for any $p, q \geq 0$ and any $V_1, V_2 \in \mathcal{A}$.
- $\mathcal{A}$ is equivalent to the direct sum of the category of finite-dimensional k -vector spaces and its abelian full subcategory $\mathcal{A}^\circ$ with objects V such that $V^{G_{F/k}} = 0$ (Lemma 4.13).
- Any object V of $\mathcal{A}^\circ$ is a quotient of a direct sum of objects (of finite length) of type $\bigotimes_F^q (\mathfrak{m}/\mathfrak{m}^s)$ for some $q, s \geq 1$ (Theorem 4.10).
- If $V \in \mathcal{A}$ is of finite type then it is of finite length and $\dim_k \text{Ext}_{\mathcal{A}}^j(V, V') < \infty$ for any $j \geq 0$ and any $V' \in \mathcal{A}$; if $V \in \mathcal{A}$ is irreducible and $\text{Ext}_{\mathcal{A}}^1(\mathfrak{m}/\mathfrak{m}^q, V) \neq 0$ for some $q \geq 2$ then $V \cong \text{Sym}_F^q \Omega_F^1$ and $\text{Ext}_{\mathcal{A}}^1(\mathfrak{m}/\mathfrak{m}^q, V) \cong k$ (Corollary 4.17).
- $\mathcal{A}^\circ$ has no projective objects (Corollary 4.14), but $\bigotimes_F^q \mathfrak{m}$ are its “projective pro-generators”: the functor $\text{Hom}_{\mathcal{C}}(\bigotimes_F^q \mathfrak{m}, -)$ is exact on $\mathcal{A}$ for any q (Corollary 4.16).

The author was supported in part by RFBR grant 02-01-22005.

To describe the objects of $\mathcal{A}$, one studies first their “restrictions” to projective groups ($\cong \mathrm{PGL}_m k$), considered as subquotients of $G_{F/k}$. It is known ([R]) that such semi-linear representations are related to homogeneous vector bundles on projective spaces.

Let $n \geq 1$ be an integer, $K_n = k(\mathbb{P}_k^n)$ be the function field of an n -dimensional projective k -space $\mathbb{P}_k^n$ and $G_n = \mathrm{Aut}(\mathbb{P}_k^n/k)$ be its automorphism group.

Fix a k -field embedding $K_n \hookrightarrow F$. We show, in particular, that if V is an admissible F -semi-linear representation of $G_{F/k}$ with no sub-objects isomorphic to F then any irreducible subquotient of the K_n -semi-linear representation $V^{G_{F/K_n}}$ of G_n is a direct summand of $\bigotimes_{K_n}^{\geq 1} \Omega_{K_n/k}^1$ (this is Theorem 2.4 and Proposition 3.4 below).

0.1. Some motivation. The study of semi-linear representations comes from the study of $\mathbb{Q}$ -linear representations of $G_{F/k}$, that are related to geometry, cf. [R].

Let $\mathcal{S}m_G$ be the category of smooth representations of $G_{F/k}$ over k . Extending of coefficients to F gives a faithful functor $F \otimes_k : \mathcal{S}m_G \rightarrow \mathcal{C}$. It is not full: if $U \subset G_{F/k}$ is an open subgroup and $\bar{f} \in (F^\times/k^\times)^U - \{1\}$ then $[\sigma] \mapsto \sigma f \cdot [\sigma]$ defines an element of $\mathrm{End}_{\mathcal{C}}(F[G_{F/k}/U])$ which is not in $\mathrm{End}_{\mathcal{S}m_G}(k[G_{F/k}/U])$. However, its restriction to the subcategory $\mathcal{I}_G \otimes k$ of “homotopy invariant” representations¹ is.

Lemma 0.1. *If $k = \bar{k}$ then the functor $\mathcal{I}_G \otimes k \xrightarrow{F \otimes_k} \mathcal{C}$ is fully faithful.*

Proof. More generally, let us show that $\mathrm{Hom}_{\mathcal{S}m_G}(W, W') = \mathrm{Hom}_{\mathcal{C}}(F \otimes_k W, F \otimes_k W')$ for any $W \in \mathcal{I}_G \otimes k$ and any $W' \in \mathcal{S}m_G$. Let $\varphi \in \mathrm{Hom}_G(W, F \otimes_k W')$ and $\varphi(w) = \sum_{j=1}^N f_j \otimes w_j$ for some $w \in W$, $w_j \in W'$, $f_j \in F$ and minimal possible $N \geq 1$. We have to show that $f_j \in k$.

Choose a smooth proper model X of $k(f_1, \dots, f_N)$ over k . If it is not a point, choose a generically finite rational dominant map π to a projective space Y over k which is

- well-defined at the generic points of the irreducible components D_α of the divisors of poles of $f_1, \dots, f_N$,
- induces on each D_α a birational map and
- separates D_α .

Then the trace $\pi_* \varphi(w)$ has poles. On the other hand, $\pi_* \varphi(w)$ is in the image of $W^{G_{F/k}(Y)} = W^{G_{F/k}}$, so $\pi_* \varphi(w) \in (F \otimes_k W')^{G_{F/k}} = k \otimes_k (W')^{G_{F/k}}$ by Lemma 7.5 of [R]. This contradiction implies that $f_j \in k$, and therefore, $\varphi(W) \subseteq k \otimes_k W' \subseteq F \otimes_k W'$. $\square$

The $\mathbb{Q}$ -linear representations of $G_{F/k}$ of particular interest are admissible representations, forming a full subcategory in $\mathcal{I}_G \otimes k$. Though tensoring with F does not transform them to admissible semi-linear representations,² there exists, at least if $k = \overline{\mathbb{Q}}$, a similar faithful functor in the opposite direction.

Namely, it is explained in Corollary 5.2 that, for any object V of $\mathcal{A}$ and any smooth k -variety Y , embedding of the generic points of Y into F determines a locally free coherent sheaf $\mathcal{V}_Y$ on Y . Any dominant morphism $X \xrightarrow{\pi} Y$ of smooth k -varieties induces an injection of coherent sheaves $\pi^* \mathcal{V}_Y \hookrightarrow \mathcal{V}_X$, which is an isomorphism if π is étale.

This gives an equivalence $\mathcal{S} : \mathcal{A} \xrightarrow{\sim} \{\text{“coherent” sheaves in the smooth topology}\}, V \mapsto (Y \mapsto \mathcal{V}_Y(Y))$. More generally, the “coherent” sheaves are contained in the category $\mathcal{F}l$ of the flat “quasi-coherent” sheaves in the smooth topology, cf. §5, p.18. For any flat

¹i.e. such that $W^{G_{F/L}} = W^{G_{F/L'}}$ for any purely transcendental extension L'/L of subfields in F

²and moreover, there are irreducible objects of $\mathcal{C}$ outside of $\mathcal{A}$ (Corollary 3.5).

“quasi-coherent” sheaf $\mathcal{V}$ in the smooth topology the space $\Gamma(Y, \mathcal{V}_Y)$ is a birational invariant of a proper Y (Lemma 5.3). Then we get a left exact functor $\mathcal{F}l \xrightarrow{\Gamma} \mathcal{S}m_G$ given by $V \mapsto \varinjlim \Gamma(Y, \mathcal{V}_Y)$, where Y runs over the smooth proper models of subfields in F of finite type over k .

The functor $\Gamma \circ \mathcal{S}$ is faithful, since $\Gamma(Y', \mathcal{V}_{Y'})$ generates the (generic fibre of the) sheaf $\mathcal{V}_{Y'}$ for appropriate finite covers Y' of Y (Lemma 5.3), if $\mathcal{V}$ is “coherent”. But it is not full, and the objects in its image are highly reducible. If $\Gamma(Y, \mathcal{V}_Y)$ has the Galois descent property then $\Gamma(V)$ is admissible. However, there is no Galois descent property in general.

0.2. Notation. Let k, F and $G_{F/k}$ be as above. For a subfield L of F we denote by $\bar{L}$ its algebraic closure in F . We fix a transcendence basis $x_1, x_2, x_3, \dots$ of F over k .

For each $n \geq 1$ set $Y_n = \mathbf{Spec} k[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \subset \mathbb{A}_k^n = \mathbf{Spec} k[x_1, \dots, x_n] \subset \mathbb{P}_k^n = \mathbf{Proj} k[X_0, \dots, X_n]$ with $x_j = X_j/X_0$, $K_n = k(x_1, \dots, x_n)$, $G_n = \text{Aut}(\mathbb{P}_k^n/k)$.

Let $\text{Aff}_n = G_n \cap \text{Aut}(\mathbb{A}_k^n/k)$ be the affine subgroup, and $(\text{Aff}_n)_u$ be its unipotent radical, i.e., the translation subgroup. Let $H_n = \text{Aff}_n \cap \text{Aut}(\mathbb{A}_k^n/\mathbb{A}_k^{n-1})$ be the subgroup fixing the coordinates $x_1, \dots, x_{n-1}$ on $\mathbb{A}_k^n$. Let $T_n \subset G_n$ be the maximal torus acting freely on Y_n .

Denote by T_n^{tors} the torsion subgroup in T_n .

For a field extension L/K we denote by $\text{Der}(L/K)$ the Lie K -algebra of derivations of L over K . For an integer $\ell \geq 2$, the group of ℓ -th roots of unity in $\bar{k}$ is denoted by μ_ℓ , and the corresponding cyclotomic number subfield in $\bar{k}$ is denoted by $\mathbb{Q}(\mu_\ell)$.

0.3. Structure of the paper. As it is mentioned above, we consider the projective groups G_n as subquotients of $G_{F/k}$. In §1 we identify irreducible subquotients of “restrictions” of objects of $\mathcal{A}$ to G_n with the generic fibres of the G_n -equivariant coherent sheaves on $\mathbb{P}_k^n$. Main ingredients there come from [BT] and [R]. In §2 we exclude some cases, thus showing that these irreducible subquotients are direct summands of $\bigotimes_{K_n}^\bullet \Omega_{K_n/k}^1$. In §3 we show that $\mathcal{A}$ is abelian and calculate Ext^1 -groups between the irreducible objects of a tannakian category $\mathfrak{SL}_u^n$ (defined at the beginning of §1, p.3) of semi-linear representations of G_n , containing “restrictions” of objects of $\mathcal{A}$ to G_n . The latter part uses [LR]. After showing principal structural results on $\mathcal{A}$ (in §4) we identify (in §5) $\mathcal{A}$ with the category of “coherent” sheaves in smooth topology. Finally (in §6), we define a descending filtration $\mathcal{A}_{>\bullet}$ of $\mathcal{A}$ by Serre “ideal” subcategories. Then we localize the quotients $\mathcal{A}/\mathcal{A}_{>m}$ for each $m \geq 0$ to get a tannakian subcategory of finite-dimensional semi-linear representations of $G_{F'/k}$ over F' for an algebraically closed extension F' of k in F of transcendence degree m .

1. EQUIVARIANTNESS OF IRREDUCIBLE PGL -SHEAVES

Let $\mathfrak{SL}_n^u$ be the category of finite-dimensional semi-linear representations of G_n over K_n whose restrictions to the maximal torus T_n in G_n are of type $K_n \otimes_k W$ for unipotent representations W of T_n (where T_n is considered as a discrete group).

Note that $V = V^{T_n^{\text{tors}}} \otimes_k K_n$ for any $V \in \mathfrak{SL}_n^u$.

In [R], for $n > 1$, a fully faithful functor $\mathfrak{SL}_n^u \xrightarrow{\mathcal{S}} \{\text{coherent } G_n\text{-sheaves on } \mathbb{P}_k^n\}$ is constructed. (A G_n -sheaf is G_n -equivariant sheaf if G_n is considered as a discrete group. In other words, $\mathcal{V}$ is a G_n -sheaf if it is endowed with a collection of isomorphisms $\alpha_g : \mathcal{V} \xrightarrow{\sim} g^*\mathcal{V}$ for each $g \in G_n$ satisfying the chain rule: $\alpha_{hg} = g^*\alpha_h \circ \alpha_g$ for any $g, h \in G_n$. The term “ G_n -equivariant” is reserved for G_n -vector bundles with algebraic G_n -action on their total

spaces.) The composition of $\mathcal{S}$ with the generic fibre functor is the identical full embedding of $\mathfrak{SL}_n^u$ into the category of finite-dimensional K_n -semi-linear G_n -representations.

In this section we show that the category $\mathfrak{SL}_n^u$ is abelian and its irreducible objects are generic fibres of irreducible coherent G_n -equivariant sheaves on $\mathbb{P}_k^n$, i.e., direct summands of $\mathrm{Hom}_{K_n}((\Omega_{K_n/k}^n)^{\otimes M}, \bigotimes_{K_n}^\bullet \Omega_{K_n/k}^1)$ for appropriate integer $M \geq 0$.

Lemma 1.1. *The category $\mathfrak{SL}_n^u$ is closed under taking K_n -semi-linear subquotients.*

Proof. Let $V \in \mathfrak{SL}_n^u$ and $0 \rightarrow V_1 \rightarrow V \xrightarrow{\pi} V_2 \rightarrow 0$ be a short exact sequence of semi-linear representations of G_n over K_n . As the k -vector space $V^{T_n^{\mathrm{tors}}}$ (of the elements in V fixed by the torsion subgroup T_n^{tors} in T_n) spans the K_n -vector space V , the k -vector space $\pi(V^{T_n^{\mathrm{tors}}}) \subseteq V_2^{T_n^{\mathrm{tors}}}$ spans the K_n -vector space V_2 .

This means that $V_2 = V_2^{T_n^{\mathrm{tors}}} \otimes_k K_n$ and $\pi(V^{T_n^{\mathrm{tors}}}) = V_2^{T_n^{\mathrm{tors}}}$.

In other words, the sequence of T_n^{tors} -invariants $0 \rightarrow V_1^{T_n^{\mathrm{tors}}} \rightarrow V^{T_n^{\mathrm{tors}}} \rightarrow V_2^{T_n^{\mathrm{tors}}} \rightarrow 0$ is exact, and extending its coefficients to K_n gives the exact sequence $0 \rightarrow V_1^{T_n^{\mathrm{tors}}} \otimes_k K_n \rightarrow V = V^{T_n^{\mathrm{tors}}} \otimes_k K_n \xrightarrow{\pi'} V_2 = V_2^{T_n^{\mathrm{tors}}} \otimes_k K_n \rightarrow 0$. As π coincides with π' , we get $V_1 = V_1^{T_n^{\mathrm{tors}}} \otimes_k K_n$.

Clearly, any subquotient of a unipotent representation of T_n is again unipotent, and thus, $V_1, V_2 \in \mathfrak{SL}_n^u$. $\square$

Lemma 1.2. *Let E be the total space of a vector bundle on $\mathbb{P}_k^n$, $\mathrm{Aut}_{\mathrm{lin}}(E)$ be the group of automorphisms of E over k inducing linear transforms between the fibres, and $\tau : G_n \rightarrow \mathrm{Aut}_{\mathrm{lin}}(E)$ be an irreducible G_n -structure on E , i.e., a discrete group homomorphism splitting the projection $\mathrm{Aut}_{\mathrm{lin}}(E) \rightarrow G_n$. Then the Zariski closure $\overline{\tau(G_n)}$ is reductive.*

Proof. Let Aut_τ be the kernel of the projection $\overline{\tau(G_n)} \xrightarrow{\pi} G_n$.

For each point $p \in \mathbb{P}_k^n$ let $\rho_p : R_p := \pi^{-1}(\mathrm{Stab}_p) \rightarrow \mathrm{GL}(E_p)$ be the natural representation.

As we suppose that E is an irreducible G_n -bundle, ρ_p is irreducible, since otherwise $B := \overline{\tau(G_n)}B_p \subset E$ is a G_n -subbundle for any proper R_p -invariant k -subspace $B_p \subset E_p$.

In particular, ρ_p is trivial on the unipotent radical of R_p . The unipotent radical of any algebraic group contains the unipotent radical of its arbitrary normal subgroup, so ρ_p is trivial on the unipotent radical of Aut_τ . As the action of Aut_τ on E is faithful, $\bigcap_p \ker \rho_p|_{\mathrm{Aut}_\tau} = \{1\}$, i.e., Aut_τ is reductive. As G_n is also reductive, so is $\overline{\tau(G_n)}$. $\square$

For a commutative finite k -algebra A denote by $R_{A/k}$ the Weil functor of restriction of scalars on A -schemes, cf. [DG], I, §1, 6.6.

We need the following particular case of Théorème 8.16 of [BT].

Theorem 1.3. *Let G be a simply connected absolutely almost simple k -group, and G' be a reductive k -group. Let $\tau : G(k) \rightarrow G'(k)$ be a homomorphism with Zariski dense image. Let $G'_1, \dots, G'_m$ be the almost simple normal subgroups of G' .*

Then there exist finite field extensions k_i/k , field embeddings $\varphi_i : k \rightarrow k_i$, a special isogeny $\beta : \prod_{i=1}^m R_{k_i/k}^{\varphi_i} G \rightarrow G'$ (here ${}^{\varphi_i} G := G \times_{k, \varphi_i} k_i$) and a homomorphism $\mu : G(k) \rightarrow Z_{G'}(k)$ such that $\beta(R_{k_i/k}^{\varphi_i} G) = G'_i$ and $\tau(h) = \mu(h) \cdot \beta(\prod_{i=1}^m \varphi_i^\circ(h))$ for any $h \in G(k)$ (here $\varphi_i^\circ : G(k) \rightarrow (R_{k_i/k}^{\varphi_i} G)(k)$ is the canonical homomorphism). $\square$

Corollary 1.4. *Under assumptions of Theorem 1.3, for any torus $T \subset G$ the Zariski closure of $\tau(T(k))$ is a torus in G' . $\square$*

Proposition 1.5. *If $n \geq 2$ then any irreducible object of $\mathfrak{SL}_n^u$ is a direct summand of $\mathrm{Hom}_{K_n}((\Omega_{K_n/k}^n)^{\otimes M}, \bigotimes_{K_n}^\bullet \Omega_{K_n/k}^1)$ for an appropriate M .*

Proof. The functor $\mathcal{S}$, mentioned in the beginning of this §, associates to an irreducible object V of $\mathfrak{SL}_n^u$ a coherent G_n -sheaf $\mathcal{V}$ on $\mathbb{P}_k^n$ with generic fibre V .

Let, as before, T_n be a maximal torus in G_n and $Y_n \subset \mathbb{P}_k^n$ be the n -dimensional T_n -orbit. As $V^{T_n^{\text{tors}}} = \Gamma(Y_n, \mathcal{V})^{T_n^{\text{tors}}}$, cf. [R], is a unipotent representation of T_n , Lemma 1.2 and Corollary 1.4 imply that $\Gamma(Y_n, \mathcal{V})^{T_n^{\text{tors}}}$ is a trivial representation of T_n .

In a k -basis of $V^{T_n^{\text{tors}}}$ the G_n -action on V determines a 1-cocycle $(g_\sigma) \in Z^1(G_n, \text{GL}_M K_n)$, where $M = \dim_{K_n} V$. There is an integer $N > n + 2$ and elements $\alpha_1, \dots, \alpha_N \in G_n$ such that the morphism $(T_n)^N \xrightarrow{\pi} G_n$, given by $(h_1, \dots, h_N) \mapsto \alpha_1 h_1 \alpha_1^{-1} \cdots \alpha_N h_N \alpha_N^{-1}$, is surjective. Namely, using the Gauß elimination algorithm, one shows that any element of G_n is a product of $\leq (n+1)^2$ elementary matrices and an element of T_n . On the other hand, it follows from the identity $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ b-a & b \end{pmatrix}$ that for any elementary matrix α the product $T_n \cdot \alpha T_n \alpha^{-1} \cdot T_n$ contains all elementary matrices of the same type as α . This gives a surjection $(T_n \times \prod_{i \neq j} \alpha_{ij} T_n \alpha_{ij}^{-1})^{(n+1)^2} \times T_n \xrightarrow{\times} G_n$, where α_{ij} is the elementary matrix with 1 in the i -th row and j -th column.

Then

$$g_\pi(h_1, \dots, h_N) = g_1(x) g'_1(\alpha_1 h_1(x)) g_2(\alpha_1 h_1 \alpha_1^{-1}(x)) g'_2(\alpha_1 h_1 \alpha_1^{-1} h_2(x)) \cdots \\ g_N(\alpha_1 h_1 \alpha_1^{-1} \cdots \alpha_{N-1} h_{N-1} \alpha_{N-1}^{-1}(x)) g'_N(\alpha_1 h_1 \alpha_1^{-1} \cdots \alpha_{N-1} h_{N-1} \alpha_{N-1}^{-1} \alpha_N h_N(x)),$$

where $g_j(x) := g_{\alpha_j}$ and $g'_j(x) := g_{\alpha_j^{-1}}$ for all $1 \leq j \leq N$. In other words, the lifting of the G_n -action on $\text{tot}(\mathcal{V})$ to $(T_n)^N$ -“coupling” via π determines a rational map $(T_n)^N \times \mathbb{P}_k^n \dashrightarrow \text{GL}_M k$. Clearly, it corresponds to a regular morphism $(T_n)^N \times \text{tot}(\mathcal{V}) \rightarrow \text{tot}(\mathcal{V})$ and factors through a regular morphism $G_n \times \text{tot}(\mathcal{V}) \rightarrow \text{tot}(\mathcal{V})$ of k -varieties, i.e., we see that $\mathcal{V}$ is equivariant.

The generic fibres of irreducible G_n -equivariant sheaves on $\mathbb{P}_k^n$ are exactly of the desired type. $\square$

REMARK. There can exist, a priori, non-equivariant irreducible coherent G_n -sheaves on $\mathbb{P}_k^n$, e.g. the extension of coefficients to $\mathcal{O}_{\mathbb{P}_k^n}$ of a non-rational representation of G_n is seemingly of this type.

2. “POSITIVITY”

In this section we show that for any admissible F -semi-linear representation V of $G_{F/k}$ any irreducible subquotient of the K_n -semi-linear representation $V^{G_{F/K_n}}$ of G is a direct summand of $\bigotimes_{K_n}^\bullet \Omega_{K_n/k}^1$.

It is shown in [R] that any finite-dimensional K_n -semi-linear G_n -representation extendable to $\text{End}(K_n/k)$, e.g. $V^{G_{F/K_n}}$, is an object of $\mathfrak{SL}_n^u$. By Proposition 1.5, we only need to eliminate the negative twists by $\Omega_{K_n/k}^n$ in irreducible subquotients of $V^{G_{F/K_n}}$.

To do that we show first that the generic fibres of irreducible coherent G_n -equivariant sheaves are determined by their restrictions to the subgroup $\text{Aff}_n = G_n \cap \text{Aut}(\mathbb{A}_k^n/k)$.

Lemma 2.1. *Let Aff_n be the group of affine transformations of an affine space $\mathbb{A}_k^n$ with the function field K_n . Then the natural morphism*

$$(1) \quad \{\text{rational } k\text{-linear } \text{Aff}_n\text{-representations}\} \xrightarrow{\otimes_k^{K_n}} \{K_n\text{-semi-linear } \text{Aff}_n\text{-representations}\}$$

transforms isomorphism classes of irreducible k -representations of Aff_n to isomorphism classes of irreducible K_n -semi-linear representations of $\text{Aff}_n^{(1)}\mathbb{Q}$, the subgroup of Aff_n consisting of $\mathbb{Q}$ -affine substitutions of $x_1, \dots, x_n$ with Jacobian equal to 1.

Proof. Let W be an irreducible k -representations of Aff_n , and $U \subset W \otimes_k K_n$ a non-zero K_n -semi-linear subrepresentation of $\text{Aff}_n^{(1)}\mathbb{Q}$. Let $\alpha = \sum_{j=1}^N w_j \alpha_j \in U$ be a non-zero element with minimal possible N , where $w_j \in W$ and $\alpha_j \in K_n$. Multiplying α by an element of K_n we may assume that all α_j are polynomials: $\alpha = \sum_I w'_I x^I$. Since W is irreducible, the elements of the unipotent radical $(\text{Aff}_n)_u$ of Aff_n , i.e., σ such that $\sigma z - z \in k$ for any linear function z on $\mathbb{A}_k^n$, act trivially on W .

Applying an appropriate composition of difference operators $\sigma - \tau$ for some σ, τ in the unipotent radical of $\text{Aff}_n^{(1)}\mathbb{Q}$ to α , we can lower the degrees of the polynomials α_j and eventually get a non-zero element of W . As $W = W_0 \otimes_{\mathbb{Q}} k$ for an irreducible representation W_0 of $\text{Aff}_n^{(1)}\mathbb{Q}$, any non-zero element of W generates $W \otimes_k K_n$, which means that $U = W \otimes_k K_n$. $\square$

Corollary 2.2. *Let Aff_n , $(\text{Aff}_n)_u$, $\mathbb{A}_k^n$ and K_n be as in Lemma 2.1. Then $V \mapsto V^{(\text{Aff}_n)_u}$ gives a natural bijection*

$$(2) \quad \left\{ \begin{array}{c} \text{isomorphism classes of} \\ \text{irreducible } G_n\text{-subrepresentations in} \\ \bigoplus_M \text{Hom}_K((\Omega_{K_n/k}^n)^{\otimes M}, \bigotimes_{K_n}^{\bullet} \Omega_{K_n/k}^1) \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} \text{isomorphism classes of} \\ \text{irreducible rational } k\text{-linear} \\ \text{Aff}_n\text{-representations} \end{array} \right\}$$

such that its composition with the morphism (1) is the inclusion map. $\square$

Let W be an $(n+1)$ -dimensional k -vector space, $L \subset W$ be a one-dimensional subspace, and $H_{\text{lin}} = \ker[\text{GL}(W, L) \rightarrow \text{GL}(W/L, L)] \cong k^\times \ltimes \text{Hom}(W/L, L)$ be the group preserving L and inducing the identity automorphism of W/L .

Lemma 2.3. *For any Young diagram λ with no columns of height $\geq n+1$ one has*

$$(S^\lambda W^\vee \otimes_k (\det W)^{\otimes s})^{H_{\text{lin}}} = \begin{cases} S^\lambda(W/L)^\vee & \text{if } s = 0, \\ 0 & \text{otherwise} \end{cases}$$

Proof. Denote by $X = AF(W) \cong \text{GL}(W)/R_u(B)$, the variety of complete affine flags in W . An affine flag is a filtration $W_\bullet = (0 = W_0 \subset W_1 \subset W_2 \subset \dots \subset W_{n+1} = W)$ with $\dim_k W_j = j$ and a collection of $l_j \in W_j/W_{j-1} - \{0\}$.

Let $Y \cong \text{GL}(W)/B$ be the variety of complete linear flags in W . Then the natural projection $X \xrightarrow{\pi} Y$ is a principal $(\mathbb{G}_m)^{n+1}$ -bundle, and there is a decomposition $\pi_* \mathcal{O}_X = \bigoplus_\mu \mathcal{M}(\mu)$ into a direct sum of invertible sheaves on Y , where μ runs over the group $\mathbb{Z}^{n+1}$ of characters of $(\mathbb{G}_m)^{n+1}$, so $\mathcal{O}(X) = \bigoplus_\mu \Gamma(Y, \mathcal{M}(\mu))$.

Set $X^\circ = \{(V_\bullet, l_\bullet) \in X \mid V_n \cap L = 0 \Leftrightarrow "l_{n+1} \in L"\}$. Then reduction modulo L defines a principal $L^{\oplus n} \times \mathbb{G}_m$ -bundle $X^\circ \rightarrow AF(W/L)$, $(l_1, \dots, l_{n+1}) \mapsto (l_1, \dots, l_n)$.

Let $X^\circ \xrightarrow{"l_{n+1}"^{-1}} L - \{0\}$ be the natural H - (or $\mathbb{G}_m$ -) equivariant map, and $\bar{l}_{n+1}$ be the composition of $"l_{n+1}"$ with a fixed isomorphism $L - \{0\} \cong \mathbb{G}_m$.

Set $SH := H_{\text{lin}} \cap \text{SL}(W) = \text{Hom}(W/L, L)$. Then $\mathcal{O}(X)^{SH} = \mathcal{O}(X) \cap \mathcal{O}(X^\circ)^{SH} = \mathcal{O}(X) \cap \mathcal{O}(AF(W/L))[\bar{l}_{n+1}, \bar{l}_{n+1}^{-1}]$, so $\mathcal{O}(X)(\mu)^{SH} = \mathcal{O}(X) \cap \mathcal{O}(AF(W/L))(\mu')\bar{l}_{n+1}^{\mu_{n+1}}$, where $\mu' \in \mathbb{Z}^n$ is the restriction of μ to the first n multiples of $(\mathbb{G}_m)^{n+1}$.

For any μ this is an irreducible representation of $\text{GL}(W/L)$, and thus, $\mathcal{O}(X)(\mu)^{SH} = \mathcal{O}(AF(W/L))(\mu')\bar{l}_{n+1}^{\mu_{n+1}}$ if $\mathcal{O}(X)(\mu) \neq 0$.

As any irreducible representation of $\mathrm{SL}(W)$ coincides with $\mathcal{O}(X)(\mu)$ for some μ , this implies that $(S^\lambda W^\vee)^{SH} = S^\lambda(W/L)^\vee$. $\square$

Theorem 2.4. *For any F -semi-linear $G_{F/k}$ -representation $V \in \mathcal{A}$ any irreducible subquotient of the K_n -semi-linear G_n -representation $V^{G_{F/K_n}}$ is a direct summand of $\bigotimes_{K_n}^\bullet \Omega_{K_n/k}^1$.*

Proof. Let $W = \mathbb{A}_k^{n+1}$ be the vector space with coordinates $x_1, \dots, x_{n+1}$, so $k(W) = K_{n+1}$. By Proposition 1.5 and Corollary 2.2, the restrictions to $\mathrm{Aff}_{n+1} = \mathrm{Aff}(W)$ of irreducible subquotients of the K_{n+1} -semi-linear G_{n+1} -representation $V^{G_{F/K_{n+1}}}$ are of type $(S^\lambda W^\vee \otimes (\det W)^{\otimes s}) \otimes_k K_{n+1}$ for a Young diagram λ with no columns of height $n+1$ and some integer s , where Aff_{n+1} acts on W via its reductive quotient $\mathrm{GL}(W)$.

Let $H \subset \mathrm{Aff}_{n+1}$ be the subgroup fixing the functionals $x_1, \dots, x_n$ in $W^\vee$ vanishing on L . Let Aff_u be the unipotent radical of Aff_{n+1} , i.e. the group of translations of W .

Then the restrictions to Aff_n of the irreducible subquotients of the K_n -semi-linear G_n -representation $V^{G_{F/K_n}}$ are contained in $((S^\lambda W^\vee \otimes (\det W)^{\otimes s}) \otimes_k K_{n+1})^H$. As $H \cap \mathrm{Aff}_u = \langle 1 \rangle_k \cong k$, we get $((S^\lambda W^\vee \otimes (\det W)^{\otimes s}) \otimes_k K_{n+1})^{H \cap \mathrm{Aff}_u} = (S^\lambda W^\vee \otimes (\det W)^{\otimes s}) \otimes_k K_n$, so $((S^\lambda W^\vee \otimes (\det W)^{\otimes s}) \otimes_k K_{n+1})^H = (S^\lambda W^\vee \otimes (\det W)^{\otimes s})^H \otimes_k K_n$.

By Lemma 2.3, $(S^\lambda W^\vee \otimes (\det W)^{\otimes s})^H$ coincides with $S^\lambda(W/L)^\vee$ if $s = 0$, and vanishes otherwise. This means that any representation of Aff_n obtained this way is a direct summand of the tensor algebra of the representation $(W/L)^\vee = (\Omega_{K_n/k}^1)^{\{\text{translations}\}}$ of $\mathrm{GL}_n k$. As any irreducible subquotient U of the K_n -semi-linear G_n -representation $V^{G_{F/K_n}}$ is determined by its restriction $U|_{\mathrm{Aff}_n}$ to Aff_n and $U|_{\mathrm{Aff}_n}$ is a direct summand of $\bigotimes_{K_n}^\bullet \Omega_{K_n/k}^1$, the same holds for U . $\square$

3. EXTENSIONS IN $\mathfrak{SL}_n^u$ AND IN $\mathcal{A}$

For an integer $\ell \geq 2$ such that $\mu_\ell \subset k$ (see §0.2), denote by $\mathrm{Aff}_n^{(\ell)} \mathbb{Q}$ the subgroup of Aff_n consisting of the $\mathbb{Q}(\mu_\ell)$ -affine substitutions of $x_1, \dots, x_n$ with Jacobian in μ_ℓ : $x_i \mapsto \sum_{j=1}^n a_{ij} x_j + b_i$, where $a_{ij}, b_i \in \mathbb{Q}(\mu_\ell) \subset k$ and $\det(a_{ij}) \in \mu_\ell$; and by $\mathrm{SAff}_n^{(\ell)} \mathbb{Q}$ the subgroup of index ℓ consisting of elements with Jacobian equal to 1: $\det(a_{ij}) = 1$.

Lemma 3.1. *Let $n, \ell \geq 2$ be integers. Assume that $\mu_\ell \subset k$. Let U_0 be the unipotent radical of $\mathrm{SAff}_n^{(\ell)} \mathbb{Q}$. Then for any object $V \in \mathfrak{SL}_n^u$ there is a rational representation W of the reductive quotient $\mathrm{SL}_n \mathbb{Q}(\mu_\ell) = \mathrm{SAff}_n^{(\ell)} \mathbb{Q} / U_0$ of $\mathrm{SAff}_n^{(\ell)} \mathbb{Q}$, and an isomorphism of semi-linear $\mathrm{SAff}_n^{(\ell)} \mathbb{Q}$ -modules $W \otimes_{\mathbb{Q}(\mu_\ell)} K_n \xrightarrow{\sim} V$.*

Irreducible rational representations of $\mathrm{SL}_n \mathbb{Q}(\mu_\ell)$ with coefficients extended to K_n are irreducible semi-linear representations of $\mathrm{SAff}_n^{(\ell)} \mathbb{Q}$ over K_n . In particular, any extension in $\mathfrak{SL}_n^u$ splits as an extension of K_n -semi-linear representations of $\mathrm{SAff}_n^{(\ell)} \mathbb{Q}$.

Proof. It is shown in Lemma 6.3 (1) of [R] that $H^0(U_0, -)$ is a fibre functor on $\mathfrak{SL}_n^u$ independent of ℓ , so $V = V^{U_0} \otimes_k K_n$, i.e. the restriction of V to $\mathrm{SAff}_n^{(\ell)} \mathbb{Q}$ is a k -linear representation V^{U_0} of $\mathrm{SL}_n \mathbb{Q}(\mu_\ell)$ with coefficients extended to K_n , for any $V \in \mathfrak{SL}_n^u$.

As it follows from Proposition 1.5, the irreducible subquotients V_α of V restricted to $\mathrm{SAff}_n^{(\ell)} \mathbb{Q}$ are of the form $W_\alpha \otimes_{\mathbb{Q}(\mu_\ell)} K_n$, where W_α are rational irreducible representations of $\mathrm{SL}_n \mathbb{Q}(\mu_\ell)$. Then the irreducible subquotients of V^{U_0} are $V_\alpha^{U_0} = W_\alpha \otimes_{\mathbb{Q}(\mu_\ell)} k$, and V_α are irreducible semi-linear representations of $\mathrm{SAff}_n^{(\ell)} \mathbb{Q}$ by Lemma 2.1.

If V^{U_0} is not semi-simple then it admits a non-semi-simple subquotient W of length 2. Let in Theorem 3.9 $\kappa = \mathbb{Q}(\mu_\ell)$, $K = k$, $G = \mathrm{SL}_n$, $\mathcal{G}$ be the Zariski closure of the

image of $\mathrm{SL}_n \mathbb{Q}(\mu_\ell)$ in $\mathrm{GL}_k(W)$ and let τ be given by the $\mathrm{SL}_n \mathbb{Q}(\mu_\ell)$ -action on W . Then the unipotent radical of $\mathcal{G}$ is commutative. As the derivations of $\kappa = \mathbb{Q}(\mu_\ell)$ are zero, we see that the k -linear representation W of $\mathrm{SL}_n \mathbb{Q}(\mu_\ell)$ is semi-simple. $\square$

REMARKS. 1. Using Theorems 1.3 and 3.9 it is not hard to show that any representation of $\mathrm{SL}_n K$ over any field of characteristic zero is semi-simple for any number field K .

2. Let $V = \Omega_{K_n}^1 / \Lambda \otimes_k K_n$, where $\Lambda \subset \Omega_k^1$ is a proper k -subspace. Let the extension $0 \rightarrow V \rightarrow U \rightarrow K_n \rightarrow 0$ be given by the cocycle $(\omega_\sigma = d \log \frac{\sigma \omega}{\omega}) \in Z^1(G_n, V)$, where $\omega = dx_1 \wedge \cdots \wedge dx_n \in \Omega_{K_n/k}^n$. Then the restriction of (ω_σ) to $\mathrm{GL}_n k$ is non-trivial.

3. The convolution with the Euler vector field $\sum_{j=1}^n x_j \partial / \partial x_j$ defines a $\mathrm{GL}_n k$ -equivariant morphism $\Omega_{K_n/k}^1 \rightarrow K_n$ given by $dx_j \mapsto x_j$. It is non-split for $n \geq 3$, since $(\Omega_{K_n/k}^1)^{\mathrm{SL}_n \mathbb{Q}} = 0$.

Lemma 3.2. *Let $n, \ell \geq 2$ and s be some integers, and λ be a Young diagram with columns of height $< n$ such that ℓ does not divide $s + \frac{|\lambda|}{n-1}$, if λ is rectangular of height $n-1$ and non-empty. Let $V = S_{K_n}^\lambda \Omega_{K_n/k}^1 \otimes_{K_n} (\Omega_{K_n/k}^n)^{\otimes s}$.*

Then $(V^{\mathcal{H}_n^{(\ell)}})^{\mathrm{Aff}_n^{(\ell)} \mathbb{Q}} = V^{\mathrm{Aff}_n^{(\ell)} \mathbb{Q}}$, where $\mathcal{H}_n^{(\ell)} := G_{K_n/K_{n-1}} \cap \mathrm{Aff}_n^{(\ell)} \mathbb{Q}$.

Proof. Let W be the k -span of $dx_1, \dots, dx_n$ in $\Omega_{K_n/k}^1$. Then $S_{K_n}^\lambda \Omega_{K_n/k}^1 = S_k^\lambda W \otimes_k K_n$ and $\Omega_{K_n/k}^n = \det_k W \otimes_k K_n$. Set $S\mathcal{H}_n^{(\ell)} = \mathcal{H}_n^{(\ell)} \cap \mathrm{SAff}_n^{(\ell)} \mathbb{Q}$. Then $\mathcal{H}_n^{(\ell)} \cong \mu_\ell \ltimes S\mathcal{H}_n^{(\ell)}$, and therefore, $V^{\mathcal{H}_n^{(\ell)}} = (V^{S\mathcal{H}_n^{(\ell)}})^{\mu_\ell}$.

One has $V^{S\mathcal{H}_n^{(\ell)}} = (S_k^\lambda W \otimes_k K_n)^{S\mathcal{H}_n^{(\ell)}} \otimes_k (\det_k W)^{\otimes s}$. As the intersection of the unipotent radical of $\mathrm{Aff}_n^{(\ell)} \mathbb{Q}$ with $S\mathcal{H}_n^{(\ell)}$ (i.e. the $\mathbb{Q}(\mu_\ell)$ -translations of x_n) acts trivially on $S_k^\lambda W$ and fixes exactly K_{n-1} in K_n , if $n \geq 1$, we get

$$V^{S\mathcal{H}_n^{(\ell)}} = (S_k^\lambda W)^{S\mathcal{H}_n^{(\ell)}} \otimes_k (\det_k W)^{\otimes s} \otimes_k K_{n-1} = S_k^\lambda (W^{S\mathcal{H}_n^{(\ell)}}) \otimes_k (\det_k W)^{\otimes s} \otimes_k K_{n-1}.$$

Then

$$\begin{aligned} V^{\mathcal{H}_n^{(\ell)}} &= S_k^\lambda (W^{S\mathcal{H}_n^{(\ell)}}) \otimes_k ((\det_k W)^{\otimes s})^{\mu_\ell} \otimes_k K_{n-1} \\ &= \begin{cases} S_{K_{n-1}}^\lambda \Omega_{K_{n-1}/k}^1 \otimes_{K_{n-1}} (\Omega_{K_{n-1}/k}^{n-1})^{\otimes s} & \text{if } \ell | s \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

On the other hand, $V^{\mathrm{Aff}_n^{(\ell)} \mathbb{Q}} = (S_k^\lambda W \otimes_k (\det_k W)^{\otimes s} \otimes_k K_n)^{\mathrm{Aff}_n^{(\ell)} \mathbb{Q}}$ coincides with $(S_k^\lambda W \otimes_k (\det_k W)^{\otimes s})^{\mathrm{Aff}_n^{(\ell)} \mathbb{Q}}$, since the unipotent radical of $\mathrm{Aff}_n^{(\ell)} \mathbb{Q}$ acts trivially on $S_k^\lambda W \otimes_k (\det_k W)^{\otimes s}$ and fixes exactly k in K_n . Thus, for $n \geq 1$, we get $V^{\mathrm{Aff}_n^{(\ell)} \mathbb{Q}} = \begin{cases} k & \text{if } \lambda = 0 \text{ and } \ell | s, \\ 0 & \text{otherwise.} \end{cases}$ This

implies that $(V^{\mathcal{H}_n^{(\ell)}})^{\mathrm{Aff}_n^{(\ell)} \mathbb{Q}} = \begin{cases} k & \text{if } \lambda \text{ is rectangular of height } n-1 \text{ and } \ell | (s + \frac{|\lambda|}{n-1}), \\ 0 & \text{otherwise} \end{cases}$ for $n \geq 2$ (assuming that empty λ is $(0 \times (n-1))$ -rectangular). $\square$

Lemma 3.3 ([R], Lemma 7.1). *Let $n > m \geq 0$ be integers and H be a subgroup of $G_{F/k}$ preserving K_n and projecting onto a subgroup of $G_{K_n/k}$ containing the permutation group of the set $\{x_1, \dots, x_n\}$. Then the subgroup in $G_{F/k}$ generated by G_{F/K_m} and H is dense. $\square$*

We note that $\mathrm{Aff}_n^{(\ell)} \mathbb{Q} \subset G_n \subset G_{K_n/k}$ does indeed contain the permutation group of the set $\{x_1, \dots, x_n\}$ for any even $\ell \geq 2$.

For any $U \in \mathcal{A}$ and $m \geq 0$ set $U_m = U^{G_{F/K_m}}$. Using smooth cochains, one defines the smooth cohomology $H_{\text{smooth}}^j(G_{F/k}, -) := \text{Ext}_{\mathcal{S}m_{G_{F/k}}}^j(\mathbb{Q}, -)$.

Proposition 3.4. *If $U \in \mathcal{A}$ and there is a subquotient of $U_n \in \mathfrak{SL}_n^u$ isomorphic to K_n then there is an embedding $F \hookrightarrow U$ in $\mathcal{A}$. One has $H_{\text{smooth}}^1(G_{F/k}, V) = 0$ for any $V \in \mathcal{A}$.*

Proof. By Lemma 3.3, $U^{G_{F/k}} = U_{n+1}^{\text{Aff}^{(\ell)}\mathbb{Q}} \cap U_n$ for any even $\ell \geq 2$. By Theorem 2.4, Lemma 3.1 and Lemma 3.2, $(U_{n+1}^{\mathcal{H}_{n+1}^{(\ell)}})^{\text{Aff}^{(\ell)}\mathbb{Q}} = U_{n+1}^{\text{Aff}^{(\ell)}\mathbb{Q}}$ for any $n \geq 1$ and any sufficiently big ℓ (where $\mathcal{H}_n^{(\ell)}$ is defined in Lemma 3.2). Then, as $U_n \subseteq U_{n+1}^{\mathcal{H}_{n+1}^{(\ell)}}$, one has $U^{G_{F/k}} = U_n^{\text{Aff}^{(\ell)}\mathbb{Q}}$ for any sufficiently big even ℓ , and thus, $U^{G_{F/k}} \neq 0$ if there is a subquotient of $U_n \in \mathfrak{SL}_n^u$ isomorphic to K_n .

Clearly, $\text{Ext}_{\mathcal{S}m_{G_{F/k}}}^j(\mathbb{Q}, -) = \text{Ext}_{\mathcal{C}}^j(F, -)$ on $\mathcal{C}$ for any $j \geq 0$,³ so we have to show that any smooth F -semi-linear extension $0 \rightarrow V \rightarrow U \rightarrow F \rightarrow 0$ splits.

Fix $u \in U$ in the preimage of $1 \in F$. The stabilizer of u contains a subgroup of type $G_{F/L}$ such that the elements of L are algebraic over K_m for some $m > 1$. Then the normalized trace $\text{tr}_{/K_m} u \in U_m$ belongs again to the preimage of $1 \in K_m$, so U_m surjects onto K_m .

By Theorem 2.4 and Lemma 3.1 the semi-linear representation U_m of $\text{Aff}_m^{(\ell)}\mathbb{Q}$ over K_m splits as $K_m \oplus V_m$, and thus, $U^{G_{F/k}}$ projects onto k . Then sending $1 \in k \subset F$ to one of its preimages in $U^{G_{F/k}}$ extends to a splitting of $U \rightarrow F$. $\square$

Corollary 3.5. *For an integer $n \geq 1$ let $H \subseteq G_{F/k}$ be a subgroup containing G_{F/K_n} such that $G_{F/\overline{K_n}}$ is a normal subgroup in H . Consider $H/G_{F/K_n}$ as a subset in the set $\{K_n \xrightarrow{/k} \overline{K_n}\}$ of field embeddings of K_n into its algebraic closure in F over k . Suppose that $H/G_{F/K_n}$ contains Aff_n . Let $V = F[G_{F/k}/H]^\circ \in \mathcal{C}$ consist of formal degree-zero F -linear combinations of elements in $G_{F/k}/H$.⁴ Then any quotient of V which lies in $\mathcal{A}$ is zero.*

Proof. V is generated by $\alpha = [1] - [\sigma] \in V_{2n}^{\langle (\text{Aff}_{2n})_u, T_{2n} \rangle}$, where σ sends x_j to x_{2n+1-j} for each $1 \leq j \leq 2n$. Any admissible semi-linear quotient of V is generated by the image of α , which is, by Propositions 1.5 and 3.4, fixed by the whole $G_{F/k}$. On the other hand, $\sigma\alpha = -\alpha$, so any admissible semi-linear quotient of V is zero. $\square$

Theorem 3.6. *The category $\mathcal{A}$ is abelian. The functor $H^0(G_{F/L}, -)$ is exact on $\mathcal{A}$ for any subfield L in F containing k .*

Proof. We have to check that $\mathcal{A}$ is stable under taking quotients. Let $V \in \mathcal{A}$ and $V \xrightarrow{\pi} V'$ be a surjection of F -semi-linear representations of $G_{F/k}$. By Proposition 3.4, for any $K \subset F$ of finite type over k and any $v \in (V')^{G_{F/K}} - \{0\}$, the extension $0 \rightarrow \ker \pi \rightarrow \pi^{-1}(F \cdot v) \rightarrow F \rightarrow 0$ of F -semi-linear representations of $G_{F/\overline{K}}$ splits. This implies that

³Any class in $\text{Ext}_{\mathcal{C}}^j(F, V)$ represented by $0 \rightarrow V \rightarrow V_j \rightarrow \cdots \rightarrow V_1 \rightarrow F \rightarrow 0$ is sent to the class of $0 \rightarrow V \rightarrow V_j \rightarrow \cdots \rightarrow V_2 \rightarrow V_1 \times_F \mathbb{Q} \rightarrow \mathbb{Q} \rightarrow 0$ in $\text{Ext}_{\mathcal{S}m_{G_{F/k}}}^j(\mathbb{Q}, V)$. Conversely, the class of $0 \rightarrow V \rightarrow U_j \rightarrow \cdots \rightarrow U_1 \rightarrow \mathbb{Q} \rightarrow 0$ in $\text{Ext}_{\mathcal{S}m_{G_{F/k}}}^j(\mathbb{Q}, V)$ is sent to the class of $0 \rightarrow V \rightarrow (U_j \otimes F)/K \rightarrow U_{j-1} \otimes F \rightarrow \cdots \rightarrow U_1 \otimes F \rightarrow F \rightarrow 0$, where K is the kernel of the surjection $\text{forget}(V) \otimes F \rightarrow V$ and $\text{forget} : \mathcal{C} \rightarrow \mathcal{S}m_{G_{F/k}}$ is the forgetful functor.

⁴In particular, if $H = G_{\{F, \overline{K_n}\}/k}$ then $V \cong F[\{L \subset F \mid L \cong \overline{K_n}\}]^\circ$ consists of formal degree-zero F -linear combinations of algebraically closed subfields in F of transcendence degree n over k .

the natural projection $V^{G_{F/K}} \xrightarrow{\pi_K} (V')^{G_{F/K}}$ is surjective, and thus, V' is also an admissible semi-linear representation.

The functor $H^0(G_{F/L}, -)$ on $\mathcal{A}$ is the composition of the forgetful functor $\Phi : \mathcal{A}_k \rightarrow \mathcal{C}_L$, the functor $H^0(G_{F/\overline{L}}, -)$ on $\mathcal{C}_L$ and the exact functor $H^0(G_{\overline{L}/L}, -)$ on $\mathcal{S}m_{G_{\overline{L}/L}}$. If L is of finite transcendence degree over k then the forgetful functor Φ factors through $\mathcal{A}_{\overline{L}}$, so the composition $H^0(G_{F/\overline{L}}, -) \circ \Phi$ is exact. If L is of infinite transcendence degree over k then $H^0(G_{F/\overline{L}}, -)$ induces an equivalence of categories $\mathcal{S}m_{G_{F/k}} \xrightarrow{\sim} \mathcal{S}m_{G_{\overline{L}/k}}$, so $H^0(G_{F/\overline{L}}, -)$ is also exact. $\square$

Corollary 3.7. $H^1_{\text{smooth}}(G_{F/k}, \Omega_{F/k, \text{closed}}^\bullet) = H^1_{\text{smooth}}(G_{F/k}, \Omega_{F/k, \text{exact}}^\bullet) = 0$.

Proof. By Proposition 3.4, $H^1_{\text{smooth}}(G_{F/k}, \Omega_{F/k}^\bullet) = 0$. Then a piece of the long cohomological sequence of the short exact sequence $0 \rightarrow \Omega_{F/k, \text{closed}}^q \rightarrow \Omega_{F/k}^q \xrightarrow{d} \Omega_{F/k, \text{exact}}^{q+1} \rightarrow 0$ looks as $H^0(G_{F/k}, \Omega_{F/k, \text{exact}}^{q+1}) \rightarrow H^1_{\text{smooth}}(G_{F/k}, \Omega_{F/k, \text{closed}}^q) \rightarrow H^1_{\text{smooth}}(G_{F/k}, \Omega_{F/k}^q) = 0$. Evidently, $H^0(G_{F/k}, \Omega_{F/k, \text{exact}}^{q+1}) = 0$, so $H^1_{\text{smooth}}(G_{F/k}, \Omega_{F/k, \text{closed}}^q) = 0$.

Clearly, $H^0(G_{F/k}, H^q_{\text{dR}/k}(F)) = 0$.⁵ A piece of the long cohomological sequence of short exact sequence $0 \rightarrow \Omega_{F/k, \text{exact}}^q \rightarrow \Omega_{F/k, \text{closed}}^q \rightarrow H^q_{\text{dR}/k}(F) \rightarrow 0$ looks as

$$H^0(G_{F/k}, H^q_{\text{dR}/k}(F)) \rightarrow H^1_{\text{smooth}}(G_{F/k}, \Omega_{F/k, \text{exact}}^q) \rightarrow H^1_{\text{smooth}}(G_{F/k}, \Omega_{F/k, \text{closed}}^q) = 0,$$

so $H^1_{\text{smooth}}(G_{F/k}, \Omega_{F/k, \text{exact}}^\bullet) = 0$. $\square$

3.1. Extensions in $\mathfrak{S}\mathcal{L}_n^u$. Now we need the following particular case of Bott's theorem.

Theorem 3.8 ([B], cf. also [D]). *If $\mathcal{V}$ is an irreducible G_n -equivariant coherent sheaf on $\mathbb{P}_k^n$ then there exists at most one $j \geq 0$ such that $H^j(\mathbb{P}_k^n, \mathcal{V}) \neq 0$. If $H^j(\mathbb{P}_k^n, \mathcal{V})^{G_n} \neq 0$ then $\mathcal{V} \cong \Omega_{\mathbb{P}_k^n/k}^j$.* $\square$

We also need the following explicit description of the homomorphisms in the case of commutative unipotent radicals of the target groups. It confirms general expectations, sketched in Remark 8.19 of [BT] and in [T], §5.1.

Theorem 3.9 ([LR], Theorem 3). *Let G be a simple simply connected Chevalley group over a field κ of characteristic zero. Let $\mathcal{G}$ be a connected algebraic group over a field extension K of κ . Let $\tau : G(\kappa) \rightarrow \mathcal{G}(K)$ be a homomorphism with Zariski dense image. Assume that the unipotent radical $\mathcal{G}_u$ of $\mathcal{G}$ is commutative and the composition $G(\kappa) \xrightarrow{\tau} \mathcal{G}(K) \rightarrow G'(K)$, where $G' = \mathcal{G}/\mathcal{G}_u$, is induced by a rational K -homomorphism $\lambda : G \times_\kappa K \rightarrow G'$.*

Then $\mathcal{G}_u$ splits over a finite field extension L/K into a direct sum of r copies of the adjoint representation of G' , so $r = \dim \mathcal{G}_u / \dim G'$.

Let $A = \kappa[\varepsilon_1, \dots, \varepsilon_r]/(\varepsilon_1^2, \dots, \varepsilon_r^2)$ and $\mathcal{H} = R_{A/\kappa}(G \times_\kappa A) \cong G \ltimes \mathfrak{g}^{\oplus r}$, where $\mathfrak{g} = \text{Lie}(G)$ is the adjoint representation of G .

Then there exist derivations $\delta_1, \dots, \delta_r : \kappa \rightarrow L$ and an L -isogeny $\mu : \mathcal{H} \times_\kappa L \rightarrow \mathcal{G} \times_\kappa L$ such that $\tau = \mu \circ \eta_\delta$, where $\eta_\delta : G(\kappa) \rightarrow \mathcal{H}(L)$ is induced by the ring embedding $\text{id} + \sum_{j=1}^r \varepsilon_j \delta_j : \kappa \rightarrow A \otimes_\kappa L$. $\square$

⁵Let $\omega \in \Omega_{A/k}^q \subset \Omega_{F/k}^q$ represent a $G_{F/k}$ -fixed element for a smooth finitely generated k -subalgebra $A \subset F$. Fix $\sigma \in G_{F/k}$ such that A and $\sigma(A)$ are algebraically independent over k . Then $\omega - \sigma\omega = d\eta$ for some $\eta \in \Omega_{B/k}^{q-1}$, where $B \subset F$ is a smooth finitely generated $(A \otimes_k \sigma(A))$ -subalgebra. Fix a k -algebra homomorphism $\varphi : \sigma(A) \rightarrow \overline{k} \subset F$ and extend $\text{id} \cdot \varphi : A \otimes_k \sigma(A) \rightarrow A \otimes_k \overline{k} \subset F$ to $\psi : B \rightarrow F$. Then ψ induces a morphism of differential graded k -algebras $\psi_* : \Omega_{B/k}^\bullet \rightarrow \Omega_{F/k}^\bullet$ identical on $\Omega_{A/k}^\bullet$, so $\omega = d\psi_*(\eta)$.

Lemma 3.10. *Let $n \geq 2$. Suppose that $\text{Ext}_{\mathfrak{S}\mathcal{L}_n^u}^1(K_n, V_\circ) \neq 0$ for some irreducible object $V_\circ$ of $\mathfrak{S}\mathcal{L}_n^u$. Then either $V_\circ \cong \Omega_{K_n/k}^1$, or $V_\circ \cong \text{Der}(K_n/k)$. One has $\text{Ext}_{\mathfrak{S}\mathcal{L}_n^u}^1(K_n, \Omega_{K_n/k}^1) = k$ and $\text{Ext}_{\mathfrak{S}\mathcal{L}_n^u}^1(K_n, \text{Der}(K_n/k)) = \text{Der}(k)$.*

Proof. Let $\mathcal{V} = \mathcal{S}(V_\circ)$ be the irreducible coherent G_n -equivariant sheaf on $\mathbb{P}(Q) = \mathbb{P}_k^n$ with the generic fibre $V_\circ$, and let $0 \rightarrow V_\circ \rightarrow V \rightarrow K_n \rightarrow 0$ be an extension in $\mathfrak{S}\mathcal{L}_n^u$.

Suppose that the short exact sequence $0 \rightarrow \mathcal{V} \rightarrow \mathcal{S}(V) \rightarrow \mathcal{O} \rightarrow 0$ of coherent sheaves on $\mathbb{P}(Q)$ splits. Let E be the total space of $\mathcal{S}(V) \cong \mathcal{O} \oplus \mathcal{V}$. Then, as $\text{Aut}(\mathcal{S}(V), \mathcal{V}) \cong (\mathbb{G}_m \times \mathbb{G}_m) \ltimes \Gamma(\mathbb{P}_k^n, \mathcal{V})$, the G_n -structure on V corresponds to a splitting of the sequence

$$(3) \quad 1 \rightarrow (\mathbb{G}_m \times \mathbb{G}_m) \ltimes \Gamma(\mathbb{P}(Q), \mathcal{V}) \rightarrow \text{Aut}_{\text{lin}}(E, \text{tot}(\mathcal{V})) \rightarrow G_n \rightarrow 1.$$

As $H^1(G_n, \mathbb{G}_m \times \mathbb{G}_m) = 1$, Theorem 3.9 (with $G = \text{SL}_{n+1}k$, $G' = G_n$ and $\mathcal{G}_u \subseteq \Gamma(\mathbb{P}_k^n, \mathcal{V})$) implies that a non-standard splitting of (3) can exist only if $\Gamma(\mathbb{P}_k^n, \mathcal{V})$ is isomorphic to the adjoint representation of G_n , i.e., if $\mathcal{V} \cong \mathcal{T}_{\mathbb{P}_k^n/k}$. The identity $\text{Ext}_{\mathfrak{S}\mathcal{L}_n^u}^1(K_n, \text{Der}(K_n/k)) = \text{Der}(k)$ follows also from Theorem 3.9.

If $\mathcal{V} \cong \Omega_{\mathbb{P}_k^n/k}^1$ then the target of the homomorphism $\text{Ext}_{\mathfrak{S}\mathcal{L}_n^u}^1(K_n, \mathcal{V}_\circ) \xrightarrow{\alpha} \text{Ext}_{\mathcal{O}}^1(\mathcal{O}_{\mathbb{P}_k^n}, \mathcal{V}) = k$ induced by the functor $\mathcal{S}$ is generated by the class of the Euler extension $0 \rightarrow \Omega_{\mathbb{P}(Q)/k}^1 \rightarrow Q^\vee \otimes_k \mathcal{O}_{\mathbb{P}(Q)}(-1) \rightarrow \mathcal{O}_{\mathbb{P}(Q)} \rightarrow 0$. Let E be the total space of the vector bundle with the sheaf of sections $Q^\vee \otimes_k \mathcal{O}(-1)$. Any G_n -structure on the middle term of this extension corresponding to an element of $\text{Ext}_{\mathfrak{S}\mathcal{L}_n^u}^1(K_n, \Omega_{K_n/k}^1)$ is a splitting of the short exact sequence

$$(4) \quad 1 \rightarrow \text{Aut}(Q^\vee \otimes_k \mathcal{O}(-1), \Omega_{\mathbb{P}(Q)/k}^1) \rightarrow \text{Aut}_{\text{lin}}(E, \text{tot}(\Omega_{\mathbb{P}(Q)/k}^1)) \rightarrow G_n \rightarrow 1.$$

As any point of $\mathbb{P}(Q)$ determines a hyperplane in $Q^\vee$, the group $\text{Aut}(Q^\vee \otimes_k \mathcal{O}(-1), \Omega_{\mathbb{P}(Q)}^1)$ coincides with the subgroup of $\text{GL}(Q)$ stabilizing all hyperplanes in $Q^\vee$, i.e., with the centre $\mathbb{G}_m$ of $\text{GL}(Q)$. Then $\text{Aut}_{\text{lin}}(E, \text{tot}(\Omega_{\mathbb{P}(Q)/k}^1))$ is a central $\mathbb{G}_m$ -extension of G_n , so the splitting of (4) is unique and corresponds to the usual G_n -equivariant structure. $\square$

Corollary 3.11. *If $k = \overline{\mathbb{Q}}$ then $\mathfrak{S}\mathcal{L}_n^u$ is equivalent to the category of G_n -equivariant vector bundles on $\mathbb{P}_k^n$.*

Proof. The category of G_n -equivariant vector bundles on $\mathbb{P}_k^n$ is a full sub-category of $\mathfrak{S}\mathcal{L}_n^u$ with the same irreducible objects. As it is mentioned at the beginning of §1, p.3, the objects of $\mathfrak{S}\mathcal{L}_n^u$ are generic fibres of coherent G_n -sheaves on $\mathbb{P}_k^n$. Suppose that $V \in \mathfrak{S}\mathcal{L}_n^u$ is the generic fibre of a non-equivariant vector bundle on $\mathbb{P}_k^n$ of minimal possible rank. Then it fits into an exact sequence $0 \rightarrow B \rightarrow V \rightarrow A \rightarrow 0$, where A, B are the generic fibres of G_n -equivariant vector bundles on $\mathbb{P}_k^n$ and A is irreducible. Let $0 \neq C \subseteq B$ be an irreducible sub-object and $D = B/C$. Then the rows in the following commutative diagram are exact:

$$\begin{array}{ccccccccc} \text{Hom}_u(A, D) & \rightarrow & \text{Ext}_u^1(A, C) & \rightarrow & \text{Ext}_u^1(A, B) & \rightarrow & \text{Ext}_u^1(A, D) & \rightarrow & \text{Ext}_u^2(A, C) \\ & & \parallel & & \parallel \text{Lemma 3.10} & & \cup & & \parallel \text{minimality of } V \uparrow \xi \\ \text{Hom}_{\text{eq}}(A, D) & \rightarrow & \text{Ext}_{\text{eq}}^1(A, C) & \rightarrow & \text{Ext}_{\text{eq}}^1(A, B) & \rightarrow & \text{Ext}_{\text{eq}}^1(A, D) & \rightarrow & \text{Ext}_{\text{eq}}^2(A, C) \end{array}$$

where subscript u refers to the category $\mathfrak{S}\mathcal{L}_n^u$ and eq refers to the category of equivariant vector bundles.

Let us show that ξ is injective for any irreducible A and C . As $C \otimes_{K_n} A^\vee$ is semi-simple (as it follows from Proposition 1.5) and $\text{Ext}_?^2(A, C) = \text{Ext}_?^2(K_n, C \otimes_{K_n} A^\vee)$, where $? = u$ or eq, we may assume that $A = K_n$ and C is still irreducible. By Bott's Theorem 3.8, $\dim_k \text{Ext}_{\text{eq}}^2(K_n, C) \leq 1$ with equality only if $C \cong \Omega_{K_n/k}^2$, so we assume further that

$C = \Omega_{K_n/k}^2$. The forgetful functor from $\mathfrak{SL}_n^u$ to the category of coherent sheaves on $\mathbb{P}_k^n$ induces a homomorphism $\text{Ext}_u^2(A, C) \rightarrow H^2(\mathbb{P}_k^n, \Omega_{\mathbb{P}_k^n/k}^2)$. Clearly, its composition with ξ is an isomorphism.

Then the 5-lemma implies that $\text{Ext}_u^1(A, B) = \text{Ext}_{\text{eq}}^1(A, B)$, and thus, V is equivariant. $\square$

4. THE CATEGORY $\mathcal{A}$ IN THE CASE $k = \overline{\mathbb{Q}}$

In this section we determine (in Theorem 4.10) the structure of the objects of $\mathcal{A}$ in the case $k = \overline{\mathbb{Q}}$, the field of algebraic numbers. The objects V of $\mathcal{A}$ are quotients of sums of representations of G over k induced by rational representations of $\text{GL}_m k$'s (considered as subquotients of G) with coefficients extended to F (cf. Lemma 4.1). Then we find (in Lemma 4.2) a supply of elements in the induced representations vanishing in V , and use them in Lemmas 4.3–4.7 to show that the objects of $\mathcal{A}$ are sums of quotients of $\bigotimes_F^\bullet \mathfrak{m}$. In §4.1 we study extensions in $\mathcal{A}$.

Let $V \in \mathcal{A}$ and $m \geq 0$ be such that $V_m \neq 0$. Then there is a non-zero morphism $F[G_{F/k}/G_{F/K_m}] \otimes_{K_m[\text{PGL}_{m+1}k]} V_m \rightarrow V$ in $\mathcal{C}$. The object V_m of $\mathfrak{SL}_m^u$ admits an irreducible sub-object $A \neq 0$. By Theorem 2.4, $A \cong S_{K_m}^\lambda \Omega_{K_m/k}^1$ for a Young diagram λ . Then $F[G_{F/k}/G_{F/K_m}] \otimes_{K_m[\text{PGL}_{m+1}k]} A \rightarrow V$ is also non-zero. Clearly, $A = B \otimes_k K_m$, where $B := A^{(\text{Aff}_m)_u} \cong (S_{K_m}^\lambda \Omega_{K_m/k}^1)^{(\text{Aff}_m)_u} \cong S_k^\lambda(k^m)$ is a rational irreducible representation of $\text{GL}_m k := \text{Aff}_m/(\text{Aff}_m)_u$.

This implies that there is a non-zero morphism $U := F[W^\circ] \otimes_{k[\text{GL}_m k]} B \xrightarrow{\varphi} V$ and a surjection $U \rightarrow S_F^\lambda \Omega_{F/k}^1$, where $W^\circ := \{K_m \xrightarrow{/k} F\}/(\text{Aff}_m)_u$ is considered as a $G_{F/k}$ -set.

As any embedding $K_m \xrightarrow{/k} F$ is determined by the images of $x_1, \dots, x_m$, one can consider W° as a subset of $(F/k)^m$ consisting of m -tuples with entries algebraically independent over k . More invariantly, let $W := \text{Hom}_k((K_m/k)^{(\text{Aff}_m)_u}, F/k) \cong (F/k)^m$ be the group (a k -vector space) generated by W° . The isomorphism is given by restriction of the homomorphisms to the basis $\{\overline{x_1}, \dots, \overline{x_m}\}$ of $(K_m/k)^{(\text{Aff}_m)_u}$. Define a homogeneous map $\varkappa : W \rightarrow \Omega_{F/k}^m \otimes_k \det_k \text{Hom}_G(F/k, W)$ of degree m by inverting the first isomorphism in the sequence $W \xleftarrow{\sim} (F/k) \otimes_k \text{Hom}_G(F/k, W) \xrightarrow{d \otimes \text{id}} \Omega_{F/k}^1 \otimes_k \text{Hom}_G(F/k, W) \rightarrow \text{Sym}_F^m(\Omega_{F/k}^1 \otimes_k \text{Hom}_G(F/k, W)) \rightarrow \Omega_{F/k}^m \otimes_k \det_k \text{Hom}_G(F/k, W)$. Then $W^\circ = \{w \in W \mid \varkappa(w) \neq 0\}$.

Let $(y_1, \dots, y_m) \mapsto [y_1, \dots, y_m]$ be the map $(F/k)^m \rightarrow \{0\} \cup W^\circ$ sending $(y_1, \dots, y_m)$ to $[x_j \mapsto y_j]$ if $y_1, \dots, y_m$ are algebraically independent over k , and to 0 otherwise. Then $[\mu y_1, \dots, \mu y_m] \otimes b = \mu^{|\lambda|} [y_1, \dots, y_m] \otimes b$ in U for any $\mu \in k$. If $y_1, \dots, y_m$ belong to the k -linear envelope of $x_1, \dots, x_M$ for some integer $M \geq 1$ then $[y_1, \dots, y_m] \otimes b \in U_M^{(\text{Aff}_M)_u}$ is a weight $|\lambda|$ eigenvector of the centre of $\text{GL}_M k$.

Let $U_M^1 \subseteq U_M^{(\text{Aff}_M)_u}$ be the $k[\text{GL}_M k]$ -envelope of $[x_1, \dots, x_m] \otimes b$ for some $b \neq 0$ (which is the same as k -envelope of all $[y_1, \dots, y_m] \otimes c$ for algebraically independent $y_1, \dots, y_m$ in the k -linear envelope of $x_1, \dots, x_M$ and all c). Clearly, $U_m^1 \cong B$ as $k[\text{GL}_m k]$ -modules, and any non-zero morphism $U \rightarrow V$ induces an embedding $U_m^1 \hookrightarrow V$.

Lemma 4.1. *If $k = \overline{\mathbb{Q}}$ then for any $V \in \mathfrak{SL}_M^u$ the representation $V^{(\text{Aff}_M)_u}$ of $\text{GL}_M k$ is rational semi-simple, and $V^{(\text{Aff}_M)_u} \otimes_k K_M = V$.*

Proof. By Corollary 3.11, V is the generic fibre of a $\text{PGL}_{M+1} k$ -equivariant vector bundle.

Then, by Lemma 6.3 (1) of [R], $V = V^{U_0} \otimes_k K_M$, where U_0 is a $\mathbb{Q}$ -lattice in $(\text{Aff}_M)_u$. The group $(\text{Aff}_M)_u$ acts rationally on V^{U_0} . As the action of the $\mathbb{Q}$ -lattice U_0 is trivial, the action of the entire $(\text{Aff}_M)_u$ is trivial, i.e., $V = V^{(\text{Aff}_M)_u} \otimes_k K_M$. Then the action of $\text{GL}_M k$ on $V^{(\text{Aff}_M)_u}$ is rational, and thus, semi-simple. $\square$

REMARK. There is no semisimplicity if k contains a transcendental element.

Indeed, let the underlying K_n -vector space of $V \in \mathfrak{S}\mathfrak{L}_n^u$ be $K_n \oplus \Omega_{K_n}^1/(\Lambda \otimes_k K_n)$ for a proper k -vector subspace $\Lambda \subset \Omega_k^1$ of finite codimension, and the G_n -action be given by $\sigma(f, \omega) = (\sigma f, \sigma\omega + \sigma f \cdot d \log(\sigma\eta/\eta))$ for any $\sigma \in G_n$, where $\eta = dx_1 \wedge \cdots \wedge dx_n \in \Omega_{K_n/k}^n$. (So V fits into a non-split exact sequence $0 \rightarrow \Omega_{K_n}^1/(\Lambda \otimes_k K_n) \rightarrow V \rightarrow K_n \rightarrow 0$.) Then $\sigma(f, \omega) = (\sigma f, \sigma\omega)$ if $\sigma \in (\text{Aff}_n)_u$, and therefore, $V^{(\text{Aff}_n)_u} = k \oplus \Omega_k^1/\Lambda$ is a non-trivial extension of trivial representations of $\text{GL}_n k$.

Lemma 4.2. *The kernel of $U_M^! \rightarrow S_F^\lambda \Omega_{F/k}^1$ is contained in the kernel of $U \xrightarrow{\varphi} V$.*

Proof. By Lemma 4.1, the image $\overline{U_M^!}$ of $U_M^!$ in V is isomorphic to $\bigoplus_{|\nu|=|\lambda|} (S_k^\nu(k^M))^{m_\nu}$. As $\overline{U_m^!} \subseteq \overline{U_M^!}^{\text{GL}_M k \cap G_{K_M/K_m}} \cong \bigoplus_{|\nu|=|\lambda|} (S_k^\nu(k^m))^{m_\nu}$ and $U_M^!$ is generated by $U_m^! \xrightarrow{\sim} S_k^\lambda(k^m) \subseteq S_k^\lambda(k^M)$, we see that $\overline{U_M^!}$ is isomorphic to $S_k^\lambda(k^M)$. Then the kernel of $U_M^! \rightarrow S_F^\lambda \Omega_{F/k}^1$ is contained in the kernel of the morphism $U \xrightarrow{\varphi} V$. $\square$

Let $W^{M^\circ} \subset W^M$ be the subset consisting of M -tuples $(y_1, \dots, y_M)$ such that $\sum_{i \in I} y_i \in W^\circ$ for any non-empty subset $I \subseteq \{1, \dots, M\}$.

Let $k[W^{M^\circ}] \rightarrow k[W^\circ] \otimes_{k[k^\times]} k(M)$ be the k -linear map sending $(y_1, \dots, y_M)$ to

$$\langle y_1, \dots, y_M \rangle := \sum_{I \subseteq \{1, \dots, M\}} (-1)^{\#I} \left[\sum_{i \in I} y_i \right] \in k[W^\circ] \otimes_{k[k^\times]} k(M).$$

Here $k(M)$ denotes a one-dimensional k -vector space with $k^\times$ -action by M -th powers. As $(y, \dots, y)$ is sent to

$$\sum_{j \geq 0} (-1)^j \binom{M}{j} j^M [y] = (t \frac{d}{dt})^M (1-t)^M|_{t=1} \cdot [y] = (-1)^M M! \cdot [y],$$

it is surjective. Clearly, $\langle y_1, \dots, y_M \rangle = \langle y_{\theta(1)}, \dots, y_{\theta(M)} \rangle$ for any permutation $\theta \in \mathfrak{S}_M$. Let $\tilde{U} := F[W^{|\lambda|^\circ}] \rightarrow U$ be the F -linear surjection sending $(y_1, \dots, y_{|\lambda|})$ to $\langle y_1, \dots, y_{|\lambda|} \rangle \otimes b$.

Lemma 4.3. *Let the k -linear map $\alpha : k[W^\circ] \rightarrow \bigotimes_k^M W$ be given by $[w] \mapsto w^{\otimes M}$. Then α factors through $k[W^\circ] \otimes_{k[k^\times]} k(M)$ and $\langle y_1, \dots, y_M \rangle \mapsto (-1)^M \sum_{\theta \in \mathfrak{S}_M} y_{\theta(1)} \otimes \cdots \otimes y_{\theta(M)}$ if $(y_1, \dots, y_M) \in W^{M^\circ}$.*

Proof. The element $\langle y_1, \dots, y_M \rangle$ is sent to

$$\sum_{I \subseteq \{1, \dots, M\}} (-1)^{\#I} \left(\sum_{i \in I} y_i \right)^{\otimes M} = \sum_{1 \leq i_1, \dots, i_M \leq M} A_{i_1, \dots, i_M} y_{i_1} \otimes \cdots \otimes y_{i_M}.$$

If $S = \{1, \dots, M\} \setminus \{i_1, \dots, i_M\}$ then $A_{i_1, \dots, i_M} = \sum_{J \subseteq S} (-1)^{M-\#J}$, so $A_{i_1, \dots, i_M} = 0$ if S is non-empty, and $A_{i_1, \dots, i_M} = (-1)^M$ if $\{1, \dots, M\} = \{i_1, \dots, i_M\}$. $\square$

Lemma 4.4. *If $M = |\lambda|$, $\mu \in k$, $y_0, y_1, y_0 + y_1 \in W^\circ$ and all coordinates of $t_2, \dots, t_M \in W$ are algebraically independent over $k(y_0, y_1)$ then*

$$\langle y_0 + y_1, t_2, \dots, t_M \rangle \otimes b \equiv \langle y_0, t_2, \dots, t_M \rangle \otimes b + \langle y_1, t_2, \dots, t_M \rangle \otimes b \pmod{\ker \varphi},$$

and $\langle \mu y_1, t_2, \dots, t_M \rangle \otimes b \equiv \mu \langle y_1, t_2, \dots, t_M \rangle \otimes b \pmod{\ker \varphi}$.

Proof. It follows from Lemmas 4.2 and 4.3, that $\langle z_0 + z_1, z_2, \dots, z_M \rangle \otimes b - \langle z_0, z_2, \dots, z_M \rangle \otimes b - \langle z_1, \dots, z_M \rangle \otimes b$ and $\langle \mu z_1, z_2, \dots, z_M \rangle \otimes b - \mu \cdot \langle z_1, \dots, z_M \rangle \otimes b$ are sent to zero by φ , where the coordinates of z_j are $x_{jm+1}, \dots, x_{jm+m}$. As the G -orbits of these elements are also sent to zero by φ , for some $u, v \in W^\circ$ with coordinates algebraically independent over the subfield in F generated over k by $y_0, y_1, t_2, \dots, t_M$, one has the following congruences modulo the kernel of φ :

$$(5) \quad \langle y_0 + y_1, t_2, \dots, t_M \rangle \otimes b \equiv \langle y_0 + y_1 + u, t_2, \dots, t_M \rangle \otimes b - \langle u, t_2, \dots, t_M \rangle \otimes b,$$

$$(6) \quad \langle y_0, t_2, \dots, t_M \rangle \otimes b \equiv \langle y_0 + u - v, t_2, \dots, t_M \rangle \otimes b - \langle u - v, t_2, \dots, t_M \rangle \otimes b,$$

$$(7) \quad \langle y_1, t_2, \dots, t_M \rangle \otimes b \equiv \langle y_1 + v, t_2, \dots, t_M \rangle \otimes b - \langle v, t_2, \dots, t_M \rangle \otimes b.$$

As $\langle y_0 + y_1 + u, t_2, \dots, t_M \rangle \otimes b \equiv \langle y_0 + u - v, t_2, \dots, t_M \rangle \otimes b + \langle y_1 + v, t_2, \dots, t_M \rangle \otimes b$, and $\langle u, t_2, \dots, t_M \rangle \otimes b \equiv \langle u - v, t_2, \dots, t_M \rangle \otimes b + \langle v, t_2, \dots, t_M \rangle \otimes b$, the left hand side of the congruence (5) is congruent to the sum of the left hand sides of the congruences (6) and (7) modulo $\ker \varphi$. $\square$

Lemma 4.5. *Let $(y_1, \dots, y_M) \in \{0\} \times W^{(M-1)^\circ} \cup W^{M^\circ}$ and let the coordinates of $t_{ij} \in W^\circ$ be algebraically independent over $k(y_1, \dots, y_M)$, where $1 \leq i \leq M$ and $2 \leq j \leq M$. Set $[0] := 0$ and $\langle 0, y_2, \dots, y_M \rangle := 0$. Then*

$$(8) \quad \langle y_1, \dots, y_M \rangle \otimes b \equiv \sum_{J \subseteq \{2, \dots, M\}} (-1)^{\#J} \langle y_1, \sum_{s \in \{1\} \cup J} t_{s2}, \dots, \sum_{s \in \{1\} \cup J} t_{sM} \rangle \otimes b \\ - \sum_{\emptyset \neq I \subseteq \{2, \dots, M\}} (-1)^{\#I} \langle y_1, y_2 + \sum_{i \in I} t_{2i}, \dots, y_M + \sum_{i \in I} t_{Mi} \rangle \otimes b \pmod{\ker \varphi}.$$

Proof. It follows from the identities

$$[\sum_{s \in J} y_s] = \sum_{\emptyset \neq I \subseteq \{2, \dots, M\}} (-1)^{\#I} \left([\sum_{s \in J} \sum_{i \in I} t_{si}] - [\sum_{s \in J} (y_s + \sum_{i \in I} t_{si})] \right) \\ - \langle \sum_{s \in J} y_s, \sum_{s \in J} t_{s2}, \dots, \sum_{s \in J} t_{sM} \rangle$$

that

$$(9) \quad \langle y_1, \dots, y_M \rangle = \sum_{\emptyset \neq I \subseteq \{2, \dots, M\}} (-1)^{\#I} \left(\langle \sum_{i \in I} t_{1i}, \dots, \sum_{i \in I} t_{Mi} \rangle \right. \\ \left. - \langle y_1 + \sum_{i \in I} t_{1i}, \dots, y_M + \sum_{i \in I} t_{Mi} \rangle \right) - \sum_{J \subseteq \{1, \dots, M\}} (-1)^{\#J} \langle \sum_{s \in J} y_s, \sum_{s \in J} t_{s2}, \dots, \sum_{s \in J} t_{sM} \rangle.$$

Then Lemma 4.4, applied to the summands containing y_1 , implies that

$$\langle y_1, \dots, y_M \rangle \otimes b \equiv \sum_{J \subseteq \{2, \dots, M\}} (-1)^{\#J} \langle y_1, \sum_{s \in \{1\} \cup J} t_{s2}, \dots, \sum_{s \in \{1\} \cup J} t_{sM} \rangle \otimes b \\ - \sum_{\emptyset \neq I \subseteq \{2, \dots, M\}} (-1)^{\#I} \langle y_1, y_2 + \sum_{i \in I} t_{2i}, \dots, y_M + \sum_{i \in I} t_{Mi} \rangle \otimes b + \langle 0, y_2, \dots, y_M \rangle \otimes b,$$

so we get (8). $\square$

Lemma 4.6. *If $M = |\lambda|$, $\mu \in k$ and $(z_j, y_2, \dots, y_M), (\sum_{i=1}^N z_i, y_2, \dots, y_M), (\mu z_1, y_2, \dots, y_M) \in W^{M^\circ}$ for all $1 \leq j \leq N$ then*

$$(10) \quad \langle \sum_{j=1}^N z_j, y_2, \dots, y_M \rangle \otimes b \equiv \sum_{j=1}^N \langle z_j, y_2, \dots, y_M \rangle \otimes b \pmod{\ker \varphi},$$

and $\langle \mu z_1, y_2, \dots, y_M \rangle \otimes b \equiv \mu \langle z_1, y_2, \dots, y_M \rangle \otimes b \pmod{\ker \varphi}$.

Proof. If $N = 2$ then (10) follows from Lemmas 4.4 and 4.5. If $N \geq 3$ then

$$\langle \sum_{j=1}^N z_j, y_2, \dots, y_M \rangle \otimes b \equiv \langle \sum_{j=3}^N z_j - u, y_2, \dots, y_M \rangle \otimes b + \langle z_1 + z_2 + u, y_2, \dots, y_M \rangle \otimes b$$

for any sufficiently general $u \in W^\circ$. By the induction assumption, this is congruent to $\sum_{j=3}^N \langle z_j, y_2, \dots, y_M \rangle \otimes b - \langle u, y_2, \dots, y_M \rangle \otimes b + \langle z_1, y_2, \dots, y_M \rangle \otimes b + \langle z_2 + u, y_2, \dots, y_M \rangle \otimes b \equiv \sum_{j=1}^N \langle z_j, y_2, \dots, y_M \rangle \otimes b$. $\square$

Lemma 4.7. *The k -linear map $k[W^{M^\circ}] \rightarrow \bigotimes_k^M W$, given by $[(y_1, \dots, y_M)] \mapsto y_1 \otimes \dots \otimes y_M$, is surjective and its kernel is spanned over k by $[(y_0, \dots, y_{j-1} + y_j, \dots, y_M)] - [(y_0, \dots, \widehat{y_{j-1}}, \dots, y_M)] - [(y_0, \dots, \widehat{y_j}, \dots, y_M)]$ and $\mu[(y_1, \dots, y_M)] - [(y_1, \dots, \mu y_j, \dots, y_M)]$ for all $y_0, \dots, y_M \in W^\circ$ and all $\mu \in k^\times$.*

Proof. By Zorn's lemma, there exists a maximal subset S in W° consisting of k -linear independent elements. If S does not generate W then the k -linear envelope of S does not contain W° , i.e., an element $y \in W^\circ$ k -linear independent over S , so $S \cup \{y\}$ is a bigger subset in W° consisting of k -linear independent elements. This contradiction shows that S is a k -basis of W .

For any $y \in W^\circ$ and any $z \in W$ there exist at most m values of $\mu \in k$ such that $y + \mu z \notin W^\circ$, since this condition is equivalent to vanishing of the $\Omega_{F/k}^m$ -valued polynomial $(dy_1 + \mu dz_1) \wedge \dots \wedge (dy_m + \mu dz_m)$ of degree $\leq m$ in μ with non-zero constant term. Let us show that the map $(k^\times S)^M \cap W^{M^\circ} \rightarrow S^M$ given by the projectivization is surjective. Indeed, let $(s_1, \dots, s_M) \in S^M$. For all but $\leq m$ values of $\mu \in k^\times$ one has $s_1 + \mu s_2 \in W^\circ$. Fix one of such μ and set $s'_2 := \mu s_2$. For all but $\leq 3m$ values of $\mu \in k^\times$ one has $s_1 + \mu s_3, s'_2 + \mu s_3, s_1 + s'_2 + \mu s_3 \in W^\circ$. Fix one of such μ and set $s'_3 := \mu s_3$. Proceeding further this way, we get an element $(s_1, s'_2, \dots, s'_M) \in ((k^\times S)^M) \cap W^{M^\circ}$ projecting onto $(s_1, \dots, s_M)$.

Fix a section of the projection $(k^\times S)^M \cap W^{M^\circ} \rightarrow S^M$. Denote by $\widetilde{S^M}$ the image of S^M under this section. Then $\widetilde{S^M}$ considered as a subset in $k[W^{M^\circ}]$ maps to a basis of $\bigotimes_k^M W$, which shows the surjectivity.

For the injectivity it suffices to check that $k[\widetilde{S^M}]$ maps onto $k[W^{M^\circ}]$ modulo the relations. For an element $w = (w_1, \dots, w_M) \in W^{M^\circ}$ set $l(w) := \sum_{j=1}^M l_j(w) \geq M$, where $l_j(w)$ is the number of non-zero coordinates of w_j in the basis S .

By induction on $l(w)$ we are going to show that $[w]$ is in the image of $k[\widetilde{S^M}]$.

If $l(w) = M$ then $w_j = \mu_j s_j$ for all $1 \leq j \leq M$, where $(s_1, \dots, s_M) \in \widetilde{S^M}$. For any sufficiently general $\nu_2, \dots, \nu_M \in k^\times$ one has

$$\begin{aligned} [w] &\equiv \nu_2^{-1} \cdots \nu_M^{-1} [(w_1, \nu_2 w_2, \dots, \nu_M w_M)] \equiv \mu_1 \nu_2^{-1} \cdots \nu_M^{-1} [(s_1, \nu_2 w_2, \dots, \nu_M w_M)] \\ &\equiv \mu_1 \mu_2 \nu_3^{-1} \cdots \nu_M^{-1} [(s_1, s_2, \nu_3 w_3, \dots, \nu_M w_M)] \equiv \cdots \equiv \mu_1 \cdots \mu_M [(s_1, \dots, s_M)]. \end{aligned}$$

The induction step: if, for instance, $l_1(w) \geq 2$ then for all but $\leq l_1(w)m + m$ values of $\mu \in k^\times$ one has $[w] \equiv \mu^{-1} [(\mu w_1, w_2, \dots, w_M)] \equiv \mu^{-1} \sum_{s \in S} [(\mu \mu_s s, w_2, \dots, w_M)]$, where $w_1 = \sum_{s \in S} \mu_s s$ is a finite sum. By the induction assumption, the summands $[(\mu \mu_s s, w_2, \dots, w_M)]$ are in the image of $k[\widetilde{S^M}]$, and thus, $[w]$ is also there. $\square$

Let $\mathfrak{m}$ be the kernel of the multiplication map $F \otimes_k F \xrightarrow{\times} F$. The map $F \otimes_k (F/k) \rightarrow \mathfrak{m}$, given by $\sum_j z_j \otimes \overline{y_j} \mapsto \sum_j z_j \otimes y_j - (\sum_j z_j y_j) \otimes 1$ is clearly an isomorphism, so we can use the notation $\mathfrak{m}$ instead of $F \otimes_k (F/k)$, and the multiplicative structure of the ideal $\mathfrak{m}$.

Lemma 4.8. *The element $\alpha_q := (x_1 \otimes 1 - 1 \otimes x_1)^{s_1} \otimes \cdots \otimes (x_q \otimes 1 - 1 \otimes x_q)^{s_q} \in \bigotimes_F^q \mathfrak{m}$ generates the sub-object $\mathfrak{m}^{s_1} \otimes_F \cdots \otimes_F \mathfrak{m}^{s_q}$.*

Proof. We need to show that for any collection of $\beta_i \in \mathfrak{m}^{s_i}$ the element $\beta_1 \otimes \cdots \otimes \beta_q$ belongs to the $F[G_{F/k}]$ -submodule generated by α_q . Set $\alpha := x_1 \otimes 1 - 1 \otimes x_1$. Then the $G_{F/k}$ -orbit of α^s contains $(\sum_{j=1}^s a_j (y_j \otimes 1 - 1 \otimes y_j))^s$ for any $a_j \in k$ and $y_j \in F$ such that $\sum_{j=1}^s a_j y_j \notin k$. The k -span of such elements with fixed $y_1, \dots, y_s$ contains $\prod_{j=1}^s (y_j \otimes 1 - 1 \otimes y_j)$. Such products generate $\mathfrak{m}^s$ as an ideal. Moreover, they generate $\mathfrak{m}^s$ as a $F \otimes_k k$ -module: $(1 \otimes b) \prod_{j=1}^s (y_j \otimes 1 - 1 \otimes y_j) = ((by_1 \otimes 1 - 1 \otimes by_1) - (y_1 \otimes 1)(b \otimes 1 - 1 \otimes b)) \prod_{j=2}^s (y_j \otimes 1 - 1 \otimes y_j) = (by_1 \otimes 1 - 1 \otimes by_1) \prod_{j=2}^s (y_j \otimes 1 - 1 \otimes y_j) - (y_1 \otimes 1)(b \otimes 1 - 1 \otimes b) \prod_{j=2}^s (y_j \otimes 1 - 1 \otimes y_j)$.

This implies that $\beta_i = \sum_{j=1}^{s_i} f_{ij} \cdot \sigma_{ij} \alpha^{s_i}$ for some $\sigma_{ij} \in G_{F/k}$ and $f_{ij} \in F$. The $G_{F/k}$ -orbit of α_q contains $\alpha' := (z_1 \otimes 1 - 1 \otimes z_1)^{s_1} \otimes \cdots \otimes (z_q \otimes 1 - 1 \otimes z_q)^{s_q}$, where $z_1, \dots, z_q \in F$ are algebraically independent over the subfield in F generated over k by all $f_{ij}, \sigma_{ij} x_1$.

For each pair (i, j) such that $1 \leq i \leq q$ and $1 \leq j \leq s_i$ there exists an element $\xi_{ij} \in G_{F/k}$ fixing all $f_{\lambda\mu}, \sigma_{\lambda\mu} x_1$ and the elements $z_{i+1}, \dots, z_q$, such that $\xi_{ij} z_\mu = z_\mu + \sigma_{ij} x_1$. Then $(\sum_{j=1}^{s_q} f_{qj} (\xi_{qj} - 1)^{s_q}) \cdots (\sum_{j=1}^{s_1} f_{1j} (\xi_{1j} - 1)^{s_1}) (\alpha') = \beta_1 \otimes \cdots \otimes \beta_q$. $\square$

Corollary 4.9. *Any homomorphism $F \otimes_k \bigotimes_k^M (F/k) \rightarrow V$ factors through $\bigotimes_F^M (\mathfrak{m}/\mathfrak{m}^s)$ for some $s \geq 1$.*

Proof. For any integer $s \geq 1$ the element $\alpha_s := (x_1 \otimes 1 - 1 \otimes x_1)^s \otimes x_2 \otimes \cdots \otimes x_M = \sum_{j=0}^s (-1)^j \binom{s}{j} x_1^{s-j} \otimes x_1^j \otimes x_2 \otimes \cdots \otimes x_M \in (\mathfrak{m}^s \otimes_k \bigotimes_k^{M-1} (F/k))_M^{(\text{Aff}_M)_u}$ is homogeneous of degree $s+M-1$. As V_M is finite-dimensional, the image of α_s in V_M is zero for all sufficiently big s . Note that α_s generates $\mathfrak{m}^s \otimes_k \bigotimes_k^{M-1} (F/k)$ as an F -semi-linear representation of G .

This implies that the image of U is a quotient of $\mathfrak{m}/\mathfrak{m}^s \otimes_k \bigotimes_k^{M-1} (F/k)$ for some $s \geq 1$, and therefore, any homomorphism $F \otimes_k \bigotimes_k^M (F/k) = \bigotimes_F^M \mathfrak{m} \rightarrow V$ factors through $\bigotimes_F^M (\mathfrak{m}/\mathfrak{m}^s)$ for some $s \geq 1$. $\square$

Theorem 4.10. *Any (finitely generated) object V of $\mathcal{A}$ is a quotient of a (finite) direct sum of objects of type $\bigotimes_F^q (\mathfrak{m}/\mathfrak{m}^s)$ for some $q, s \geq 1$ and F , if $k = \overline{\mathbb{Q}}$. In particular, any irreducible object of $\mathcal{A}$ is a direct summand of the tensor algebra $\bigotimes_F^\bullet \Omega_{F/k}^1$.*

Proof. V is generated by V_m for some $m \geq 0$. By Lemma 4.1, $V_m^{(\text{Aff}_m)_u}$ is a semi-simple $\text{GL}_m k$ -module generating V . As it is explained at the beginning of this section, V is a

quotient of a direct sum of U 's corresponding to irreducible direct summands of $V_m^{(\text{Aff}_m)_u}$. By Lemmas 4.6 and 4.7, V is a quotient of a direct sum of $F \otimes_k \bigotimes_k^M (F/k)$ for some $M \geq 0$. Then the conclusion follows from Corollary 4.9 and the identities $\mathfrak{m}^j/\mathfrak{m}^{j+1} = \text{Sym}_F^j(\mathfrak{m}/\mathfrak{m}^2)$ and $\mathfrak{m}/\mathfrak{m}^2 = \Omega_{F/k}^1$. $\square$

Corollary 4.11. *Any finitely generated object of $\mathcal{A}$ is of finite length, if $k = \overline{\mathbb{Q}}$.* $\square$

4.1. Ext's in $\mathcal{A}$.

Lemma 4.12. $\text{Hom}_C(\bigotimes_F^r \mathfrak{m}, \bigotimes_F^q (\mathfrak{m}/\mathfrak{m}^{N+1})) = \text{Hom}_{\mathcal{A}}(\bigotimes_F^r (\mathfrak{m}/\mathfrak{m}^{N+1}), \bigotimes_F^q (\mathfrak{m}/\mathfrak{m}^{N+1}))$ admits a natural k -basis identified with the set $P = P(q, r, N)$ of the surjections $\{1, \dots, r\} \rightarrow \{1, \dots, q\}$ with fibres of cardinality $\leq N$, if $N \geq 1$, $q, r \geq 0$ and $q + r \geq 1$. In particular (take $r \geq q = 1$), any subobject of $\mathfrak{m}/\mathfrak{m}^{N+1}$ is of type $\mathfrak{m}^r/\mathfrak{m}^{N+1}$.

EXAMPLE. $P = \emptyset$ if $q > r$, or if $r > qN$; $\#P = q!$ if $q = r$; $\#P = 1$ if $N \geq r \geq q = 1$.

Proof. By Lemma 4.8, $\mathfrak{m}^{s_1} \otimes_F \dots \otimes_F \mathfrak{m}^{s_r}$ is generated by the element $\otimes_{j=1}^r (x_j \otimes 1 - 1 \otimes x_j)^{s_j} \in (\bigotimes_{j=1}^r \mathfrak{m}^{s_j})_r^{(\text{Aff}_r)_u}$ of weight $(s_1, \dots, s_r)$ with respect to $(k^\times)^r \subseteq \text{GL}_r k := \text{Aff}_r/(\text{Aff}_r)_u$. The central weights of $(\bigotimes_F^q (\mathfrak{m}/\mathfrak{m}^{N+1}))_r^{(\text{Aff}_r)_u}$ are contained in the interval $[q, qN]$, so $\bigotimes_{j=1}^r \mathfrak{m}^{s_j}$ is mapped to 0 if $\sum_{j=1}^r s_j \notin [q, qN]$. In particular, the morphisms factor through $\bigotimes_F^r (\mathfrak{m}/\mathfrak{m}^{qN-r+2})$, and are zero if $r > qN$.

The elements $\pi_\varphi := \bigotimes_{i=1}^q \prod_{\varphi(u)=i} (x_u \otimes 1 - 1 \otimes x_u)$ for all surjections $\varphi \in P$ span the $(\underbrace{1, \dots, 1}_r)$ -eigenspace of $(\bigotimes_F^q (\mathfrak{m}/\mathfrak{m}^{N+1}))_r^{(\text{Aff}_r)_u}$. Any morphism of $\bigotimes_F^r \mathfrak{m}$ is determined by the image $\sum_{\varphi \in P} \lambda_\varphi \pi_\varphi$ of the generator $\bigotimes_{j=1}^r (x_j \otimes 1 - 1 \otimes x_j)$ for some collection of $\lambda_\varphi \in k$. $\square$

Lemma 4.13. $\mathcal{A}$ splits as $\text{Vec}_k \oplus \mathcal{A}^\circ$, where Vec_k is the category of finite-dimensional k -vector spaces and $\mathcal{A}^\circ$ is the full subcategory of $\mathcal{A}$ with objects V such that $V^{G_{F/k}} = 0$.

Proof. For any $V \in \mathcal{A}$ set $V^\circ := \bigcap_{\varphi \in \text{Hom}_C(V, F)} \ker \varphi$. It follows from Theorem 4.10 and Lemma 4.12 that $V = (V^{G_{F/k}} \otimes_k F) \oplus V^\circ$, and $\text{Ext}_{\mathcal{A}}^*(F, \mathcal{A}^\circ) = \text{Ext}_{\mathcal{A}}^*(\mathcal{A}^\circ, F) = 0$. The equivalence is given by $V \mapsto (V^{G_{F/k}}, V^\circ)$. $\square$

Define the following decreasing “weight” filtration on the objects V of $\mathcal{A}$: $W^q V$ is the sum of the images of all morphisms to V from $\bigotimes_F^{\geq q} \mathfrak{m}$. Clearly, $W^\bullet$ is functorial and multiplicative. By Theorem 4.10, $gr_W^q V$ is a finite direct sum of direct summands of $\bigotimes_F^q \Omega_F^1$.

Corollary 4.14. $\mathcal{A}^\circ$ has no non-zero projective objects.

Proof. Let $P \in \mathcal{A}^\circ$ be a projective object and $\xi_2 : P \rightarrow S_F^\lambda \Omega_F^1$ be its irreducible quotient for a Young diagram λ , where $|\lambda|$ is minimal such that $W^{|\lambda|+1} P \neq P$. Then, for any $s \geq 2$, there is a lifting $\xi_s : P \rightarrow S_F^\lambda (\mathfrak{m}/\mathfrak{m}^s)$ of ξ_2 . By Theorem 4.10, there exist $q, a \geq 1$ and a morphism $\bigotimes_F^q (\mathfrak{m}/\mathfrak{m}^a) \rightarrow P$ such that its composition with ξ_2 is non-zero. Then its composition with any ξ_s is also non-zero. By Lemma 4.12, $\text{Hom}_{\mathcal{A}}(\bigotimes_F^q (\mathfrak{m}/\mathfrak{m}^a), S_F^\lambda (\mathfrak{m}/\mathfrak{m}^N)) = 0$ for any $N \geq a + q$, leading to contradiction. $\square$

Lemma 4.15. One has $\text{Ext}_{\mathcal{A}}^1(\bigotimes_F^q (\mathfrak{m}/\mathfrak{m}^N), V) = 0$ for any $V \in \mathcal{A}$ of finite length, $q \geq 1$ and $N >$ the maximal weight of V .

Proof. Induction on the length of V reduces the problem to the case of irreducible V . Let $0 \rightarrow V \rightarrow E \xrightarrow{\pi} \bigotimes_F^q (\mathfrak{m}/\mathfrak{m}^N) \rightarrow 0$ be an extension. By Theorem 4.10, there

is a surjection of a direct sum of objects of type $\bigotimes_F^p(\mathfrak{m}/\mathfrak{m}^a)$ onto E . By Lemma 4.12, $\mathrm{Hom}_{\mathcal{A}}(\bigotimes_F^{\neq q}(\mathfrak{m}/\mathfrak{m}^a), \bigotimes_F^q(\mathfrak{m}/\mathfrak{m}^2)) = 0$, so there is a morphism of a direct sum of objects of type $\bigotimes_F^q(\mathfrak{m}/\mathfrak{m}^a)$ to E surjective over $\bigotimes_F^q(\mathfrak{m}/\mathfrak{m}^2)$. As the latter is semi-simple, there is a morphism of $\bigoplus_{|\lambda|=q} S_F^\lambda(\mathfrak{m}/\mathfrak{m}^a)$ to E surjective over $\bigotimes_F^q(\mathfrak{m}/\mathfrak{m}^2)$. By Lemma 4.12, its composition with π is surjective, and therefore, the weights of its kernel are $\geq N$, so it does not intersect V . In other words, the extension splits. $\square$

Corollary 4.16. *The following pro-representable functor on $\mathcal{A}$*

$$\mathrm{Hom}_{\mathcal{C}}(\mathfrak{m}^{s_1} \otimes_F \cdots \otimes_F \mathfrak{m}^{s_q}, -) = \varinjlim \mathrm{Hom}_{\mathcal{A}}((\mathfrak{m}^{s_1}/\mathfrak{m}^N) \otimes_F \cdots \otimes_F (\mathfrak{m}^{s_q}/\mathfrak{m}^N), -)$$

is exact if and only if $s_1 = \cdots = s_q = 1$.

Proof. Let $V \rightarrow V'$ be a surjection in $\mathcal{A}$ and $\xi : \bigotimes_F^q \mathfrak{m} \rightarrow V'$ be a morphism in $\mathcal{C}$. We have to show that ξ factors through V . By Lemma 4.8, the image of ξ is cyclic. Let V'' be the cyclic sub-object of V generated by a pre-image of a generator of the image of ξ . Then the kernel K of $V'' \rightarrow \mathrm{Im}(\xi)$ is of finite length. As ξ factors through $\bigotimes_F^q(\mathfrak{m}/\mathfrak{m}^N)$ for some $N \gg 0$, and Lemma 4.15 implies that $\mathrm{Ext}_{\mathcal{A}}^1(\bigotimes_F^q(\mathfrak{m}/\mathfrak{m}^N), K) = 0$, ξ factors through V .

The rest follows from the fact that the projection $\mathfrak{m}^s \rightarrow \mathfrak{m}^s/\mathfrak{m}^{N+s}$ does not lift to $\mathfrak{m}^s \rightarrow \bigotimes_F^s(\mathfrak{m}/\mathfrak{m}^{N+1})$, if $s \geq 2$: neither non-zero morphism $\bigotimes_F^s \mathfrak{m} \rightarrow \bigotimes_F^s(\mathfrak{m}/\mathfrak{m}^{N+1})$ factors through $\mathfrak{m}^s$, if $N \geq 2$. $\square$

Corollary 4.17. *If $V \in \mathcal{A}$ is of finite type then $\dim_k \mathrm{Ext}_{\mathcal{A}}^j(V, V') < \infty$ for any $j \geq 0$ and any $V' \in \mathcal{A}$. If $V \in \mathcal{A}$ is irreducible and $\mathrm{Ext}_{\mathcal{A}}^1(\mathfrak{m}/\mathfrak{m}^q, V) \neq 0$ for some $q \geq 2$ then $V \cong \mathfrak{m}^q/\mathfrak{m}^{q+1}$ and $\mathrm{Ext}_{\mathcal{A}}^1(\mathfrak{m}/\mathfrak{m}^q, V) \cong k$.*

Proof. If $V \in \mathcal{A}$ is of finite type then, by Theorem 4.10, it admits a resolution $\cdots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0$ whose terms are finite direct sums of objects of type $\bigotimes_F^s \mathfrak{m}$. By Lemma 4.8, the terms of the complex $\mathrm{Hom}_{\mathcal{C}}(P_\bullet, V')$ are finite-dimensional over k and, by Corollary 4.16, it calculates $\mathrm{Ext}_{\mathcal{A}}^\bullet(V, V')$. $\square$

Corollary 4.18. *The filtration $W^\bullet$ is strictly compatible with the surjections.*

Proof. Let $V \rightarrow V'$ be a surjection in $\mathcal{A}$. Then, by Corollary 4.16, any morphism $\bigotimes_F^q \mathfrak{m} \rightarrow V'$ factors through V . $\square$

5. “COHERENT” SHEAVES IN SMOOTH TOPOLOGY

Let $\mathfrak{S}m_k$ be the category of locally dominant morphisms of smooth k -schemes. Consider on $\mathfrak{S}m_k$ the (pre-)topology, where the covers are surjective smooth morphisms. Clearly, the covers are stable under the base changes.

By definition, the structure presheaf $\mathcal{O}$ of $\mathfrak{S}m_k$ associates to any $Y \in \mathfrak{S}m_k$ its k -algebra of regular functions $\mathcal{O}(Y)$. Clearly, $\mathcal{O}$ is a sheaf in this topology.

A sheaf $\mathcal{F}$ on $\mathfrak{S}m_k$ is “(quasi-)coherent” if its values $\mathcal{F}(Y)$ are endowed with $\mathcal{O}(Y)$ -module structures and its restriction to the small étale site of Y is a (quasi-)coherent sheaf for any $Y \in \mathfrak{S}m_k$.

Lemma 5.1. *Let $X \rightarrow Y$ be an étale morphism of smooth varieties over k sending a point $q \in X$ to a point $p \in Y$. Then $\mathfrak{m}_q^s/\mathfrak{m}_q^N = \mathcal{O}_q \otimes_{\mathcal{O}_p} (\mathfrak{m}_p^s/\mathfrak{m}_p^N)$ for any $s \leq N$, where $\mathfrak{m}_q := \ker(\mathcal{O}_q \otimes_k \mathcal{O}_q \xrightarrow{\times} \mathcal{O}_q)$.*

Proof. One has $\mathfrak{m}_q/\mathfrak{m}_q^2 = \mathcal{O}_q \otimes_{\mathcal{O}_p} (\mathfrak{m}_p/\mathfrak{m}_p^2)$, so applying $\mathrm{Sym}_{\mathcal{O}_q}^s$ we get $\mathfrak{m}_q^s/\mathfrak{m}_q^{s+1} = \mathrm{Sym}_{\mathcal{O}_q}^s(\mathfrak{m}_q/\mathfrak{m}_q^2) = \mathcal{O}_q \otimes_{\mathcal{O}_p} \mathrm{Sym}_{\mathcal{O}_p}^s(\mathfrak{m}_p/\mathfrak{m}_p^2) = \mathcal{O}_q \otimes_{\mathcal{O}_p} (\mathfrak{m}_p^s/\mathfrak{m}_p^{s+1})$. The induction on $N - s$ gives the conclusion:

$$\begin{array}{ccccccc} 0 & \rightarrow & \mathfrak{m}_q^{s+1}/\mathfrak{m}_q^N & \rightarrow & \mathfrak{m}_q^s/\mathfrak{m}_q^N & \rightarrow & \mathfrak{m}_q^s/\mathfrak{m}_q^{s+1} \rightarrow 0 \\ & & \parallel & & \cup & & \parallel \\ 0 & \rightarrow & \mathcal{O}_q \otimes_{\mathcal{O}_p} (\mathfrak{m}_p^{s+1}/\mathfrak{m}_p^N) & \rightarrow & \mathcal{O}_q \otimes_{\mathcal{O}_p} (\mathfrak{m}_p^s/\mathfrak{m}_p^N) & \rightarrow & \mathcal{O}_q \otimes_{\mathcal{O}_p} (\mathfrak{m}_p^s/\mathfrak{m}_p^{s+1}) \rightarrow 0 \end{array}$$

□

Corollary 5.2. *The category $\mathcal{A}$ is equivalent to the category of “coherent” sheaves on $\mathfrak{S}m_k$, if $k = \overline{\mathbb{Q}}$.*

Proof. Fix an embedding over k of the function field of each connected component of each smooth k -variety into F . Then, for any $V \in \mathcal{A}$, $Y \in \mathfrak{S}m_k$ and a point $q \in Y$ define an $\mathcal{O}_q$ -lattice $\mathcal{V}_q \subset V^{G_{F/k(Y)}}$ as follows. Let $\mathcal{O}_p \subseteq \mathcal{O}_q$ be an étale extension of a local subring in F of a closed point p of a projective space.

Any object V of $\mathcal{A}$ is a quotient of a direct sum of objects of type $\bigotimes_F^s (\mathfrak{m}/\mathfrak{m}^N)$. Then, as it is true for $\bigotimes_F^s (\mathfrak{m}/\mathfrak{m}^N)$ (Lemma 5.1), it follows that the module $\mathcal{V}_p \subset V$ provided by the exact functor $\mathcal{S}$, cf. §1, is independent of the choice of the projective space, and $\mathcal{V}_q := \mathcal{O}_q \otimes_{\mathcal{O}_p} \mathcal{V}_p \subset V$ is independent of $\mathcal{O}_p$.

This determines a locally free coherent sheaf $\mathcal{V}_Y$ on Y with the generic fibre $V^{G_{F/k(Y)}}$.

It follows also that, for any dominant morphism $X \xrightarrow{\pi} Y$ of smooth k -varieties, the inclusion of the generic fibres $k(X) \otimes_{k(Y)} V^{G_{F/k(Y)}} \subseteq V^{G_{F/k(X)}}$ induces an injection of the coherent sheaves $\pi^* \mathcal{V}_Y \hookrightarrow \mathcal{V}_X$ on X , which is an isomorphism if π is étale.

To check that $\mathcal{V}$ is a sheaf on $\mathfrak{S}m_k$, we need to show that for any surjective smooth morphism $X \rightarrow Y$ the sequence $0 \rightarrow \mathcal{V}(Y) \xrightarrow{\beta} \mathcal{V}(X) \xrightarrow{p_1^* - p_2^*} \mathcal{V}(X \times_Y X)$ is exact. As $\mathcal{V}_X$ is a sheaf in Zariski topology on X , it suffices to treat the case of affine X and Y . In the case $V = \bigotimes_F^s (\mathfrak{m}/\mathfrak{m}^N)$, which is sufficient by Theorem 4.10 and Lemma 4.12, this amounts to the exactness of the sequence $0 \rightarrow \bigotimes_B^s (\mathfrak{m}_B/\mathfrak{m}_B^N) \rightarrow \bigotimes_A^s (\mathfrak{m}_A/\mathfrak{m}_A^N) \rightarrow \bigotimes_{A \otimes_B A}^s (\mathfrak{m}_{A,B}/\mathfrak{m}_{A,B}^N)$, where B is a smooth k -algebra of finite type, A is a smooth B -algebra of finite type, $\mathfrak{m}_C := \ker(C \otimes_k C \xrightarrow{\times} C)$ for any k -algebra C , and $\mathfrak{m}_{A,B} := \mathfrak{m}_{A \otimes_B A}$. But this is clear.

Conversely, a “coherent” sheaf $\mathcal{V}$ on $\mathfrak{S}m_k$ is sent to the object $\varinjlim \mathcal{V}(U)$, where U runs over the spectra of regular subalgebras in F of finite type over k . (As F is the union of its regular subalgebras of finite type over k , $\varinjlim \mathcal{V}(U)$ is an $(F = \varinjlim \mathcal{O}(U))$ -module. The action of an element $\sigma \in G$ comes as the limit of isomorphisms $\sigma^* : \mathcal{V}(U) \xrightarrow{\sim} \mathcal{V}(U')$, where $U = \mathbf{Spec}(A)$ and $U' = \mathbf{Spec}(\sigma(A))$ induced by the isomorphism $U' \xrightarrow{\sim} U$.) □

Lemma 5.3. *For any “quasi-coherent” flat (as $\mathcal{O}$ -module) sheaf $\mathcal{V}$ on $\mathfrak{S}m_k$ the k -space $\mathcal{V}(Y)$ is a birational invariant of proper Y . If $\mathcal{V}$ is “coherent” then $\mathcal{V}(Y')$ generates the (generic fibre of the) sheaf $\mathcal{V}_{Y'}$ for appropriate finite covers Y' of Y .*

Proof. According to Hironaka, for any pair of smooth proper birational k -varieties Y, Y'' there is a smooth proper k -variety Y' and birational k -morphisms $Y' \xrightarrow{\pi} Y$ and $Y' \rightarrow Y''$. Let $Z \subset Y$ be the subset consisting of points z such that $\pi : \pi^{-1}(z) \rightarrow z$ is not an isomorphism. It is a subvariety of codimension ≥ 2 . As $\mathcal{V}$ is torsion-free, one has $\mathcal{V}(Y) \rightarrow \mathcal{V}(Y') \xrightarrow{i^*} \mathcal{V}(U)$,⁶ where $U := Y - Z \xrightarrow{i} Y'$ is the section of π . It suffices

⁶To show that i^* is also injective, choose an affine covering $\{U_j\}$ of Y' , and a dense affine subset $U' \subseteq U$. As sum of ample divisors is ample, any intersection of open affine subsets is again affine, so $\{U_j \cap U'\}$ is an

to check that for any affine Y one has $\mathcal{V}(Y) \xrightarrow{\sim} \mathcal{V}(U)$. Choose an affine covering $\{U_j\}$ of U . Then $0 \rightarrow \mathcal{V}(U) \rightarrow \bigoplus_j \mathcal{O}(U_j) \otimes_{\mathcal{O}(Y)} \mathcal{V}(Y) \rightarrow \bigoplus_{i,j} \mathcal{O}(U_i \cap U_j) \otimes_{\mathcal{O}(Y)} \mathcal{V}(Y)$ is exact, so, as $0 \rightarrow \mathcal{O}(U) = \mathcal{O}(Y) \rightarrow \bigoplus_j \mathcal{O}(U_j) \rightarrow \bigoplus_{i,j} \mathcal{O}(U_i \cap U_j)$ is also exact, we get $\mathcal{V}(Y) = \mathcal{V}(U)$. $\square$

REMARK. If $\mathcal{V} : Y \mapsto \Omega_{k(Y)}^j / \Omega^j(Y)$ then the sequence $0 \rightarrow \Omega^j(Y) \rightarrow \Omega_{k(Y)}^j \rightarrow \mathcal{V}(Y) \rightarrow H^1(Y, \Omega_Y^j) \rightarrow 0$ is exact, so $\mathcal{V}(Y)$ is birationally invariant if and only if $j = 0$: for any closed smooth $Z \subset \mathbb{P}^{j+1} = Y$ of codimension 2 such that $\Omega^{j-1}(Z) \neq 0$ one has $H^1(Y', \Omega_{Y'}^j) \cong H^1(Y, \Omega_Y^j) \oplus \Omega^{j-1}(Z)$, where Y' is the blow-up of Y along Z .

Then, using Lemma 5.3, we get a left exact (non faithful) functor (with faithful restriction to the subcategory of “coherent” sheaves)

$$\{\text{flat “quasi-coherent” sheaves on } \mathfrak{S}m_k\} \xrightarrow{\Gamma} \{\text{smooth representations of } G_{F/k} \text{ over } k\}$$

given by $\mathcal{V} \mapsto \varinjlim \Gamma(Y, \mathcal{V}_Y)$, where Y runs over the smooth proper models of subfields in F of finite type over k . This functor is not full, and the objects in its image are highly reducible, e.g., $\Gamma(\Omega_{F/k}^1) \cong \bigoplus_A (A(F)/A(k)) \otimes_{\text{End}(A)} \Gamma(A, \Omega_{A/k}^1)$, where A runs over the set of isogeny classes of simple abelian varieties over k . If $\mathcal{V}$ is “coherent” and $\Gamma(Y, \mathcal{V}_Y)$ has the Galois descent property then $\Gamma(\mathcal{V})$ is admissible. However, there is no Galois descent property in general.

EXAMPLE. Let Y' be a smooth projective hyperelliptic curve $y^2 = P(x)$, considered as a 2-fold cover of the projective line Y . Then, for $\mathcal{V}_Y = (\Omega_{Y/k}^1)^{\otimes 2}$, the section $y^{-2}(dx)^2 = P(x)^{-1}(dx)^2$ is a Galois invariant element of $\Gamma(Y', \mathcal{V}_{Y'})$, which is not in $\Gamma(Y, \mathcal{V}_Y) = 0$.

6. $\mathcal{A}/\mathcal{A}_{>m}$

The only finite-dimensional objects of $\mathcal{A}$ are direct sums of copies of F , so the category $\mathcal{A}$ is far from being tannakian. However, $\mathcal{A}$ admits a decreasing filtration by Serre subcategories $\mathcal{A}_{>m}$ such that all $\mathcal{A}/\mathcal{A}_{>m}$ are again abelian tensor categories and their objects are finite-dimensional. The category $\mathcal{A}/\mathcal{A}_{>m}$ is not rigid.

Let $\mathcal{A}_{>m}$ be the full subcategory of $\mathcal{A}$ with objects V such that $V_m = 0$. Clearly, $\mathcal{A}_{>m}$ is a Serre subcategory of $\mathcal{A}$. Moreover, it is an “ideal” in $\mathcal{A}$ in the sense that the tensor product functor $\mathcal{A}_{>m} \times \mathcal{A} \rightarrow \mathcal{A}$ factors through $\mathcal{A}_{>m}$, so the quotient abelian category $\mathcal{A}/\mathcal{A}_{>m}$ carries a tensor structure.

By definition, the objects of $\mathcal{A}/\mathcal{A}_{>m}$ are the objects of $\mathcal{A}$, but the morphisms are defined by $\text{Hom}_{\mathcal{A}/\mathcal{A}_{>m}}(V, V') = \text{Hom}_{\mathcal{A}}(\langle V_m \rangle, V' / (V')_{>m}) = \text{Hom}_{\mathcal{A}/\mathcal{A}_{>m}}(V, \langle V'_m \rangle)$, where $\langle V_m \rangle$ denotes the semi-linear subrepresentation of V generated by V_m and $(V')_{>m}$ is the maximal subobject of V' in $\mathcal{A}_{>m}$.⁷ In particular, $V \cong \langle V_m \rangle$ in $\mathcal{A}/\mathcal{A}_{>m}$.

EXAMPLE. $\mathcal{A}/\mathcal{A}_{>0}$ is equivalent to the category of finite-dimensional k -vector spaces.

affine covering of U' . Then the diagram

$$\begin{array}{ccccc} & \mathcal{V}(Y') & \hookrightarrow & \bigoplus_i \mathcal{V}(U_i) & \\ & \downarrow i^* & & \downarrow \varphi & \\ \mathcal{V}(U) & \xrightarrow{\quad} & \mathcal{V}(U') & \hookrightarrow & \bigoplus_i \mathcal{V}(U_i \cap U') \end{array}$$

is commutative, and φ is injective since $\mathcal{V}(U_i \cap U') = \mathcal{O}(U_i \cap U') \otimes_{\mathcal{O}(U_i)} \mathcal{V}(U_i)$ and $\mathcal{V}$ is torsion-free.

⁷The functor $\mathcal{A} \rightarrow \mathcal{A}_{>m}$, $V \mapsto (V)_{>m}$ is right adjoint to inclusion functor $\mathcal{A}_{>m} \rightarrow \mathcal{A}$. In particular, it is left exact.

The functor $\mathcal{A}/\mathcal{A}_{>m} \rightarrow \mathfrak{S}\mathcal{L}_m^u$, $V \mapsto V_m$ is exact, faithful and tensor. Note also that the objects of $\mathcal{A}/\mathcal{A}_{>m}$ are *finite-dimensional*. Namely, $\bigwedge^{\dim_{K_m} V_m + 1} V = 0$.

Let Φ be a monoid of one-dimensional objects of $\mathcal{A}/\mathcal{A}_{>m}$, such as $(\Omega_{F/k}^m)^{\otimes N}$ for any $N \geq 0$. The set Φ is partially ordered: $\omega \leq \eta$ if there is $\xi \in \mathcal{A}/\mathcal{A}_{>m}$ such that $\eta \cong \omega \otimes \xi$. In particular, $\omega \leq \omega \otimes \eta$ and $\eta \leq \omega \otimes \eta$. If $k = \overline{\mathbb{Q}}$ then Φ consists of some (symmetric) F -tensor powers of $\Omega_{F/k}^m$.

Lemma 6.1. *The k -vector space $\text{Hom}_{\mathcal{A}/\mathcal{A}_{>m}}(V \otimes \omega, V' \otimes \omega)$ is finite-dimensional and independent of $\omega \in \Phi$ for ω sufficiently big.*

Proof. For any $\omega, \eta \in \Phi$ such that $\omega \leq \eta$ (i.e., $\eta \cong \omega \otimes \xi$ for some $\xi \in \mathcal{A}/\mathcal{A}_{>m}$) the twist by ξ defines a canonical inclusion $\text{Hom}_{\mathcal{A}/\mathcal{A}_{>m}}(V \otimes \omega, V' \otimes \omega) \subseteq \text{Hom}_{\mathcal{A}/\mathcal{A}_{>m}}(V \otimes \eta, V' \otimes \eta)$. The k -vector spaces

$$\begin{aligned} \text{Hom}_{\mathcal{A}/\mathcal{A}_{>m}}(V, V') &= \text{Hom}_{\mathcal{A}}(\langle V_m \rangle, V'/(V')_{>m}) \rightarrow \text{Hom}_{\mathcal{A}/\mathcal{A}_{>m}}(V \otimes U, V' \otimes U) \\ &= \text{Hom}_{\mathcal{A}}(\langle V_m \otimes_{K_m} U_m \rangle, (V' \otimes U)/(V' \otimes U)_{>m}) \subseteq \text{Hom}_{K_m \langle G_{K_m/k} \rangle}(V_m \otimes_{K_m} U_m, V'_m \otimes_{K_m} U_m) \end{aligned}$$

are finite-dimensional. On the other hand,

$$\begin{aligned} \text{Hom}_{\mathcal{A}}(\langle V_m \rangle, V'/(V')_{>m}) &\subseteq \text{Hom}_{K_m \langle G_{K_m/k} \rangle}(V_m, V'_m) \\ &\subseteq \text{Hom}_{K_m \langle G_{K_m/k} \rangle}(V_m \otimes_{K_m} U_m, V'_m \otimes_{K_m} U_m) \end{aligned}$$

for any $U \in \mathcal{C}$ with $U_m \neq 0$, where the second equality takes place if and only if $\dim_{K_m} U_m = 1$, e.g., for $U \in \Phi$. $\square$

Let $Ob(\mathcal{A}_{\Phi,m}^+) := Ob(\mathcal{A})$ and $\text{Hom}_{\mathcal{A}_{\Phi,m}^+}(V, V') := \text{Hom}_{\mathcal{A}/\mathcal{A}_{>m}}(V \otimes \omega, V' \otimes \omega)$ for sufficiently big $\omega \in \Phi$. Then $\otimes \omega : \mathcal{A}_{\Phi,m}^+ \rightarrow \mathcal{A}_{\Phi,m}^+$ is a fully faithful functor, so we can invert objects in Φ to get a category $\mathcal{A}_{\Phi,m} := \mathcal{A}_{\Phi,m}^+[\Phi^{-1}]$. If Φ is the set of all one-dimensional objects of $\mathcal{A}/\mathcal{A}_{>m}$ then $\mathcal{A}_m := \mathcal{A}_{\Phi,m}$ is tannakian.

Acknowledgement. I would like to thank Uwe Jannsen for many inspiring discussions, and in particular for suggesting (in Spring 2003) that smooth representations can be considered as sheaves in some topology. I am indebted to Dmitry Kaledin for thorough reading of a previous version of this paper and proposing numerous clarification. I am grateful to the Max-Planck-Institut für Mathematik in Bonn and to Regensburg University for their hospitality, and to the Max-Planck-Institut and to Alexander von Humboldt-Stiftung for the financial support.

REFERENCES

- [B] R.Bott, *Homogeneous vector bundles*, Ann. of Math. (2) 66 (1957), 203–248.
- [BT] A.Borel, J.Tits, *Homomorphismes "abstraites" de groupes algébriques simples*, Ann. of Math. (2) 97 (1973), 499–571.
- [D] M.Demazure, *A very simple proof of Bott's theorem*, Invent. Math. 33 (1976), no. 3, 271–272.
- [DG] M.Demazure, P.Gabriel, *Groupes algébriques*. Tome I. Masson; North-Holland, 1970.
- [LR] L.Lifschitz, A.Rapinchuk, *On abstract homomorphisms of Chevalley groups with nonreductive image*. I, J. Algebra 242 (2001), no. 1, 374–399.
- [R] M.Rovinsky, *Semi-linear representations of PGL*, math.RT/0306333, to appear in Selecta Math.
- [T] J.Tits, *Homomorphismes "abstraites" de groupes de Lie*, Symposia Mathematica, Vol. XIII (Convegno di Gruppi e loro Rappresentazioni, INDAM, Rome, 1972), pp. 479–499. Academic Press, London, 1974.

Independent University of Moscow
121002 Moscow
B.Vlasievsky Per. 11
`marat@mccme.ru`

and Institute for Information
Transmission Problems
of Russian Academy of Sciences