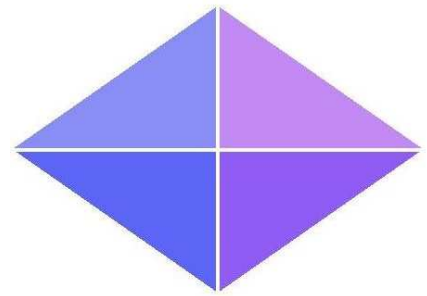


**DFG-Forschergruppe
Regensburg/Leipzig**
Algebraische Zyklen und
L-Funktionen



Slice filtration on motives and
the Hodge conjecture
(with an appendix by J. Ayoub)

Annette Huber

Preprint Nr. 13/2005

Slice filtration on motives and the Hodge conjecture (with an appendix by J. Ayoub*)

Annette Huber**

¹ Mathematisches Institut, Augustusplatz 10/11, 04109 Leipzig, Germany

Key words motives, slice filtration, weight filtration, generalized Hodge conjecture

MSC (2000) Primary: 14F42; Secondary: 14C30, 14C25, 18E30

We clarify the expected properties of the slice filtration on triangulated motives from the point of view of the generalized Hodge conjecture. In the appendix, J. Ayoub proves unconditionally that the slice filtration does not respect geometric motives.

Copyright line will be provided by the publisher

Introduction

The slice filtration on Voevodsky's triangulated category of motives is defined by effectivity conditions. It is constructed and studied in [HK]. An analogous filtration on the homotopy category was introduced by Voevodsky.

For Artin-Tate motives, the weight filtration agrees with the slice filtration (see [HK]). In the abelian (as opposed to derived) category of Hodge structures it is possible to reconstruct the weights from the slice filtration and its dual. So I had hoped to define the weight filtration on motives from the slice filtration. B. Kahn pointed out that my construction was assuming a number of nice (maybe too nice) properties of the slice filtration.

In this note we try to get a - conjectural - picture of these properties by systematic use of the realization functor to the derived category of Hodge structures. A key ingredient is Grothendieck's Generalized Hodge Conjecture about the analogous filtration on pure Grothendieck motives.

This approach is successful, even if the answers are contrary to what I had hoped for. (In fact, the application to the weight filtration does not work.) An old example of Griffiths's allows to deduce - using deep but standard conjectures - the following (see Proposition 5.3):

1. The slice filtration does *not* respect the subcategory of geometrical motives.
2. The slice filtration does *not* commute with the weight filtration.
3. The induced slice filtration functors on the (conjectural) abelian category of mixed motives are left exact but *not* exact.

Moreover, the induced filtration on the (conjectural) abelian category of pure motives agrees with the coniveau filtration. As a byproduct of our considerations, Grothendieck's Generalized Hodge Conjecture is generalized to triangulated motives. The generalization is implied by the same set of conjectures.

J. Ayoub communicated a non-conditional argument for property 1. to me. It is given in an appendix. This may be read as a confirmation for the conjectural picture we have of the theory of motives.

The note was written in context of the joint project with B. Kahn on the slice filtration and its properties, see [HK]. I would like to thank him heartily for many interesting discussions. Several people helped me in my hunt for a good example. I am indebted to H. Esnault, B. Herzog, U. Jannsen, B. Moonen, A. Mukherjee and C. Voisin. It is a pleasure to thank them. I would also like to thank J. Ayoub for writing the Appendix and the referee for suggesting that it be written.

* e-mail: ayoub@math.jussieu.fr

** e-mail: huber@mathematik.uni-leipzig.de,¹

1 Definition of the slice filtration

We review the construction of the slice filtration as constructed in [HK]. For the purpose of the present article it suffices to work over the field of complex numbers $\mathbb{C}$. Most results extend to all fields of characteristic zero. We restrict to a $\mathbb{Q}$ -rational theory.

Let $DM_{gm} = DM_{gm}(\mathbb{C}) \otimes \mathbb{Q}$ be Voevodsky's category of geometrical motives ([V] section 2.1), $DM_{gm}^{eff} = DM_{gm}^{eff}(\mathbb{C}) \otimes \mathbb{Q}$ the full subcategory of effective motives ([V] Definition 2.1.1). Let DM_{-}^{eff} the category of bounded above complexes of Nisnevich sheaves with transfers which have homotopy invariant cohomology, i.e., Voevodsky's category of motivic complexes ([V] section 3.1). Let DM_{-} be the category obtained from DM_{-}^{eff} by formally inverting the Tate object. Quasi-invertibility (e.g. [HK] Prop. A.1) can be used to show that there is a natural full embedding

$$\iota : DM_{-}^{eff} \rightarrow DM_{-}.$$

In all, there is a commutative diagram of full embeddings

$$\begin{array}{ccc} DM_{gm}^{eff} & \longrightarrow & DM_{gm} \\ \downarrow & & \downarrow \\ DM_{-}^{eff} & \xrightarrow{\iota} & DM_{-} \end{array}$$

Lemma 1.1 ([HK], Prop. 1.1) *The functor ι has a right adjoint τ , i.e., $\tau : DM_{-} \rightarrow DM_{-}^{eff}$ and a natural transformation $\tau \rightarrow \text{id}$ s.t.*

$$\text{Hom}_{DM_{-}}(\iota N, M) = \text{Hom}_{DM_{-}^{eff}}(N, \tau(M))$$

for all $N \in DM_{-}^{eff}$, $M \in DM_{-}$.

Proof. Let M, N as in the lemma. By definition, $M(m)$ is effective for m big enough. We put

$$\tau(M) := \lim_{m \rightarrow \infty} \underline{\text{Hom}}_{DM_{-}^{eff}}(\mathbb{Q}(m), M(m))$$

where $\underline{\text{Hom}}$ is internal Hom in DM_{-}^{eff} ([V] 3.2). By quasi-invertibility of $\mathbb{Z}(1)$, the limit stabilizes: if $M = \tilde{M}(-m)$ with $\tilde{M} \in DM_{-}^{eff}$, $m \geq 0$, then

$$\tau(M) = \underline{\text{Hom}}_{DM_{-}^{eff}}(\mathbb{Q}(m), \tilde{M}).$$

The universal property is easy to check. □

For $n \in \mathbb{Z}$ let $DM_{-}^{\geq n} = DM_{-}^{eff}(n)$. There is a sequence of functors

$$\nu^{\geq n} : DM_{-} \rightarrow DM_{-}^{\geq n}$$

right adjoint to the embedding. Explicitly:

$$\nu^{\geq n}(M) = \tau(M(-n))(n).$$

Definition 1.2 ([HK] (1.1)) The sequence of transformations

$$\nu^{\geq n} \rightarrow \nu^{\geq n-1} \rightarrow \dots \rightarrow \text{id}$$

is called the *slice filtration*.

The same type of filtration is also considered by Voevodsky in terms of homotopy theory of schemes.

Let $DM_{\leq n}^{\leq n}$ be the category of motives on which $\nu^{\geq n+1}$ vanishes. Quite formally one deduces from the adjunction properties of the slice filtration the existence of a sequence of functors

$$\nu_{\leq n} : DM \rightarrow DM_{\leq n}^{\leq n}$$

sitting in natural distinguished triangles

$$\nu^{\geq n} \rightarrow \text{id} \rightarrow \nu_{\leq n-1} \rightarrow \nu^{\geq n}[1].$$

Example 1.3 Let $M = \mathbb{Q}(n)$. Then

$$\tau(M) = \begin{cases} \underline{\text{Hom}}(\mathbb{Q}, \mathbb{Q}(n)) = \mathbb{Q}(n) & n \geq 0, \\ \underline{\text{Hom}}(\mathbb{Q}(-n), \mathbb{Q}) = 0 & n < 0. \end{cases}$$

Lemma 1.4 ([HK] Prop. 1.7) Let $M = M^c(X)$ where X is a variety of dimension at most d . Then

$$\nu^{\geq n} M = \begin{cases} M & n \leq 0, \\ 0 & n > d. \end{cases}$$

Proof. The first part only says that M is effective. The second assertion follows from duality and the known facts on motivic cohomology with values in $\mathbb{Q}$. For details, see [HK], Prop. 1.7. $\square$

This means that the slice filtration is finite, separated and exhaustive on geometrical motives. However, we do not know:

Question 1.5 If $M \in DM_{\text{gm}}$, is it true that all $\nu^{\geq n} M$ are geometric?

Contrary to my original hope, I expect the answer to be no in general, see 5.3 below for an argument relying on conjectures. A non-conditional argument by Ayoub is given in the appendix.

2 Slice filtration on mixed Hodge structures

In order to understand what the slice filtration means let us first consider the toy model of mixed Hodge structures. We denote $\mathcal{H}$ the category of mixed polarizable $\mathbb{Q}$ -Hodge structures. A Hodge structure is called *effective* if its non-zero Hodge numbers are concentrated in the first quadrant. The category of effective Hodge structure is denoted $\mathcal{H}^{\text{eff}}$. Note that this category is stable under subquotients and extensions. A mixed Hodge structure is effective if and only if its simple subquotients are effective.

Lemma 2.1 The inclusion $\iota : \mathcal{H}^{\text{eff}} \rightarrow \mathcal{H}$ has a left adjoint τ , i.e., $\tau : \mathcal{H} \rightarrow \mathcal{H}^{\text{eff}}$ and natural transformation $\text{id} \rightarrow \tau$ s.t.

$$\text{Hom}_{\mathcal{H}}(N, \iota M) = \text{Hom}_{\mathcal{H}^{\text{eff}}}(\tau N, M)$$

for all $N \in \mathcal{H}$, $M \in \mathcal{H}^{\text{eff}}$.

Proof. This is just linear algebra. Let H be an object of $\mathcal{H}$. Then we define τH as the biggest quotient of H which is effective. We have to verify existence of this biggest quotient. Consider the set of all effective quotients of H ordered by the natural projections. This is an Artinian category. Suppose $p_1 : H \rightarrow H^1$ and $p_2 : H \rightarrow H^2$ are two effective quotients. Let K^i be the kernel of p_i and $K = K^1 \cap K^2$. We consider $H \rightarrow H/K$. It dominates H^1 and H^2 . As subobject of $H^1 \oplus H^2$ the quotient H/K is effective. Hence the category of effective quotients has a unique maximal object. It is functorial. It is easy to check the universal property. $\square$

Remark 2.2 We have switched from right adjoints in motives to left adjoints in Hodge structures. This corresponds to the fact that the Hodge realization functor is contravariant.

Let $\mathcal{H}^{\geq n} = \mathcal{H}^{\text{eff}}(-n)$. There is a sequence of functors

$$\nu^{\geq n} : \mathcal{H} \rightarrow \mathcal{H}^{\geq n}$$

left adjoint to the embedding. Explicitly:

$$\nu^{\geq n}(H) = \tau(H(n))(-n) .$$

Definition 2.3 The sequence of transformations

$$\text{id} \rightarrow \dots \rightarrow \nu^{\geq n-1} \rightarrow \nu^{\geq n}$$

is called the *slice cofiltration*.

If $\tau H = 0$, this does *not* mean that H has Hodge numbers only outside the first quadrant. It is easy to write down a simple, pure Hodge structure of weight 0 with Hodge type $\{(-1, 1), (0, 0), (1, -1)\}$. This Hodge structure has no effective quotient! This effect only occurs with $\mathbb{Q}$ -Hodge structures as every simple $\mathbb{R}$ -Hodge structure has Hodge type of the form $\{(p, q), (q, p)\}$ or $\{(p, p)\}$. Indeed, the slice functors become exact on $\mathbb{R}$ -Hodge structures.

Lemma 2.4 The functors $\nu^{\geq n}$ are right exact but not exact on $\mathcal{H}$.

Proof. It suffices to consider τ . The functor τ is right exact because it is a left adjoint of an exact functor. Assume now that τ is exact. Let H be a simple polarizable Hodge structure of positive weight which is *not* effective and $H^\vee$ its dual. Note that $H^\vee$ is not effective either. Let E be a non-trivial extension

$$0 \rightarrow \mathbb{Q}(0) \rightarrow E \rightarrow H \rightarrow 0 .$$

They are classified by

$$\text{Ext}_{\mathcal{H}}^1(H, \mathbb{Q}(0)) = \text{Ext}_{\mathcal{H}}^1(\mathbb{Q}(0), H^\vee) = \text{Coker}(H_{\mathbb{Q}}^\vee \oplus F^0 H_{\mathbb{C}}^\vee \rightarrow H_{\mathbb{C}}^\vee) \neq 0 .$$

In fact, this is an infinite dimensional $\mathbb{Q}$ -vector space because $H^\vee$ is not effective. Hence E exists. We apply τ to the sequence and get

$$0 \rightarrow \mathbb{Q}(0) \rightarrow \tau E \rightarrow 0 \rightarrow 0$$

because $\mathbb{Q}(0)$ is effective and H is not effective but simple. The isomorphism $\mathbb{Q}(0) \rightarrow \tau E$ together with the projection $E \rightarrow \tau E$ splits the original sequence, contradiction. $\square$

3 Hodge conjecture

Recall ([H1] 2.3.5, [H2]) that there is a *Hodge realization functor*

$$\underline{R}_{\mathcal{H}} : D\mathcal{M}_{\text{gm}} \longrightarrow D^b(\mathcal{H}) .$$

We write $\underline{H}_{\mathcal{H}}(M) = \bigoplus H^i(\underline{R}_{\mathcal{H}}(M)) \in \mathcal{H}$.

If X is a smooth variety, then by construction $\underline{H}_{\mathcal{H}}(M(X))$ is singular cohomology $H^*(X(\mathbb{C}), \mathbb{Q})$ of the complex manifold $X(\mathbb{C})$ with the Hodge structure defined by Deligne [D1] Theorem 3.2.5 (iii).

If M is effective, then $\underline{H}_{\mathcal{H}}(M)$ is also effective. What about the converse? This is the set-up of the generalized Hodge conjecture.

Let $\mathcal{M}$ be Grothendieck's category of pure motives up to homological equivalence, see e.g. [S] 1.4.

Conjecture 3.1 (Hodge) The functor $\underline{H}_{\mathcal{H}} : \mathcal{M} \rightarrow \mathcal{H}$ is fully faithful.

In more down to earth terms this says something about (p, p) -cycles. There was also a more general conjecture by Hodge for (p, q) -cycles. It was “false for trivial reasons” as Grothendieck pointed out. The corrected version is:

Conjecture 3.2 (GHC Grothendieck [G]) *The Hodge conjecture holds and a pure motive $M \in \mathcal{M}$ is effective if and only if $\underline{H}_{\mathcal{H}}(M)$ is effective.*

This usually goes by the name of generalized Hodge conjecture. I propose to extend the conjecture to DM_{gm} .

Conjecture 3.3 (GHC for triangulated motives) *The Hodge conjecture holds and an object $M \in DM_{\text{gm}}$ is effective if and only if its Hodge realization is effective.*

Why should this be true?

Today's standard conjectures

- GHC for pure Grothendieck motives up to homological equivalence (3.2).
- DM_{gm} admits a t -structure τ^{mot} . Its heart $\mathcal{MM}$ (mixed motives) contains $\mathcal{M}$ as full subcategory. For each object of DM_{gm} the filtration induced by the truncation functors $\tau_{\leq n}^{\text{mot}}$ is finite, separated and exhaustive.
- There are weight filtration functors $W_{\leq n}$ on DM_{gm} which commute with the t -structure and such that the pure objects in $\mathcal{MM}$ are in $\mathcal{M}$. For each object of DM_{gm} the filtration induced by the truncation functors $W_{\leq n}$ is finite, separated and exhaustive.
- The functor $\underline{H}_{\mathcal{H}}$ is compatible with t -structure and weights.

The cohomological functor of the motivic t -structure τ^{mot} is denoted H^i . Note that the Hodge realization is contravariant. This implies that $\underline{H}_{\mathcal{H}}(H^i(X)) = \underline{H}_{\mathcal{H}}^{-i}(X)$. We normalize the weight filtration such that

$$\underline{H}_{\mathcal{H}}(W_{\leq n}M) = \underline{H}_{\mathcal{H}}(M)/W_{-(n+1)}\underline{H}_{\mathcal{H}}(M),$$

i.e., a pure motive of weight n is mapped to a pure Hodge structure of weight $-n$. Note that this means $M(X)$ has cohomology in non-positive degrees and non-positive weights. If X is a smooth proper variety the conjectures imply that $H^i(X)$ is pure of weight i .

Proposition 3.4 *We assume the above conjectures. Then:*

1. *The functor $\underline{H}_{\mathcal{H}}$ is conservative on DM_{gm} , i.e., if $\underline{H}_{\mathcal{H}}(M) = 0$ then $M = 0$.*
2. *A pure Grothendieck motive is effective in $\mathcal{M}$ if and only if it is effective in DM_{gm} .*
3. *An object $M \in DM_{\text{gm}}$ is effective if and only if all $H^i(M)$ are effective in $\mathcal{MM}$ and if and only if all $\text{Gr}_j H^i(M)$ are effective in $\mathcal{M}$.*
4. *GHC holds for triangulated motives, i.e., conjecture 3.3 is true.*

Proof. We start with assertion 1. By today's standard conjectures, the H^i and Gr_j^W are conservative and commute with $\underline{H}_{\mathcal{H}}$. This reduces the question to pure Grothendieck motives. In this case it is the faithfulness part of the Hodge conjecture.

Now consider assertion 2. Suppose M is an effective object of $\mathcal{M}$. By the Hodge conjecture, $\mathcal{M}$ is a full subcategory of the semi-simple category of polarizable pure Hodge structures, hence semi-simple. Without restriction we may assume that M is pure of weight $-i$. By definition this means that it is a direct summand of $H^{-i}(X)$ for a smooth projective variety X . By the Hodge conjecture, the $H^{-i}(X)$ satisfy hard Lefschetz. By a general argument of Deligne (see [D2]) this implies that in DM_{gm}

$$M(X) = \bigoplus H^{-i}(X)[i].$$

Hence M is a direct summand of $M(X)[-i]$. As $DM_{\text{gm}}^{\text{eff}}$ is pseudo-abelian, this implies that M is also effective viewed as object of DM_{gm} . Conversely, if M is in $\mathcal{M} \cap DM_{\text{gm}}^{\text{eff}}$, then its Hodge realization is effective. By GHC for pure motives, M is an object of M^{eff} .

For property 3, note that $\underline{H}_{\mathcal{H}}(DM_{\text{gm}}^{\text{eff}}) \subset \mathcal{H}^{\text{eff}}$ and that $DM_{\text{gm}}^{\text{eff}}$ is stable under triangles: if two vertices of a triangle are effective, then so is the third. Now let M be a triangulated motive such that $\underline{H}_{\mathcal{H}}(M) =$

$\bigoplus \underline{H}_{\mathcal{H}}(H^{-i}(M))$ is effective. Hence all $H^{-i}(\mathrm{Gr}_j^W M)$ are effective in $\mathcal{M}$ by the GHC for $\mathcal{M}$. By the considerations above this implies that all $H^{-i}(\mathrm{Gr}_j^W M)$ are effective in DM_{gm} . The motive M is successive extension of effective objects, hence effective.

This implies the non-trivial part of GHC for triangulated motives. The remaining statements follows from GHC for DM_{gm} . $\square$

Lemma 3.5 *If the motivic t -structure and the weight filtration exist on DM_{gm} , they extend to a t -structure and weight filtration on DM_- .*

Proof. We use two facts on DM_- .

- Every object is third vertex in a distinguished triangle where the other two are (infinite) direct sums of geometrical motives.
- For direct sums of geometrical motives

$$\mathrm{Hom}_{DM_-}(\bigoplus_{i \in I} M_i, \bigoplus_{j \in J} N_j) = \prod_{i \in I} \bigoplus_{j \in J} \mathrm{Hom}_{DM_{\mathrm{gm}}}(M_i, N_j).$$

Consider the smallest full subcategory $\tau_{\leq n}^{\mathrm{mot}} DM_-$ of DM_- which contains $\tau_{\leq n}^{\mathrm{mot}} M$ for all geometrical motives M , is closed under direct sums and such that if $M_1 \rightarrow M_2 \rightarrow M_3$ is a distinguished triangle with M_1, M_3 in $\tau_{\leq n}^{\mathrm{mot}} DM_-$, then M_2 in $\tau_{\leq n}^{\mathrm{mot}} DM_-$. Dually we define $\tau_{\geq n+1}^{\mathrm{mot}} DM_-$. We claim that this is a t -structure. The vanishing of morphisms and behaviour under shifts follows from the above facts. We also need to check that for all $M \in DM_-$ there are $\tau_{\leq n}^{\mathrm{mot}} M \in \tau_{\leq n}^{\mathrm{mot}} DM_-$ and $\tau_{\geq n+1}^{\mathrm{mot}} M \in \tau_{\geq n+1}^{\mathrm{mot}} DM_-$ such that there is a distinguished triangle

$$\tau_{\leq n}^{\mathrm{mot}} M \rightarrow M \rightarrow \tau_{\geq n+1}^{\mathrm{mot}} M.$$

This holds for geometrical motives and extends to direct sums of geometrical motives. Let $M_1 \rightarrow M_2 \rightarrow M_3$ be a distinguished triangle and assume that truncation is defined on M_2 and M_3 . Let

$$\tilde{H}^n(M_3) = \mathrm{Im}(H^n(M_2) \rightarrow H^n(M_3)).$$

A modified truncation of M_3 is defined by the diagram of triangles

$$\begin{array}{ccccc} \tau_{\leq n-1}^{\mathrm{mot}} M_3 & \longrightarrow & \tau_{\leq n}^{\mathrm{mot}} M_3 & \longrightarrow & H^n(M_3)[-n] \\ \uparrow & & \uparrow & & \uparrow \\ \tau_{\leq n-1}^{\mathrm{mot}} M_3 & \longrightarrow & \tau'_{\leq n} M_3 & \longrightarrow & \tilde{H}^n(M_3)[-n] \end{array}$$

By construction the map $\tau_{\leq n}^{\mathrm{mot}} M_2 \rightarrow \tau_{\leq n}^{\mathrm{mot}} M_3$ factors through $\tau'_{\leq n} M_3$. We define $\tau_{\leq n}^{\mathrm{mot}} M_1$ by the distinguished triangle

$$\tau_{\leq n}^{\mathrm{mot}} M_1 \rightarrow \tau_{\leq n}^{\mathrm{mot}} M_2 \rightarrow \tau'_{\leq n} M_3.$$

Define $\tau'_{\geq n+1} M_3$ and $\tau_{\geq n+1}^{\mathrm{mot}} M_1$ as third vertices in the distinguished triangles

$$\begin{aligned} \tau'_{\leq n} M_3 &\rightarrow M_3 \rightarrow \tau'_{\geq n+1} M_3, \\ \tau_{\leq n}^{\mathrm{mot}} M_1 &\rightarrow M_1 \rightarrow \tau_{\geq n+1}^{\mathrm{mot}} M_1. \end{aligned}$$

We automatically get a distinguished triangle

$$\tau_{\geq n+1}^{\mathrm{mot}} M_1 \rightarrow \tau_{\geq n+1}^{\mathrm{mot}} M_2 \rightarrow \tau'_{\geq n+1} M_3.$$

It is easy to see that $\tau_{\leq n}^{\mathrm{mot}} M_1 \in \tau_{\leq n}^{\mathrm{mot}} DM_-$ and $\tau_{\geq n+1}^{\mathrm{mot}} M_1 \in \tau_{\geq n+1}^{\mathrm{mot}} DM_-$.

We skip the arguments for the weight filtration, which are simpler. $\square$

We now can use GHC in order to get a conjectural understanding of the slice filtration. The functor $\nu^{\geq n}$ on $D\mathcal{M}_{\text{gm}}$ defines $H^0\nu^{\geq n}$ on $\mathcal{MM}$.

Proposition 3.6 *Assume again today's standard conjectures. An object $M \in \mathcal{MM}$ is in $\mathcal{MM}^{\text{eff}}(n)$ if and only if $H^0\nu^{\geq n}M = M$. The functors $H^0\nu^{\geq n}$ and $H^0\nu_{\leq n}$ are left exact on $\mathcal{MM}$. The functor $H^0\nu^{\geq n}$ is right adjoint to the inclusion $\iota : \mathcal{MM}^{\text{eff}}(n) \rightarrow \mathcal{MM}$.*

One should think of $\nu^{\geq 0}$ as the derived functor of $H^0\nu^{\geq 0}$.

Proof. It suffices to consider $n = 0$. Let $M \in \mathcal{MM}$. That the motive M is effective means $M = \nu^{\geq 0}M$, in particular $H^i(\nu^{\geq 0}M) = 0$ for $i \neq 0$. Conversely, assume $M = H^0(\nu^{\geq 0}M)$. Clearly $\nu^{\geq 0}M$ is effective and hence also its H^0 .

For left exactness let $M \in \mathcal{MM}$ be a mixed motive. Consider the distinguished triangle

$$\nu^{\geq 0}M \rightarrow M \rightarrow \nu_{>0}M.$$

We first want to show that $H^i(\nu_{<0}M)$ is effective for $i \neq 0$. Consider the long exact sequence with respect to H^i . It yields isomorphisms

$$H^i\nu_{<0}M \rightarrow H^{i+1}\nu^{\geq 0}M$$

for $i \neq -1, 0$. The object on the right is effective hence so is the object on the left. The same sequence yields

$$0 \rightarrow H^{-1}\nu_{<0}M \rightarrow H^0\nu^{\geq 0}M \rightarrow M \rightarrow H^0\nu_{<0}M \rightarrow H^1\nu^{\geq 0}M \rightarrow 0.$$

It remains to show that subobjects of effective motives in $\mathcal{MM}$ are effective: this is true by GHC for triangulated motives because the dual is true in $\mathcal{H}$. Hence by adjunction

$$\text{Hom}(\tau_{<0}^{\text{mot}}\nu_{<0}M, \nu_{<0}M) = \text{Hom}(\tau_{<0}^{\text{mot}}\nu_{<0}M, \nu_{\geq 0}\nu_{<0}M) = 0,$$

i.e., $\tau_{<0}^{\text{mot}}M = 0$. In particular, $H^0\nu_{<0}$ is left exact. By the above isomorphisms this also implies that $H^0\nu^{\geq 0}$ is left exact.

Let $M \in \mathcal{MM}$, $N \in \mathcal{MM}^{\text{eff}}$. Then

$$\begin{aligned} \text{Hom}_{\mathcal{MM}}(N, M) &= \text{Hom}_{D\mathcal{M}_{\text{gm}}}(N, M) = \text{Hom}_{D\mathcal{M}_{\text{eff}}}(N, \nu^{\geq 0}M) \\ &= \text{Hom}_{D\mathcal{M}_{\text{eff}}}(N, H^0\nu^{\geq 0}M) = \text{Hom}_{\mathcal{MM}}(N, H^0\nu^{\geq 0}M). \end{aligned}$$

The crucial third equality holds because N and $\nu^{\geq 0}M$ have cohomology concentrated in degree 0 and in non-negative degrees respectively. $\square$

Remark 3.7 Recall that the t -structure on $D\mathcal{M}_{\text{gm}}$ extends to a t -structure on $D\mathcal{M}_{-}$. The proposition implies that $\nu^{\geq n}$ is also left exact on $D\mathcal{M}_{-}$.

Question 3.8 *Does the slice filtration commute with the weight filtration?*

I think that the answer is no, see 5.3 below for an argument relying on conjectures.

4 Coniveau filtration

In this section we concentrate on pure motives. As a left exact functor, $H^0\nu^{\geq n}$ respects the category of pure motives $\mathcal{M}$. In fact, on a simple pure motive it is either the zero or the full object. We are going to review Grothendieck's coniveau filtration. Note that we have to reverse all arrows because we use covariant motives whereas his setting was contravariant.

Definition 4.1 (compare [G]) Let X be a smooth proper variety. The *coniveau filtration* on $M = H^{-i}(X)$ is defined as

$$F^p M = M / \text{Im} \left(\bigoplus H^{-i}(U) \right)$$

where the sum runs over all open subvarieties $U \subset X$ such that $T = X \setminus U$ has codimension at least p .

Alternatively, $F'^p M$ can be described as the smallest quotient of M such that all Gysin morphisms $H^{-i}(X) \rightarrow H^{-i+2q}(\tilde{T})(q)$ for all $\tilde{T} \rightarrow X$ with $\tilde{T}$ smooth, projective, $\dim X - \dim \tilde{T} = q \geq p$ factor through $F'^p M$.

Proposition 4.2 *Assume today's standard conjectures. Let X be smooth and proper, $M = H^{-i}(X)$. Then the composition*

$$H^0 \nu^{\geq p} M \rightarrow M \rightarrow F'^p M$$

is an isomorphism, hence the slice filtration provides a splitting of the coniveau filtration.

Proof. The key observation is that the $H^{-i+2q}(\tilde{T})(q)$ of the alternative description are in $DM^{\geq q} \subset DM^{\geq p}$. Hence a simple constituent of M which is not in $DM^{\geq p}$ is also mapped to zero in $F'^p M$. Conversely, let $M' \subset M$ be a simple direct summand which is in $DM^{\geq p}$. It is direct summand of some $H^{-i+2p}(Y)(p)$, with Y smooth and projective. The projection $M \rightarrow M'$ is induced by a morphism of motives $\phi : H^{-i}(X) \rightarrow H^{-i+2p}(Y)(p)$. We assume that pure motives in $\mathcal{MM}$ are Grothendieck motives, hence this morphism is represented by a cycle T in $X \times Y$ with $\dim X - \dim T = p$. Let $\tilde{T}$ be a desingularization of T . Then ϕ is the composition $H^{-i}(X) \rightarrow H^{-i+2p}(\tilde{T})(p) \rightarrow H^{-i+2p}(Y)(p)$. As M' is a direct summand of $H^{-i+2p}(Y)(p)$, it is also a direct summand of $H^{-i+2p}(\tilde{T})(p)$. This implies that M' does not vanish in $F'^p M$ either. $\square$

GHC for pure motives can be formulated as saying that $H^0 \nu^{\geq p}$ commutes with the Hodge realization functor. We ask if this can be extended to the triangulated case.

Question 4.3 *Does $\nu^{\geq n}$ on DM_- commute with the Hodge realization?*

In order for this question to make sense, we first have to extend the Hodge realization to a functor on DM_- . It will have values in $D^+(\text{Pro-}\mathcal{H})$ where $\text{Pro-}\mathcal{H}$ is the pro-category of Hodge structures. The question can be reduced to the case of $M \in \mathcal{M}$. However, I do not have a guess for the answer.

5 The counterexample

Let X be a smooth projective variety and Z a cohomologically trivial cycle of codimension 2. By the Abel-Jacobi map it induces an extension of mixed Hodge structures

$$0 \rightarrow \underline{H}_{\mathcal{H}}^3(X) \rightarrow H_Z \rightarrow \mathbb{Q}(-2) \rightarrow 0.$$

Lemma 5.1 *Let X be a generic quintic in $\mathbb{P}^4$, $H = \underline{H}_{\mathcal{H}}^3(X)$. Then H is a simple Hodge structure of weight 3 with $H^{3,0} \neq 0$. The image of the Abel-Jacobi map in $\text{Ext}^1(\mathbb{Q}(-2), \underline{H}_{\mathcal{H}}^3(X))$ is not finite dimensional.*

Proof. Quintics in $\mathbb{P}^4$ are simply connected Calabi-Yau threefolds and very well studied. In particular, their H^3 is primitive and has Hodge type

$$(3, 0), (2, 1), (1, 2), (0, 3).$$

By [PS] Corollary 18 it is simple for a generic X . The Abel-Jacobi map on homologically trivial cycles was studied by Griffiths and Clemens in this example. By [C1] Theorem 0.2 or [C2] Theorem 6 its image in $\text{Ext}^1(\mathbb{Q}(-2), \underline{H}_{\mathcal{H}}^3(X))$ is not finite dimensional. $\square$

I would like to thank C. Voisin and B. Moonen for pointing these arguments out to me.

Corollary 5.2 *Assume today's standard conjectures. Let X be as in the lemma, $M = H^{-3}(X)^{\vee}(2)$. Then $H^0 \nu^{\geq 0} M = 0$ and $\text{Ext}_{\mathcal{MM}}^1(\mathbb{Q}(0), M)$ is infinite dimensional.*

Proof. $M = H^3(X)^{\vee}(2)$ is simple because its Hodge realization is simple. Moreover, its Hodge realizations is of type $(-1, 2), (0, 1), (1, 0), (2, -1)$, i.e., M is not effective. Hence it does not have any effective subobjects. By duality and functoriality

$$\text{Ext}_{\mathcal{MM}}^1(\mathbb{Q}(0), M) \cong \text{Ext}_{\mathcal{MM}}^1(H^{-3}(X), \mathbb{Q}(2)) \rightarrow \text{Ext}_{\mathcal{H}}^1(\mathbb{Q}(-2), \underline{H}_{\mathcal{H}}^3(X)).$$

The Abel-Jacobi map factors through this map. By the Lemma the dimension of the Ext-group has to be infinite. $\square$

Proposition 5.3 (see Questions 1.5 and 3.8) *Assume today's standard conjectures. Then $H^0\nu^{\geq n}$ is not exact, the functors $\nu^{\geq n}$ do not respect geometrical motives and do not commute with the weight filtration.*

Proof. It suffices to consider $n = 0$. Let M be as in the corollary. Consider a non-trivial extension

$$0 \rightarrow M \rightarrow E \rightarrow \mathbb{Q}(0) \rightarrow 0.$$

The long exact sequence for $H^i\nu^{\geq 0}$ starts

$$0 \rightarrow 0 \rightarrow H^0\nu^{\geq 0}E \rightarrow \mathbb{Q}(0) \rightarrow H^1\nu^{\geq 0}M \rightarrow \dots$$

If $H^0\nu^{\geq 0}$ was exact, then $H^0\nu^{\geq 0}E \cong \mathbb{Q}(0)$. If $\nu^{\geq 0}$ commuted with the weight filtration, then $H^1\nu^{\geq 0}M$ would be pure of weight 1. The boundary map would vanish and again $H^0\nu^{\geq 0}E \cong \mathbb{Q}(0)$. In both cases this isomorphism together with the inclusion $H^0\nu^{\geq 0}E \rightarrow E$ would split the sequence and we would have a contradiction.

Now assume that $\nu^{\geq 0}M$ is geometric. We have

$$\mathrm{Hom}_{D\mathcal{M}_{\mathrm{gm}}}(\mathbb{Q}, M[1]) = \mathrm{Hom}_{D\mathcal{M}_{\mathrm{eff}}}(\mathbb{Q}, \nu^{\geq 0}M[1]).$$

As M is simple and non-effective, $H^0\nu^{\geq 0}M = 0$. By assumption $H^1\nu^{\geq 0}M$ is geometric, hence

$$\mathrm{Hom}_{D\mathcal{M}_{\mathrm{gm}}}(\mathbb{Q}, M[1]) = \mathrm{Hom}_{D\mathcal{M}_{\mathrm{eff}}}(\mathbb{Q}, H^1\nu^{\geq 0}M).$$

For pure motives, we have

$$\mathrm{Hom}_{\mathcal{M}}(\mathbb{Q}, N) = \mathrm{Hom}_{\mathcal{H}}(\underline{H}_{\mathcal{H}}(N), \mathbb{Q})$$

by the Hodge conjecture, in particular this is a finite dimensional vector space. Hence this is also true for mixed motives. This contradicts the infinite dimensionality established in corollary 5.2. $\square$

Remark 5.4 Our results are over the complex numbers. The situation may be different over a number field, where the group of cohomologically trivial cycles is expected to be *finite dimensional*. In this special case, the slice filtration might still respect geometric motives. The other two assertions of the proposition depend on the existence of just some non-trivial element in this group.

A The slice filtration on $\mathrm{DM}(k)$ does not preserve geometric motives

by Joseph Ayoub

In this appendix we give an unconditional argument for the following (un)-property of the slice filtration on $\mathrm{DM}(k)$:

Proposition A.0.1 *The slice filtration on $\mathrm{DM}(k)$ does not preserve geometric motives.*

Recall (Definition 1.2) that the slice filtration is a sequence of transformations:

$$\nu^{\geq n} \rightarrow \nu^{\geq n-1} \rightarrow \dots \rightarrow \mathrm{id}$$

where $\nu^{\geq n}(M) = \tau(M(-n))(n)$ with $\tau : \mathrm{DM}(k) \rightarrow \mathrm{DM}_{\mathrm{eff}}(k)$ the right adjoint to the full embedding $\mathrm{DM}_{\mathrm{eff}}(k) \subset \mathrm{DM}(k)$. When M is effective (e.g., the motive $M(X)$ of a smooth projective variety X) we have by the proof of Lemma 1.1 $\tau(M(-n)) = \underline{\mathrm{Hom}}_{\mathrm{eff}}(\mathbb{Z}(n), M)$ where $\underline{\mathrm{Hom}}_{\mathrm{eff}}$ stands for the internal hom in $\mathrm{DM}_{\mathrm{eff}}(k)$. We will prove the following:

Proposition A.0.2 *Assume that k is big enough. There exists a smooth projective k -variety X such that $\underline{\mathrm{Hom}}_{\mathrm{eff}}(\mathbb{Z}(1), M(X))$ is not a geometric motive.*

We will implicitly assume k of characteristic zero and algebraically closed. We also work with rational coefficients for simplicity.

A.1 Compactness in $\mathbf{DM}(k)$

Recall the following classical notions (see [N]):

Definition A.1.1 Let $\mathcal{T}$ be a triangulated category with arbitrary infinite sums. An object $U \in \mathcal{T}$ is called compact if the functor $\mathrm{hom}_{\mathcal{T}}(U, -) : \mathcal{T} \rightarrow \mathcal{A}b$ commutes with sums. The category $\mathcal{T}$ is compactly generated, if there exists a set $\underline{G}$ of compact objects in $\mathcal{T}$ such that the family of triangulated functors $\mathrm{hom}_{\mathcal{T}}(U[n], -)$, where $U \in \underline{G}$ and $n \in \mathbb{Z}$, is conservative (that is detects isomorphisms).

If $\mathcal{T}$ is compactly generated by $\underline{G}$ then the subcategory $\mathcal{T}_{\mathrm{comp}}$ of compact objects is the pseudo-abelian envelop of the triangulated sub-category of $\mathcal{T}$ generated by $\underline{G}$.

Let $(A_n)_{n \in \mathbb{N}}$ be an inductive system in $\mathcal{T}$. Its homotopy colimit is the cône of:

$$(\mathrm{id} - s) : \bigoplus_{n \in \mathbb{N}} A_n \rightarrow \bigoplus_{n \in \mathbb{N}} A_n$$

where s is the composition $A_{n_0} \rightarrow A_{n_0+1} \rightarrow \bigoplus_{n \in \mathbb{N}} A_n$ on the factor A_{n_0} . It is denoted by $\mathrm{hocolim}_{n \in \mathbb{N}} A_n$. We have the following lemma:

Lemma A.1.2 If $U \in \mathcal{T}$ is compact, then $\mathrm{hom}_{\mathcal{T}}(U, -)$ commutes with $\mathbb{N}$ -indexed homotopy colimits.

The following proposition is well-known. It follows immediately from the commutation of Nisnevich hypercohomology with infinite sums of complexes:

Proposition A.1.3 The category $\mathbf{DM}_{\mathrm{eff}}(k)$ is compactly generated by the set of $\mathbf{M}(X)$ with X in a set representing isomorphism classes of smooth k -varieties. Moreover the sub-category $\mathbf{DM}_{\mathrm{eff}}^{\mathrm{gm}}(k)$ is the sub-category of compact objects of $\mathbf{DM}_{\mathrm{eff}}(k)$.

A.2 Finite generation in $\mathbf{HI}(k)$

Recall that $\mathbf{DM}_{\mathrm{eff}}(k)$ admits a natural t -structure whose heart $\mathbf{HI}(k)$ is the category of homotopy invariant Nisnevich sheaves with transfers. For an object $M \in \mathbf{DM}_{\mathrm{eff}}(k)$ we denote $h_i(M)$ the truncation with respect to this t -structure. Recall that $h_i(M)$ is simply the i -th homology sheaf of the complex M . We will also write $h_i(X)$ for $h_i(\mathbf{M}(X))$ when X is a smooth k -variety. We make the following definition:

Definition A.2.1 A sheaf $F \in \mathbf{HI}(k)$ is called finitely generated if there exists a smooth variety X and a surjection $h_0(X) \twoheadrightarrow F$.

It is clear that the property of being finitely generated is stable by quotients. It is also stable by extensions. Indeed, let $F \subset G$ in $\mathbf{HI}(k)$ such that F and G/F are finitely generated and chose surjections $a : h_0(X) \twoheadrightarrow F$ and $b : h_0(Y) \twoheadrightarrow G/F$. There exists a Nisnevich cover $U \rightarrow Y$ such that $b|_U$ lifts to $b' : h_0(U) \twoheadrightarrow G$. We get in this way a surjection $a \amalg b' : h_0(X \amalg U) \twoheadrightarrow G$.

Assuming that k is countable we say that a sheaf F is countable if for any smooth k -variety X the set $F(X)$ is countable. Note the following technical lemma:

Lemma A.2.2 Let F be a sheaf in $\mathbf{HI}(k)$ which is countable. There exists a chain $(S_n)_{n \in \mathbb{N}}$ of finitely generated sub-sheaves of F such that $F = \bigcup_{n \in \mathbb{N}} S_n$.

Proof. Consider the set $\mathcal{S}$ whose elements are the finitely generated sub-sheaves of F . This set is countable as every finitely generated sub-sheaf of F is the image of a map $a : h_0(X) \rightarrow F$ with X a smooth k -variety and $a \in F(X)$. Fix a bijection $b : \mathbb{N} \xrightarrow{\sim} \mathcal{S}$ and denote $S_n = \sum_{i=0}^n b(i)$. We clearly have that $F = \bigcup_{n \in \mathbb{N}} S_n$. $\square$

As a corollary we have the following:

Proposition A.2.3 Let F be a countable sheaf in $\mathbf{HI}(k)$. Suppose that $\mathrm{hom}_{\mathbf{HI}(k)}(F, -)$ commutes with $\mathbb{N}$ -indexed colimits. Then F is finitely generated.

Proof. By lemma A.2.2 we can write $F = \mathrm{colim}_{n \in \mathbb{N}} (S_n)$ with S_n finitely generated sub-sheaves of F . Using $\mathrm{hom}(F, F) = \mathrm{colim} \mathrm{hom}(F, S_n)$ one can find $n_0 \in \mathbb{N}$ such that the identity of F factors through the inclusion $S_{n_0} \subset F$. This implies that $F = S_{n_0}$. $\square$

Remark A.2.4 By working a little bit more, one shows under the hypothesis of A.2.3 that F is finitely presented in the sense that there exists an exact sequence:

$$h_0(X_2) \longrightarrow h_0(X_1) \longrightarrow F \longrightarrow 0$$

with X_1 and X_2 two smooth k -varieties.

A.3 conclusion

Using propositions A.1.3 and A.2.3 we can prove the following:

Theorem A.3.1 *Let M be a geometric motive in $\mathbf{DM}_{\text{eff}}(k)$. Suppose that $h_i(M) = 0$ for $i < 0$. Then $h_0(M)$ is finitely generated⁶.*

Proof. The motive M being geometric, it is defined over a finitely generated field (in particular a countable one). Hence, we may assume our base field k countable. It follows that the sheaves $h_i(M)$ are countable. This can be proved by reducing to the case $M = M(X)$ with X a smooth k -variety and using Voevodsky's identification $M(X) = C_*\mathbb{Z}_{\text{tr}}(X)$ with C_* the Suslin-Voevodsky complex.

By A.2.3 we need only to check that $\text{hom}_{\mathbf{HI}(k)}(h_0(M), -)$ commutes with $\mathbb{N}$ -colimits. Let $(A_n)_{n \in \mathbb{N}}$ be an inductive system and denote A its colimit. First, remark that A is also the homotopy colimit of $(A_n)_{n \in \mathbb{N}}$ in $\mathbf{DM}_{\text{eff}}(k)$. Indeed, one has an exact triangle:

$$\oplus A_n \xrightarrow{\text{id} - s} \oplus A_n \longrightarrow \text{hocolim} A_n \longrightarrow$$

It is easy to see that the morphism of sheaves $\text{id} - s$ is injective. It follows that $\text{hocolim}_{n \in \mathbb{N}} A_n$ is the co-kernel of $\text{id} - s$ which is canonically isomorphic to A .

Having this in mind, we can write:

$$\begin{aligned} \text{hom}_{\mathbf{HI}(k)}(h_0(M), \text{colim} A_n) &\stackrel{1}{=} \text{hom}_{\mathbf{DM}_{\text{eff}}(k)}(h_0(M), \text{hocolim} A_n) \\ &\stackrel{2}{=} \text{hom}_{\mathbf{DM}_{\text{eff}}(k)}(M, \text{hocolim} A_n) \stackrel{3}{=} \text{colim} \text{hom}_{\mathbf{DM}_{\text{eff}}(k)}(M, A_n) \\ &\stackrel{4}{=} \text{colim} \text{hom}_{\mathbf{HI}(k)}(h_0(M), A_n) \end{aligned}$$

Equality (1) follows from the above discussion. Equalities (2) and (4) follow from the condition $h_i(M) = 0$ for $i < 0$. Equality (3) is the compactness of M . This proves the theorem. $\square$

Let X be a smooth projective variety of dimension d . Using [V], Theorem 4.2.2 and Proposition 4.2.3 we have:

- the sheaf $h_i(\text{Hom}_{\text{eff}}(\mathbb{Z}(1)[2], M(X)))$ is zero for $i < 0$,
- the sheaf $h_0(\text{Hom}_{\text{eff}}(\mathbb{Z}(1)[2], M(X)))$ is canonically isomorphic to the Nisnevich sheaf $\text{CH}_{/X}^{d-1}$ associated to the pre-sheaf: $U \rightsquigarrow \text{CH}^{d-1}(U \times_k X)$.

To prove A.0.2 it suffices by A.3.1 to find a smooth projective variety X of dimension $d = 3$ such that $\text{CH}_{/X}^{d-1}$ is not finitely generated. To do this, we will construct a quotient of $\text{CH}_{/X}^{d-1}$ which is constant but not finitely generated.

Definition A.3.2 *Let U be a smooth k -scheme. A cycle $[Z] \in \text{CH}^{d-1}(U \times_k X)$ is said to be U -algebraically equivalent to zero if there exist a smooth and connected U -scheme $V \rightarrow U$, a finite correspondence of degree zero $\sum_i n_i [T_i] \in \text{Cor}(V/U)$ (i.e. $n_i \in \mathbb{Z}$ and $t_i : T_i \rightarrow U$ are finite and surjective) and a cycle $[W] \in \text{CH}^{d-1}(V \times_k X)$ such that $[Z]$ is rationally equivalent to $\sum_i n_i (t_i \times \text{id}_X)_* [W \cup (T_i \times X)]$.*

We denote $\text{NS}^{d-1}(U \times_k X)_U$ the quotient of $\text{CH}^{d-1}(U \times_k X)$ with respect to the U -algebraic equivalence. We let also $\text{NS}_{/X}^{d-1}$ be the Nisnevich sheaf associated to the pre-sheaf $U \rightsquigarrow \text{NS}^{d-1}(U \times_k X)_U$.

⁶ In fact $h_0(M)$ is even finitely presented (see A.2.4)

We have clearly a surjective morphism $\mathrm{CH}_{/X}^{d-1} \rightarrow \mathrm{NS}_{/X}^{d-1}$. The latter sheaf is constant (because our base field k is algebraically closed). Indeed, for any finitely generated extension $k \subset K$ we have $\mathrm{NS}_{/X}^{d-1}(K) = \mathrm{NS}^{d-1}(X \otimes_k K)$. It is a well-known fact that the Neron-Severi group is invariant by extensions of an algebraically closed field.

Now, it is easy to see that a constant sheaf is finitely generated if and only if its group of sections over k is a finite dimensional $\mathbb{Q}$ -vector space (using that a map from $h_0(X)$ to a constant sheaf factors through $\mathbb{Q}_{\mathrm{tr}}(\pi_0(X))$ with $\pi_0(X)$ the set of connected components of the variety X). We are done since $\mathrm{NS}^2(X)$ is not finite dimensional for a generic quintic in $\mathbb{P}^4$ (see [C1] Theorem 0.2).

References

- [C1] H. Clemens, Homological equivalence, modulo algebraic equivalence, is not finitely generated. *Inst. Hautes Études Sci. Publ. Math.* No. 58, (1983), 19–38 (1984).
- [C2] H. Clemens, Some results about Abel-Jacobi mappings. in: *Topics in transcendental algebraic geometry. Proceedings of a seminar held at the Institute for Advanced Study, Princeton, N.J., during the academic year 1981/1982.* Edited by Phillip Griffiths. *Annals of Mathematics Studies*, 106. Princeton University Press, Princeton, NJ, 1984.
- [D1] P. Deligne, Théorie de Hodge II, *Publ. Math. de IHES* 40 (1971), 5–57.
- [D2] P. Deligne, Décompositions dans la catégorie dérivée. *Motives* (Seattle, WA, 1991), 115–128, *Proc. Sympos. Pure Math.*, 55, Part 1, Amer. Math. Soc., Providence, RI, 1994.
- [G] A. Grothendieck, Hodge’s general conjecture is false for trivial reasons. *Topology* 8 (1969), 299–303.
- [H1] A. Huber, Realization of Voevodsky’s motives. *J. Algebraic Geom.* 9 (2000), no. 4, 755–799.
- [H2] A. Huber, Corrigendum: Realization of Voevodsky’s motives, *J. Algebraic Geom.* 13 (2004), no. 1, 195–207.
- [HK] A. Huber, B. Kahn, The Slice filtration and mixed Tate motives, Preprint 2005, to appear *Compositio Math.*
- [N] A. Neeman, *Triangulated categories*, *Annals of Mathematics Studies*, 148. Princeton University Press, Princeton, NJ, 2001.
- [PS] C. Peters, J. Steenbrink, *Monodromy of Variations of Hodge structure, Monodromy and differential equations* (Moscow, 2001). *Acta Appl. Math.* 75 (2003), no. 1-3, 183–194.
- [S] A.J. Scholl, *Classical motives. Motives* (Seattle, WA, 1991), 163–187, *Proc. Sympos. Pure Math.*, 55, Part 1, Amer. Math. Soc., Providence, RI, 1994.
- [V] V. Voevodsky, *Triangulated categories of motives over a field. Cycles, transfers, and motivic homology theories*, 188–238, *Ann. of Math. Stud.*, 143, Princeton Univ. Press, Princeton, NJ, 2000.

Annette Huber

Math. Institut, Augustusplatz 10/11, Universität Leipzig, 04109 Leipzig

huber@mathematik.uni-leipzig.de

Joseph Ayoub

Inst. de Math. de Jussieu, 175–179 rue du Chevaleret, 75013 Paris, France ayoub@math.jussieu.fr